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The Computation of Urban Land Use Equilibria*

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1. INTRODUCTION

In the past decade a small group of economists (e.g., Beckmann [2], Mills [8], Muth [11], Solow [21]) have developed purely analytic models which attempt to explain certain pervasive characteristics of urban land use. These models deal primarily with monocentric cities with radial transportation networks, and emphasize the determinants of residential location. The outputs of such models usually take the form of rent or housing price gradients and population density gradients. The chief weaknesses of these models are the drastic simplifying assumptions which are made in order to obtain analytic solutions. Identical preferences and/or incomes across all consumers, Cobb–Douglas utility and production functions, constant returns to scale, and absence of externalities and public sector activities are assumptions which are frequently made in these models.

Recently very powerful methods for computing fixed points of well-behaved mappings have been developed by Scarf [18] and others. It is the purpose of this paper to show how these fixed point computational methods can be used in simulating urban land use equilibria.

The natural question to ask is why simulate such equilibria, since one cannot prove any qualitative theorems with such an approach. The main reason is that qualitative theorems may be virtually impossible to obtain when more realistic features are introduced into the current crop of purely analytic models. When builders of purely analytic models such as Solow and Muth have recently sought to incorporate more realistic features in their models, such as transportation congestion, they have found it necessary to use ad hoc numerical methods to obtain solutions (see [12, 19, 20]).

Given that one must resort to simulation, the fixed point methods appear

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* Provision of computer time by the Computer Research Center, National Bureau of Economic Research in Cambridge, Massachusetts, is gratefully acknowledged. FIX-POINT, an interactive computer system developed at the Computer Research Center, was used in conjunction with a computer program written by the author in performing the simulations reported in this paper.

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to have a number of advantages over such ad hoc numerical methods. First, they are guaranteed to converge, whereas many of the ad hoc methods are not. Second, we strongly suspect that they can accommodate broader characterizations of technology and tastes. Third, it seems reasonable to develop a computational framework which can handle a wide variety of problems, rather than creating a separate ad hoc method for each different problem investigated.

The first published paper dealing with the application of fixed point methods to urban economics was the ingenious paper by MacKinnon [5]. Our algorithm differs substantially from his, especially in the manner in which consumers are allocated across space. Loosely speaking, given land prices in various parts of the urban area, MacKinnon allocates his consumers so that excess demand for land is zero, even if this procedure prohibits some consumers from living at their most preferred location. Then his algorithm finds a price vector where the allocation is also in accord with consumer preferences. We essentially do the reverse—allocating consumers on the basis of their preferences, and letting the algorithm find a price vector which fixes up excess demands. King [4] describes a fixed point algorithm for a constant returns to scale urban economy whose treatment of the spatial allocation of consumers is more in the spirit of our approach than MacKinnon's. However, King's algorithm transforms what economists normally regard as ordinary consumption decisions into production decisions, and introduces a number of "artificial" commodities which consumers purchase. Our algorithm does not require any such transformation, nor is it restricted to constant returns to scale technologies. Arnott and MacKinnon [1] have exploited the structure of a more specialized urban economy to obtain an efficient fixed point algorithm. Their contribution is discussed more fully in Section 6. The procedure developed in our paper appears to be far less cumbersome than MacKinnon's [5], and more general than any of the models alluded to above, especially in integrating residential location and production.

Fixed point algorithms have been used to compute general equilibrium prices in an Arrow–Debreu–McKenzie economy. The attributes of such an economy that the algorithms crucially exploit are the Walras law (the sum of the value of excess demand over all goods whose prices are endogenous is nonpositive for all possible price vectors) and the homogeneity of degree zero of excess demands in the vector of endogenous prices. These two attributes are missing from many of the analytic urban models in the literature, which frequently assume certain prices and consumer income are exogenous. In Section 2 we develop a description of an urban economy which does satisfy the Walras law and homogeneity postulate. This is done in the context of a static long-run equilibrium model which is deliberately kept simple for illustrative purposes, with more complicated models reserved for a later section. Unlike most of the analytic urban models, land is treated as a discrete commodity rather than a continuous variable. Incorporation of consumer locational choice is the significant analytical difference between our model and the more conventional A–D–M world.

Section 3 describes the basic Scarf algorithm, which underlies the computational routine for urban land use equilibria described in Section 4. Section 5 relaxes some of the assumptions of the model described in Section 2. In particular we indicate how urban models which do not satisfy the Walras law and homogeneity postulate can be treated in our framework, and we also sketch how our algorithm can be used to simulate phenomena such as zoning, racial discrimination, multiple workplaces, and other more realistic features of an urban economy. Section 6 presents a numerical example of our algorithm, and Section 7 contains some concluding comments.

2. Basic Assumptions and Definitions

(A) *Geography*

We deal with the traditional monocentric city composed of a circular central business district (CBD) surrounded by an annular zone in which land is used for residential purposes, the transportation network, and other production activities. There is a fixed amount of land available, which we divide into γ circular rings. The first ring, of width ρ_0, must contain the CBD. Thus ρ_0 is an upper bound on the radius of the CBD. The remaining $(\gamma - 1)$ circular rings, each of width ρ, exhaust the rest of the total land area.

(B) *Transportation*

All travel is radial travel to and from the CBD. The amount of land devoted to the transportation network in each ring is exogenous. We assume that travel time to and from the center of the city is a function only of distance and that there is no congestion.

(C) *Production*

There are two classes of nonland commodities. There are n_0 goods (class I goods) which can only be produced using a contiguous area of land closest to the city center within ring 1. We shall say that these goods are produced in the CBD. The clustering of the activities producing such goods is usually explained by appealing to various types of externalities, which will be exogenous to our model.

There are n_1 class II goods, whose defining trait is that their production is not restricted to land nearest the center. Examples would be agricultural goods and housing in particular rings. All nonproduced goods such as labor are also included in this class. We initially assume that labor is an input only in activities producing class I goods, and that all goods except labor can be transported to any location in the city at zero real cost.[1]

Let $s = (s_1, \ldots, s_\eta)$, where $\eta = \gamma + n_0 + n_1$. s_j (for $j = 1, \ldots, \gamma$) denotes the input of land from ring j, s_j (for $j = \gamma + 1, \ldots, \gamma + n_0$) denotes the net production of a class I good, and s_j (for $j = \gamma + n_0 + 1, \ldots, \eta$) denotes the net production of a class II good. Negative coordinates represent inputs. Technically feasible production of class I goods is described by a production set Y_{I}, which is assumed to satisfy:

(C.1) $s \in Y_{\mathrm{I}}$ implies $s_j \leqq 0$ for all $j \in \{1, \ldots, \gamma\}$, and if $s \nleq 0$, then $s_k > 0$ for at least one $k \in \{\gamma + 1, \ldots, \gamma + n_0\}$, $s_1 < 0$, and $\exists \bar{s} \in Y_{\mathrm{I}}$ with $\bar{s}_k = s_k$ for $k \notin \{2, \ldots, \gamma\}$ and $\bar{s}_k = 0$ for $k \in \{2, \ldots, \gamma\}$.

This assumption says that land is not an output and insures that any efficient production of class I goods will involve the use of land in ring 1 only. This is one way of recognizing the clustering of activities while keeping the causes of such clustering exogenous to the model.[2]

Y_{I} describes production which is restricted to ring 1. Technically feasible production activity which is not restricted to ring 1 (though it may choose to locate there) is described by the production set Y_{II}, which is assumed to satisfy:

(C.2) $s \in Y_{\mathrm{II}}$ implies $s_j \leqq 0$ for all $j \in \{1, \ldots, \gamma + n_0\}$, and $s \nleq 0$ implies $s_k < 0$ for some $k \in \{1, \ldots, \gamma\}$.

Condition (C.2) reflects the restriction of class I goods production to ring 1, though allows class I goods to be used as inputs in class II goods production. It also states that land is a necessary input in production.[3] Class II goods can be given a wide variety of interpretations. In particular, some of them may be agricultural goods. Given our assumption of costless transport, agriculture would tend to use the land most distant from the city center, thus determining the outer boundary of the residential zone of the city.[4]

---

[1] Alternatively we could incorporate costly transportation by assuming that all goods whose transport is costly are marketed at the center of the city (see footnote 4, below).

[2] Alternatively, in our description of production possibilities we could have required fewer inputs (such as transportation inputs) per unit output of class I goods for land closer to the center; thus encouraging use of land closest to the center for production of these goods without explicitly restricting their production to such an area.

[3] *Such an assumption is quite reasonable at this level of aggregation.*

[4] *One way of incorporating costly transport for certain goods would be to assume that the market for these goods is in the CBD. Then, for example, we could define one of the* class II goods as "transport services." The amount of input of this good needed in the production of some output would then be tied to the ring from which the land input for that output is drawn. These transport services could reflect the costs of transporting inputs from the CBD and/or outputs to the CBD.

We have summarized production by means of the two aggregate production sets Y_{I} and Y_{II}, but it would be straightforward to construct them from a more disaggregated description of production. The aggregate production set for the urban economy is $Y = Y_{\text{I}} + Y_{\text{II}}$. Y is assumed to satisfy the following properties:

(C.3) $0 \in Y$.

(C.4) Y is closed, convex and exhibits free disposal.[5]

Finally, all producers are assumed to be competitive profit maximizers. All profits are distributed to households according to some exogenous formula.

(D) *Households*

There are m different types of households, with a large number r_i ($i = 1,..., m$) of the ith type. All households of the same type have the same tastes and initial endowments. Each household has one worker who commutes to the CBD.

Each household of type i has an initial semipositive endowment η-vector w^i/r_i, where the total endowment vector $w = (w_1,..., w_\eta)$ equals $\sum_{i=1}^{m} w^i$. We assume that only land which is not devoted to the transportation network is privately owned, so that w_j (for $j = 1,..., \gamma$) equals the amount of land in ring j which is available for residential or production purposes.[6]

If a household of type i lives in ring j, its preferences are defined by a utility function $u_j^i(y^{ij})$ defined on a consumption set X_j^i. y^{ij} is a η-vector representing the consumption bundle of a type i household that lives in ring j. The kth coordinate of y^{ij} corresponds to the kth good as defined by the vectors belonging to the production set Y.[7]

---

[5] We could have imposed the convexity assumption on each of the sets Y_{I} and Y_{II} and let Y inherit this property. However, one would like to allow Y_{I} to be nonconvex, in order to include the notion that increasing returns to scale has something to do with the formation of a CBD. But unless there is a reasonably general economic interpretation of the types of sets Y_{I} and Y_{II}, with Y_{I} nonconvex, that are compatible with convexity of Y, we are not much better off than we would be by simply assuming Y_{I} and Y_{II} are convex. If we assume that one of the reasons for the existence of a CBD is that land located closer to the center has a higher productivity in the production of class I goods (an approach which Mills [8], in part, uses), then the convexity assumption on Y_{I} becomes more plausible. As more class I goods are produced, land of lower productivity must be used (where implicitly the definition of Y_{I} assumes that land in ring 1 closest to the center is used first). Nevertheless, one should recognize that the description of CBD production is somewhat unsatisfactory.

[6] It would be easy to make alternative assumptions about land ownership, such as public ownership with rents being redistributed to consumers.

[7] The subscript j associated with u_j^i allows us to incorporate a number of aspects of the problem directly in the definition of the utility function. In particular, u_j^i would give heavier weight to components of y^{ij} representing goods identified with ring j. For example,

For simplicity of exposition we assume that leisure time is a constant, so that time spent working and commuting is also a constant. Neither commuting nor working time is assumed to affect utility directly. The only cost of commuting will be time costs. Let τ_j ($j = 1,..., \gamma$) denote the round trip commuting time between ring j and the city center.[8] Then we adopt the convention that $y^{ij}_{\gamma+1}$ represents the time spent commuting by a household of type i that lives in ring j. Hence we require that $y^{ij} \in X_j^i$ imply $y^{ij}_{\gamma+1} = \tau_j$ and that $y^{ij}_{\gamma+1}$ not affect utility directly. $s_{\gamma+1}$ represents the use of labor in production, and $w^i_{\gamma+1}/r_i$ represents the total time available for commuting and working to a type i household.[9]

We make the following additional assumptions:

(D.1) For all i, j

(a) u_j^i is continuous.

(b) There exist parameters $\bar{\zeta}_k^{ij}$ ($k = 1,..., \eta$) such that u_j^i is nonincreasing in the kth coordinate of $y^{ij} \in X_j^i$ if $y_k^{ij} \geq \bar{\zeta}_k^{ij}$.[10] (We use the convention that $\bar{\zeta}^{ij}_{\gamma+1} = \tau_j$.)

(c) If $u_j^i(y) > u_j^i(\bar{y})$, then $u_j^i(\lambda y + (1 - \lambda)\bar{y}) > u_j^i(\bar{y})$ for all $\lambda \in (0, 1)$.

(D.2) X_j^i is a closed, convex subset of R^η (the nonnegative orthant of Euclidean η-space) for all i, j.

(D.3) There exists a commodity h such that for all i, j

---

land or housing in ring j would presumably get a heavier weight in u_j^i than land or housing in a different ring.

[8] Each household has one member who commutes to the CBD, and as an approximation we assume he commutes to the center of the CBD. Commuting time is the same from any residential location in ring j. This is a reasonable approximation if the widths of rings 2 through γ are chosen sufficiently small, and the width of the residential zone in ring 1 (which is determined endogenously) is small.

[9] Many of the above assumptions could easily be relaxed. For example, real costs of transportation could be incorporated in the definition of X_j^i, by requiring certain levels of consumption of goods (such as gas) which are used as inputs in the transportation process. One way to treat alternative travel modes would be to parameterize the τ_j and X_j^i by the mode. Thus choice of a faster mode might be associated with more costly restrictions on the consumption set. In the context of models which do not satisfy the Walras law (see Section 5), one could parameterize income by mode and ring in dealing with real costs and alternative modes. Also, labor–leisure tradeoffs could be treated in the framework developed in this paper, and disutility of commuting could be reflected in the functional form for u_j^i.

[10] The $\bar{\zeta}_k^{ij}$ can be chosen arbitrarily large, so that this satiation assumption is not very restrictive. This assumption is used to guarantee that various sets of utility maximizing demands used later in the paper are bounded. The boundedness of these sets greatly simplifies the exposition.

(a) u_j^i is increasing in y_h^{ij} up to the level $\bar{\zeta}_h^{ij}$, and $\bar{\zeta}_h^{ij}$ strictly bounds the hth coordinate of vectors $s + w$ in the set $\{s + w \mid s \in Y, s + w \geqslant 0\}$. (Assumptions (C.1)–(C.4) imply this set is bounded.)

(b) If $y^{ij} \in X_j^i$ and $\tilde{y}_k^{ij} = y_k^{ij}$ for $k \neq h$ and $y_h^{ij} < \tilde{y}_h^{ij} \leqslant \bar{\zeta}_h^{ij}$, then $\tilde{y}^{ij} \in X_j^i$.

(c) There exists $\bar{y}^{ij} \in X_j^i$ such that $\bar{y}^{ij} \leqslant w^i/r_i$ and $\bar{y}_h^{ij} < w_h^i/r_i$.

(D.4) For every good k there exists a household type i_k for which (for all j) there exists $\hat{y}^{i_k j} \in X_j^{i_k}$ with $\hat{y}^{i_k j} \leqslant w^{i_k}/r_{i_k}$ and strict inequality holding in the kth coordinate.[11]

Finally we define $C_j^i = \{y \in X_j^i \mid y \leqslant b/r_i\}$, where b is an η-vector with positive coordinates which strictly bounds from above the set $\{s + w \mid s \in Y, s + w \geqslant 0\}$, and $b_h = \min_{i,j} r_i \bar{\zeta}_h^{ij}$.[12] Let $\bar{U}_j^i = \max_{y \in C_j^i} u_j^i(y)$ and $\bar{U}^i = \max_j \bar{U}_j^i$. (It will become apparent later that $\bar{U}^i$ represents a strict upper bound on the equilibrium utility level for type i households.)

Then we assume:

(D.5) For each i, $\max_{y \in X_j^i} u_j^i(y) > \bar{U}^i$ for all $j = 1, \ldots, \gamma$.[13]

Given the above conventions, the budget constraint for a household of type i which lives in ring j can be written as follows. Let p denote the η-vector of prices, and $\psi^i(\pi)/r_i$ denote the profits distributed to a type i household, where ψ^i is a function of total profits π (which in turn is a function of p). ψ^i is continuous and $\sum_{i=1}^{m} \psi^i(\pi) = \pi$. Then the budget constraint is

$$p y^{ij} \leqslant p \frac{w^i}{r_i} + \frac{\psi^i(\pi)}{r_i} . \tag{1}$$

Because of the restriction on $y_{\gamma+1}^{ij}$, this budget constraint includes the time costs of commuting.

---

[11] Assumptions (D.3) and (D.4) are designed to rule out the so-called "exceptional case," which can lead to discontinuous demand correspondences. Note that (D.4) implies that all coordinates of w are positive, but it does not imply the same for w^i. In the present context it does not seem reasonable to use the analog of Debreu's [3] assumption—that there exists $\bar{y}^{ij} \in X_j^i$ with each coordinate of $\bar{y}^{ij}$ less than the corresponding coordinate of w^i, since this would imply all coordinates of w^i are positive. It is unlikely that each household would own land in every ring.

[12] We can choose b_h in this way because of (D.3a). Also, since b is a strict upper bound, it follows that $b_k > w_k$ and hence $b_k > w_k^i$ for $k = 1, \ldots, \eta$. Thus C_j^i is nonempty (from D.3c) and closed and bounded (by (D.2)).

[13] The left-hand side of the inequality is well defined because of (D.1) and (D.2). This assumption is designed to rule out satiation equilibrium consumptions. It is likely to be satisfied if there are some goods whose marginal utilities are positive (up to large quantities) and diminish sufficiently slowly.

(E) *Allocations*

An arbitrary partition of the set of all households into the γ rings of land is summarized by the set $\{r_{ij}\}$, where r_{ij} is the number of households of type i that reside in ring j (for $i = 1,\ldots, m$ and $j = 1,\ldots, \gamma$). (Some of the rings may be empty.)

An *allocation* is a specification of a partition $\{r_{ij}\}$, a production η-vector s, and consumption bundles $\{y^{ij}\}$ for type i households living in ring j, where $y^{ij} = 0$ if $r_{ij} = 0$. All households of the same type living in the same ring receive identical bundles.

DEFINITION 2.1. The allocation $\{r_{ij}\}$, $\{y^{ij}\}$, s is an *attainable allocation* iff $y^{ij} \in X_j{}^i$ if $r_{ij} > 0$, $s \in Y$, and $\sum_{i,j} r_{ij} y^{ij} - s = w$.

In an attainable allocation the consumption vectors of all households living in a particular ring must belong to their respective consumption sets, production must be technically feasible, and total consumption minus total supply must equal total endowment.

(F) *The Equilibrium Concept*

DEFINITION 2.2. An *urban land use equilibrium* is a price vector p^* and an attainable allocation $\{r_{ij}^*\}$, $\{y^{*ij}\}$, s^* such that

(a) (profit maximization) $p^*s^* \geq p^*s \ \forall s \in Y$;

(b) (utility maximization) If $r_{ij}^* > 0$, y^{*ij} maximizes $u_j{}^i(y^{ij})$ on

$$\left\{ y^{ij} \in X_j{}^i \mid p^*y^{ij} \leq p^* \frac{w^i}{r_i} + \frac{\psi^i(p^*s^*)}{r_i} \right\}.$$

(c) (mobility equilibrium) If $r_{ij}^* > 0$, $u_j{}^i(y^{*ij}) \geq u_{\mathbf{j}}{}^i(y^{i\mathbf{j}}(M))$ for $\mathbf{j} = 1,\ldots, \gamma$, where $y^{i\mathbf{j}}(M)$ maximizes $u_{\mathbf{j}}{}^i(y^{i\mathbf{j}})$ on

$$\left\{ y^{i\mathbf{j}} \in X_{\mathbf{j}}{}^i \mid p^*y^{i\mathbf{j}} \leq p^* \frac{w^i}{r_i} + \frac{\psi^i(p^*s^*)}{r_i} \right\}.$$

Condition (b) requires that each household maximize its utility over bundles which belong to its consumption set for the ring in which it chooses to live, subject to the appropriate budget constraint. Condition (c) states that no household wishes to move to a ring different from the one it is assigned to in the equilibrium allocation. Specifically the condition says that the equilibrium utility a type i household living in ring j achieves is not less than the level it could get by moving to ring $\mathbf{j}$ and maximizing over the consumption set and budget constraint relevant for that ring. A simple consequence of this condition is that all households of the same type receive the same equilibrium utility level, even though their equilibrium consumption bundles may be different if they reside in different rings.

The assumptions we have made are sufficient (though much stronger than needed) to guarantee that the urban land use equilibrium of Definition 2.2 is Pareto efficient.

3. The Scarf Algorithm

In this section we briefly describe the algorithm which underlies our computational routines for urban land use equilibria. An excellent brief reference is Scarf [17]. A fuller development can be found in Scarf and Hansen [18].

The Scarf algorithm involves a systematic search among points lying on the $(\eta - 1)$-dimensional unit simplex $T = \{x \geqslant 0 \mid \sum_{i=1}^{\eta} x_i = 1\}$. A large number of vectors $x^{\eta+1},\ldots, x^{k}$ from Euclidean η-space which lie in the interior of T are initially selected. These vectors, together with the additional vectors $x^1,\ldots, x^{\eta}$, which represent the η sides of the simplex (x^i representing the side along which the ith coordinate is zero), constitute the *grid* on the simplex.

Each x^i in the grid is assigned a η-dimensional vector *label* a^i. The particular choice of a^i depends upon the specific problem to which the algorithm is applied. We can form a $\eta \times \bar{k}$ matrix A from these vector labels, with a^j as the jth column. The labels comprising the columns of A must satisfy the following restrictions:

(i) The label a^i $(i = 1,\ldots, \eta)$, which corresponds to the ith side of the simplex, must be the η-vector whose ith coordinate is 1 and whose other coordinates are zero.

(ii) $\{v \geqslant 0 \mid Av = b\}$ is bounded, where b is a nonnegative η-vector whose definition depends upon the particular problem one is trying to solve.

Given the above assumptions, Scarf's result [18, Theorem 4.2.3] is:

Theorem 3.1 (Scarf). *There exists a set of η points $x^{j_1},\ldots, x^{j_\eta}$ from the grid which are close together and whose labels $a^{j_1},\ldots, a^{j_\eta}$ form a feasible basis (in the linear programming sense) for $Av = b$.*

Scarf's algorithm starts in a corner of the simplex with a collection of η points from the grid which are close together and systematically traces out a sequence of such collections of points from the grid until the desired set of points as described in the theorem is found. The closeness of this final collection of points depends on the fineness of the grid used. The particular sequence traced out depends crucially upon the choice of labels for the various points of the grid.

A detailed description of the operations involved in moving along the simplex is not needed for the purposes of this paper. Scarf shows there is a finite upper bound on the number of iterations needed to find the desired collection of points. What is important to note is that in practice the algorithm usually finds the desired collection in far fewer iterations than this theoretical upper bound.[14]

MacKinnon [5] uses a procedure which assigns each point on the grid an integer label from 1 to η rather than a vector label, with the grid point corresponding to the ith side of the simplex assigned an integer different from i. The theorem he uses asserts the existence of a collection of η points from the grid which are close together such that each point is labeled with a different integer. This theorem is a special case of the Scarf theorem stated above.[15]

4. COMPUTATIONAL ROUTINE FOR URBAN LAND USE EQUILIBRIA

In this section we use the Scarf theorem described above in deriving a computational routine for the equilibria of Definition 2.2. We search over the price simplex $P = \{p \geq 0 \mid \sum_{i=1}^{\eta} p_i = 1\}$.

As described in Section 3, we begin by selecting a large number of vectors p^l to comprise the grid on the simplex P, with the sides of the simplex represented by the vectors p^l for $l = 1,..., \eta$. p^l corresponds to the side along which the price of the lth good is zero. We refer to these first η points of the grid as side vectors, and to their corresponding labels (the unit vectors) as side columns.

Next we must construct vector labels for the remaining points p^l in the grid (i.e., for $l > \eta$). These labels will essentially correspond to excess demand vectors.

In creating these excess demand vectors we shall confine the supply vectors to a certain convex, closed, and bounded subset $\overline{Y}$ of the production set.

DEFINITION 4.1. Let $b = (b_1 ,..., b_\eta) \gg 0$[16] be a vector which strictly bounds from above the set $\{s + w \mid s \in Y, s + w \geq 0\}$, where $b_h \geq r_i \bar{\zeta}_h^{ij}$ for all i, j.[17] Let $\overline{Y} = \{s \in Y \mid -b \leq s \leq b\}$.

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[14] An extension of the Scarf algorithm due to Merrill [7] was used in the computations presented in Section 6. It considerably enhances the efficiency of the Scarf algorithm and is similar to the "sandwich method" of Kuhn and MacKinnon as described in [5].

[15] Substitute for the integer label k the vector label (0, 0,..., 1,..., 0), where the 1 is the kth coordinate. Letting $b = (1, 1,..., 1)$, it is easy to show that the Scarf theorem stated above implies the one used by MacKinnon.

[16] For $x, y \in R^k$, $x \gg y$ if $x_i > y_i$ for all i, $x \geq y$ if $x_i \geq y_i$ for all i, $x > y$ if $x \geq y$ and $x \neq y$.

[17] This vector b was defined in the text between assumptions (D.4) and (D.5) and is repeated here for completeness.

$$\begin{tikzcd} & \left\{\begin{array}{ll} s^l & \text{net production} \\ p^l s^l & \text{profits} \end{array}\right. \\ p^l \arrow[ur] \arrow[dr] & \\ & \left\{\begin{array}{ll} \{r^l_{ij}\} & \text{partition} \\ y^{(ij)l} & \text{demands} \end{array}\right. \end{tikzcd}$$

FIGURE 1

For any p^l in the grid which is not a side vector we create an excess demand vector from the various components illustrated in Fig. 1. Corresponding to the given p^l we derive a profit maximizing production vector $s^l \in \bar{Y}$ and the associated profit level $p^l s^l$. Next we assign all households of type i to their most preferred ring. Such a ring is determined by comparing the utility levels they could achieve by maximizing u_j^i over

$$\left\{ y^{ij} \in X_j^i \mid p^l y^{ij} \leq p^l \frac{w^i}{r_i} + \frac{\psi^i(p^l s^l)}{r_i} \right\}$$

for $j = 1, \ldots, \gamma$.[18] Note that the choice set depends upon the ring (in particular the travel time) through X_j^i, on the given p^l and on the profit level $p^l s^l$ just determined. Since $0 \in \bar{Y}$, $\psi^i(p^l s^l) \geq 0$, and so by (D.3c) the choice set is nonempty. Since $p^l \gg 0$, the choice set is compact. Since u_j^i is continuous, these maximizations are well defined for all i and j. Thus for each household type i we can find a most preferred ring $\mathbf{j}$.[19] We then set $r^l_{i\mathbf{j}} = r_i$ and $r^l_{ij} = 0$ for $j \neq \mathbf{j}$. In this way we derive the partition $\{r^l_{ij}\}$ indicated in Fig. 1.

Finally we set $y^{(ij)l} = 0$ for $j \neq \mathbf{j}$, and set $y^{(i\mathbf{j})l}$ equal to an optimal solution of the utility maximization problem for ring $\mathbf{j}$. If there is more than one optimal solution, any one is chosen which has the property that its kth coordinate is less than or equal to $\bar{\zeta}_k^{il}$ (for $k = 1, \ldots, \eta$). A solution with such a property always exists because of assumption (D.1b).

Then we take these basic ingredients and form an excess demand vector $(\sum_{i,j} r^l_{ij} y^{(ij)l} - s^l - w)$. The label for p^l (for $l > \eta$) is $[\sum_{i,j} r^l_{ij} y^{(ij)l} - s^l - w + b]$. Definition 4.1 and the property that $y^{(ij)l} \geq 0$ imply that these labels are positive vectors.

As shown below, Scarf's theorem enables us to determine a price vector whose corresponding excess demand vector is (approximately) zero. This is precisely what we want for an urban land use equilibrium. The excess demands have the utility and profit maximization requirements of Definition 2.2 built into them, and the mobility requirement would be satisfied from the derivation of the partition.

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[18] We arbitrarily break a tie for the largest maximum utility level among two or more rings by assigning the household type to the largest numbered of the tied rings.

[19] In this assignment all households of the same type reside in the same ring. However, this will not necessarily be a characteristic of the equilibrium partition we will determine. (For example, if there were only one household type, one would not expect all the households to crowd into one ring in equilibrium.)

We must first check to see that the assumptions of Scarf's theorem are satisfied. In our application the upper bound vector b will be used as the right-hand side vector in the Scarf theorem. Then $\{v \geqslant 0 \mid Av = b\}$, where A is the matrix of labels, is obviously bounded, since all the non-side columns of A are positive. Thus all the assumptions of Scarf's theorem are satisfied, and hence there exists a collection of η points $\{p^{j_1},..., p^{j_\eta}\}$ from the grid which are close together such that

$$\sum_l v_l \left\{\sum_{i,j} r_{ij}^l y^{(ij)^l} - s^l - w + b\right\} \leqslant b \qquad \text{with } v_l \geqslant 0, \tag{2}$$

where the summation of the v_l is taken over all l corresponding to columns which comprise the labels for the nonside vectors in the set $\{p^{j_1},..., p^{j_\eta}\}$. (The expression "for all l" will refer to l with this correspondence.) In (2) the kth inequality is an equality unless the kth side vector is a member of $\{p^{j_1},..., p^{j_\eta}\}$.

Next we take a sequence of finer and finer grids on the simplex. With each grid in the sequence is associated:

(A) a desired collection of η grid points $\{p^l\}$ as described by the Scarf theorem,

(B) a set of partitions $\{\{r_{ij}^l\}\}_l$ corresponding to the nonside vectors in $\{p^l\}$;

(C) a set of non-side labels $\{\sum_{i,j} r_{ij}^l y^{(ij)^l} - s^l - w + b\}$ corresponding to the non-side vectors in $\{p^l\}$;

(D) a set of weights $\{v_l\}$ corresponding to the non-side labels.

In the limit, as the number of points comprising the grid becomes infinite and dense on the simplex, a subsequence of these grids can be chosen such that:

(A′) the non-side vectors in the desired collections of grid points $\{p^l\}$ converge to a single point p^*;

(B′) the sets of partitions $\{\{r_{ij}^l\}\}_l$ converge to a set of partitions $\{\{r_{ij}^{*l}\}\}_l$;

(C′) the sets of non-side labels $\{\sum_{i,j} r_{ij}^l y^{(ij)^l} - s^l - w + b\}$ converge to a set of vectors $\{\sum_{i,j} r_{ij}^{*l} y^{*ij^l} - s^{*l} - w + b\}$;[20]

(D′) the sets of weights $\{v_l\}$ converge to the set $\{v_l^*\}$.

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[20] Since all the s^l lie in the bounded set $\bar{Y}$, s^{*l} exists. All the $y^{(ij)^l}$ vectors in the sequence lie in the bounded set $\{y^{ij} \in X_j^i \mid y_k^{ij} < \zeta_k^{ij}$ for $k = 1,..., \eta\}$, and hence y^{*ij^l} exists.

In the limit, expression (2) becomes

$$\sum_l v_l^* \left\{\sum_{i,j} r_{ij}^{*l} y^{*ij^l} - s^{*l} - w + b\right\} \leqslant b \qquad \text{with } v_l^* \geqslant 0, \tag{3}$$

where the kth inequality is an equality if $p_k^* > 0$. Taking the scalar product of (3) and p^* yields

$$\sum_l v_l^* \left\{\sum_{i,j} r_{ij}^{*l} p^* y^{*ij^l} - p^* s^{*l} - p^* w\right\} = p^* b \left(1 - \sum_l v_l^*\right). \tag{4}$$

We obtain an equality in (4) because of the complementary slackness relationship of p^* and the inequalities in (3).

$r_{ij}^{*l} = r_i$ for one $j \in \{1,..., \gamma\}$ and zero for all other j. For the particular j for which $r_{ij}^{*l} = r_i$.

$$p^* y^{*ij^l} \leqslant p^* \frac{w^i}{r_i} + \frac{\psi^i(p^* s^{*l})}{r_i} \tag{5}$$

Expression (5) follows from the property that $r_{ij}^l > 0$ implies $y^{(ij)^l}$ satisfies the budget constraint $p^l y^{(ij)^l} \leqslant p^l(w^i/r_i) + \psi^i(p^l s^l)/r_i$, and the continuity of ψ^i.

Using (5) we have that

$$\sum_{i,j} r_{ij}^{*l} p^* y^{*ij^l} \leqslant \sum_i r_i \left\{p^* \frac{w^i}{r_i} + \frac{\psi^i(p^* s^{*l})}{r_i}\right\} = p^* w + \sum_i \psi^i(p^* s^{*l})$$
$$= p^* w + p^* s^{*l},$$

the last equality following from the definition of ψ^i. Hence the expression in braces on the left-hand side of (4) is no greater than zero, which implies

$$\sum_l v_l^* \geqslant 1. \tag{6}$$

Expressions (3) and (6) imply

$$\sum_{i,j} \sum_l v_l^* r_{ij}^{*l} y^{*ij^l} - \sum_l v_l^* s^{*l} - w \leqslant 0. \tag{7}$$

Since $s^{*l} \in \bar{Y}$ and $\bar{Y}$ is convex, $\sum_l v_l^* s^{*l} / \sum_l v_l^* \in \bar{Y} \subset Y$. Since $r_{ij}^{*l} y^{*ij^l} \geqslant 0$, (7) implies $\sum_l v_l^* s^{*l} / \sum_l v_l^* + w \geqslant 0$. Hence by Definition 4.1, $\sum_l v_l^* s^{*l} / \sum_l v_l^* + w \ll b$, which implies by (7) that

$$\sum_l v_l^* \sum_i \sum_j r_{ij}^{*l} y^{*ij^l} \Big/ \sum_l v_l^* \ll b. \tag{8}$$

It can be shown that (8) implies that if $r_{ij}^{*l} > 0$, y^{*ij^l} does not correspond to a satiation consumption.[21] Hence by local nonsatiation (which follows from (D.1c)) it follows that (5) holds with equality if $r_{ij}^{*l} > 0$, since the utility maximizing vectors $y^{(ij)^l}$ in the sequence which converged to y^{*ij^l} would have satisfied their respective budget constraints with equality. Since (5) holds with equality, we can strengthen (6) to

$$\sum_l v_l^* = 1. \tag{9}$$

Then (3) and (9) imply

$$\phi^* = \sum_{i,j} \sum_l v_l^* r_{ij}^{*l} y^{*ij^l} - \sum_l v_l^* s^{*l} - w \leqslant 0, \tag{10}$$

with strict inequality in the kth relation only if $p_k^* = 0$. Let

$$r_{ij}^* = \sum_l v_l^* r_{ij}^{*l}. \tag{11}$$

Then $\{r_{ij}^*\}$ defines a partition.[22] (Note that the partition $\{r_{ij}^*\}$ does not necessarily assign all households of the same type to the same ring.)[23] If $r_{ij}^* > 0$ we define

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[21] Using (8), there exists an l, call it $\bar{l}$, such that $\sum_j r_{ij}^{*\bar{l}} y^{*ij^{\bar{l}}} \leqslant b$ for all i. Since $r_{ij}^{*\bar{l}} = r_i$ for one j, call it $\bar{j}$, and 0 for all other j, it follows that $r_{i\bar{j}}^{*\bar{l}} > 0$ implies $y^{*i\bar{j}^{\bar{l}}} \leqslant b/r_i$, and hence that $y^{*i\bar{j}^{\bar{l}}} \in C_{\bar{j}}^i$. (See the text between assumptions (D.4) and (D.5) in Section 2D for the definition of $C_{\bar{j}}^i$.) Thus if $r_{i\bar{j}}^{*\bar{l}} > 0$, $u_{\bar{j}}^i(y^{*i\bar{j}^{\bar{l}}}) \leqslant \bar{U}^i$. From the continuity of the utility functions, for all $l \neq \bar{l}$, $r_{ij}^{*l} > 0$ implies $u_j^i(y^{*ij^l}) = u_{\bar{j}}^i(y^{*i\bar{j}^{\bar{l}}})$. Then by (D.5), y^{*ij^l} does not represent a satiation consumption.

[22] *Proof.* Every set of partitions $\{(r_{ij}^l)\}_l$ in the subsequence has the property that r_{ij}^l equals r_i for one $j \in \{1, \ldots, \gamma\}$ and 0 for all other j. Thus $\sum_j r_{ij}^l = r_i$ and hence $\sum_j r_{ij}^{*l} = r_i$. Therefore, $\sum_j \sum_l v_l^* r_{ij}^{*l} = r_i$ using (9) and hence $\sum_j r_{ij}^* = r_i$. r_{ij}^* may not be an integer. However, since we have chosen r_i "large," it is obvious that a partition obtained by rounding off the r_{ij}^* to adjacent integers will correspond to an approximate equilibrium, in which excess demands are small but not necessarily zero. It should not be surprising that the assumption of a large number of households plays a role in this discussion. There is an indivisibility or nonconvexity present in the household location problem (in the sense that a given household must live in only one place). As recent work in mathematical economics has shown, nonconvexities tend to lose their sting in the presence of large numbers of economic agents. It would be easy to relax the convexity assumptions on preferences and the consumption sets. However, we use these assumptions to simplify notation (they allow us to focus on attainable allocations in which households of the same type living in the same ring are assigned identical consumption bundles), and because we want to highlight the locational nonconvexity. By letting the number of households of each type approach infinity, we could guarantee integral partitions. However, since our primary concern in this paper is with computation, we will not detail these limiting arguments here. Unlike the limiting arguments with respect to grid size, which facilitate the exposition of why the Scarf theorem can be used to obtain an approximate equilibrium, the limiting arguments with respect to the number of consumers would perform no similar service, and would add a great deal of technical complication.

$$y^{*ij} = \sum_l v_l^* r_{ij}^{*l} y^{*ij^l} \Big/ r_{ij}^* . \tag{12}$$

If $r_{ij}^* = 0$ we define

$$y^{*ij} = 0. \tag{13}$$

What we show below is that p^*, $\{r_{ij}^*\}$, $\{y^{*ij}\}$, and $s^* = \sum_l v_l^* s^{*l} + \phi^*$ (where ϕ^* is defined by (10)) is an urban land use equilibrium.

Since $(\sum_l v_l^* s^{*l}) \in Y$, $\phi^* \leq 0$ (by (10)), and Y exhibits free disposal, $s^* \in Y$. Since X_j^i is closed, $y^{(ij)^l} \in X_j^i$ if $r_{ij}^l > 0$, and $y^{(ij)^l} \to y^{*(ij)^l}$, it follows that $y^{*(ij)^l} \in X_j^i$ if $r_{ij}^{*l} > 0$. Since X_j^i is convex, it follows from (11) and (12) that $y^{*ij} \in X_j^i$ if $r_{ij}^* > 0$.

Using the definitions of r_{ij}^* and y^{*ij} ((11)–(13)) we can rewrite (10) as

$$\sum_{i,j} r_{ij}^* y^{*(ij)} - \sum_l v_l^* s^{*l} - w \leqslant 0. \tag{14}$$

Since ϕ^* equals the left-hand side of (14) (since (10) and (14) are equivalent), it follows that $\sum_{i,j} r_{ij}^* y^{*ij} - s^* - w = 0$. Thus from Definition 2.1, $\{r_{ij}^*\}$, $\{y^{*ij}\}$, s^* is an attainable allocation.

Next we verify that s^* maximizes profits on Y at the prices p^*. It is straightforward to show that s^{*l} maximizes profits on $\bar{Y}$ at the prices p^* for all l. Hence $\sum_l v_l^* s^{*l}$ also maximizes profits on $\bar{Y}$ given p^*. We have shown above (in our derivation of (8)) that

$$0 \leq \sum_l v_l^* s^{*l} + w \ll b. \tag{15}$$

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[23] For neighboring price vectors in the grid, the corresponding partitions may be quite different. It is this "knife-edged" characteristic of location choice which implies that r_{ij}^{*l} may differ from $r_{ij}^{*l'}$ for $l \neq l'$, which in turn implies that not all households of the same type reside in the same ring in (what we show to be) an equilibrium partition $\{r_{ij}^*\}$. All of the partitions used in connection with Fig. 1, and hence all the partitions $\{r_{ij}^{*l}\}$, are "pure" partitions in the sense that all households of a given type are assigned to the same ring. Thus it may seem a little mysterious that at the end of the algorithm out pops a partition $\{r_{ij}^*\}$ which may not be pure in this sense. But if $\{r_{ij}^*\}$ is not a pure partition, it means that there is more than one most preferred ring for some household type. From the standpoint of insuring equal utilities for all households of this type, any arbitrary distribution among the most preferred rings will do, though the resulting total demand vector for these households as a whole will clearly be sensitive to the particular distribution. Furthermore, it is easy to see that any partition of the households among those most preferred rings can be written as a convex combination of the pure partitions, i.e., those which assign all the households of this type to the same most preferred ring. What the algorithm is doing is finding the appropriate weights v_l^* on the pure partitions $\{r_{ij}^{*l}\}$ so that excess demands will be zero.

Since $b \gg w$ (which follows from Definition 4.1 and the fact that $0 \in Y$) and $w > 0$, (15) implies

$$-b \leqslant \sum_l v_l^* s^{*l} \leqslant b. \tag{16}$$

Using the definition of $\overline{Y}$ and the convexity of Y, (16) implies by a standard argument that $\sum_l v_l^* s^{*l}$ maximizes profits on Y at the prices p^*. From the definition of s^*, $p^*s^* = p^* \sum_l v_l^* s^{*l} + p^*\phi^*$. $p^*\phi^*$ equals the left-hand side of (4) (using (10)), and hence equals zero since $\sum_l v_l^* = 1$. Thus $p^*s^* = p^* \sum_l v_l^* s^{*l}$, and hence s^* maximizes profits on Y given p^*.

Next we verify that the utility maximization condition of Definition 2.2 is satisfied. Assumptions (D.3) and (D.4) allow us to rule out the "exceptional case."[24] Hence we conclude, using the continuity of the utility function and closedness of the choice set, that $r_{ij}^{*l} > 0$ implies that $y^{*(ij)^l}$ maximizes $u_j^i(y^{ij})$ on $\{y^{ij} \in X_j^i \mid p^*y^{ij} \leqslant p^*(w^i/r_i) + \psi^i(p^*s^{*l})/r_i\}$. Then using (12), quasi-concavity of u_j^i, and $p^*s^{*l} = p^*s^*$, we immediately conclude that $r_{ij}^* > 0$ implies that y^{*ij} maximizes u_j^i on $\{y^{ij} \in X_j^i \mid p^*y^{ij} \leqslant p^*(w^i/r_i) + \psi^i(p^*s^*)/r_i\}$.

Finally, the continuity of the utility functions and the derivation of the partitions which led to $\{r_{ij}^{*l}\}$ and hence $\{r_{ij}^*\}$ imply straightforwardly that condition (c) of Definition 2.2 is satisfied. Hence we conclude that p^*, $\{r_{ij}^*\}$, $\{y^{*ij}\}$, s^* constitute an urban land use equilibrium.[25]

Above we have employed a limiting process in finding an equilibrium. In practice of course, we must work with a finite grid, and we use a weighted average of the η price vectors with which the Scarf algorithm terminates in computing an approximate equilibrium. Loosely speaking, in such an approximate equilibrium, excess demands will be small, profits will be close to the maximum possible, and the level of utility that a household can obtain by residing in a ring different from the one to which it is assigned cannot be much greater. This is an appropriate sense of approximation in that excess demands will be sufficiently small so as not to induce a revision of prices,

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[24] If $p_h^* > 0$, then (D.3c) and $\psi^i(p^*s^{*l}) \geqslant 0$ imply that there exists $\bar{y}^{ij} \in X_j^i$ such that $p^*\bar{y}^{ij} < p^*(w^i/r_i) + \psi^i(p^*s^{*l})/r_i$. Thus if we can show $p_h^* > 0$ we will have ruled out the exceptional case. Suppose $p_h^* = 0$. Since $p^* > 0$, there exists k such that $p_k^* > 0$. Then by (D.4) there exists i_k and $\hat{y}^{i_k j} \in X_j^{i_k}$ such that $p^*\hat{y}^{i_k j} < p^*(w^{i_k}/r_{i_k}) + \psi^{i_k}(p^*s^{*l})/r_{i_k}$ for all l and j. In particular this holds for l and j such that $r_{i_k j}^{*l} > 0$. Hence for household type i_k we conclude that for all l and j such that $r_{i_k j}^{*l} > 0$, $y^{*(i_k j)^l}$ maximizes $u_j^{i_k}$ on $\{y^{i_k j} \in X_j^{i_k} \mid p^*y^{i_k j} \leqslant p^*(w^{i_k}/r_{i_k}) + \psi^{i_k}(p^*s^{*l})/r_{i_k}\}$. But then if $p_h^* = 0$, (D.3) implies $y_h^{*(i_k j)^l} = \zeta_h^{i_k j}$ for all l and j with $r_{i_k j}^{*l} > 0$. This contradicts (8), since $\sum_l v_l^* \sum_j r_{ij}^{*l} = r_i$ and $r_{i_k} \zeta_h^{i_k j} > b_h$. Hence $p_h^* > 0$, which is sufficient, as argued above, to rule out the exceptional case.

[25] If $r_{i1}^* > 0$ for some i, we can arbitrarily assign these households to the outermost land in the ring, since commuting time to the city center is assumed to be the same from any point in the ring. Thus ring 1 will consist of a circular CBD which, if it doesn't exhaust the land in the ring, will be surrounded by a residential zone.

producers will have no compelling incentive to search for higher profits, and households will have no compelling incentive to move.

5. Extensions of the Model

In this section we sketch how our algorithm can be applied to models which allow for various types of zoning and discrimination, multiple workplaces, and other common features of the urban environment. We also comment on how the algorithm can be used in urban models in which the Walras law and homogeneity postulate, as described in Section 1, do not hold. Our primary aim is simply to indicate the flexibility of the computational technique, not to give a detailed treatment of such topics.

Incorporating various types of discrimination, such as income or racial, is straightforward. For example, if we define household types on the basis of race, we can simply exclude from consideration certain rings when assigning households of a particular type to their most preferred ring in creating the vector labels derived in Fig. 1.

Zoning certain rings for certain activities can be included in the description of the production set. For example, residential zoning can be incorporated by excluding from the production set all vectors which use inputs of land from certain rings. For a slightly more complicated illustration, consider the case of a zoning law which only permits single family dwellings and specifies a minimum lot size. Suppose a household's utility function has housing services in the various rings, but not land, as arguments. Also suppose that housing services can be produced with a constant returns to scale technology using capital and land as inputs. Then in creating the partition and demands illustrated in Fig. 1, we would first find the cost-minimizing amount of land used per unit output of housing services in each of the rings to which the zoning law applies. We would then divide the minimum lot size by this amount of land to obtain the minimum number of units of housing services which must be consumed by each household in that ring. This constraint on the consumption of housing services would then be imposed on the utility maximization problems which are solved to generate the partition and demands of Fig. 1. In this manner each household would in effect be constrained to consume indirectly a certain amount of land.

There is a variety of ways of introducing public sector activities and many other endogenous externalities. The techniques which can be used in conjunction with the framework in this paper in incorporating such phenomena are very similar to those which we have described elsewhere [14–16].

In many applications it may be convenient to work with a model which does not satisfy the Walras law and homogeneity postulate. For example, the rental rate on capital may be determined in a national capital market, and hence may be exogenous to a given urban area. Some of the land in a given urban area may be owned by outside landlords, or a portion of a household's income may be derived from sources outside of the urban area. In such cases the endogenous prices can no longer be normalized to sum to 1, nor will the Walras law, which was crucially exploited in our proof in Section 4, hold.[26]

However, by making assumptions which ensure that the total value of excess demands for the endogenous goods is negative if the sum of the endogenous prices becomes sufficiently large, we can still obtain a computational routine based on the Scarf theorem. Excess demands are calculated essentially the same as before (only now they are possibly parameterized by some exogenous prices and income). The procedure used is sketched in [14], and is similar to one used by MacKinnon [6] in a different context.

The generalization of our methodology which we wish to highlight in this section deals with the existence of multiple workplaces. Instead of assuming one CBD, we now assume there are a fixed number $\bar{e}$ of commercially zoned locations, together with a set of γ locations which are available for residential use. The demand for inputs, in particular labor and land, and the quantities of outputs produced at each commercial location are endogenous. It is assumed that the round trip commuting time τ_{jk} from any potential residential location j to any potential employment center k can be specified exogenously. (Examples of manageable models in which this can be done are described later.)

Each household will have a two-pronged choice – where to live and where to work. The basic tradeoffs involved in this choice will be the possibly different wage rates in the various employment centers and the commuting times between various residential locations and employment centers. Here, as in the model of Section 2, we are emphasizing the accessibility to employment opportunities as a prime determinant in the residential location decision while essentially ignoring the location of consumer markets as a determinant of employment location.

In modifying our algorithm to accommodate multiple workplaces we will treat labor used in different employment centers as different goods. Whereas previously $s_{\gamma+1}$ represented labor employed in the CBD, we now let $s_{\gamma+k}$ $(k = 1,..., \bar{e})$ represent the employment in the kth employment center. If household i lives in residential location j and commutes to employment center k, it faces the consumption set X^i_{jk}. Analogous to the convention in the model of Section 2 that $y^{ij}_{\gamma+1} = \tau_j$ if $y^{ij} \in X_j{}^i$, we now require that

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[26] By introducing a "rest of the world" consumer and similar devices it is conceivable that the exogenous aspects described above could be incorporated in a model satisfying the Walras law and homogeneity postulate. But it may be more convenient to take a more direct approach.

$y^{ij}_{\gamma+k} = \tau_{jk}$ and $y^{ij}_{\gamma+\bar{k}} = 0$ (for $\bar{k} \neq k$, $\bar{k} = 1,\ldots, \bar{e}$) if $y^{ij} \in X^i_{jk}$. Thus if household i chooses to live in residential location j and commute to employment center k, he must consume τ_{jk} of his nonleisure hours in commuting.

Given a grid point p^l, we calculate a profit maximizing vector s^l over an appropriate production set. Next we create both a residential partition and an employment partition of the households and derive a vector y^{ijl} by letting each household pick its residential location and employment location that jointly maximize utility subject to the appropriate budget constraint and consumption set. For a type i household that chooses to work in employment center k, we define $w^{il}_{\gamma+k}/r_i$ to be its endowment of nonleisure hours (analogous to $w^i_{\gamma+1}/r_i$ of the earlier model), and let $w^{il}_{\gamma+\bar{k}} = 0$ for $\bar{k} \neq k$, $\bar{k} = 1,\ldots, \bar{e}$. The initial endowments w_j^i for $j \neq \gamma + 1,\ldots, \gamma + \bar{e}$ are exogenously determined as in the model of Section 2.

Then we form the excess demand vectors corresponding to p^l as before, the only novel element being that the rows corresponding to goods $\gamma + 1,\ldots,$ $\gamma + \bar{e}$ have an l superscript on the total endowment of that good, i.e., $w_j^l = \sum_{i=1}^{m} w_j^{il}$ for $j = \gamma + 1,\ldots, \gamma + \bar{e}$. The limiting arguments go through essentially the same as before. Because of the conventions chosen for y_k^{ijl} and w_k^{il} for $k = \gamma + 1,\ldots, \gamma + \bar{e}$, an excess demand of zero for these goods amounts to the statement that the total amount of commuting time to center k by its workers plus the total employment hours there equals the total quantity of nonleisure hours available by household members choosing to work in center k.

A few examples of possible characterizations of residential and employment location are given below. We are constrained to using characterizations in which it is possible to conveniently specify the commuting times τ_{jk}. One such characterization would involve a straightforward generalization of the model of Section 2. Employment opportunities would be confined not simply to the CBD but also to a number of the other rings. For example, one or more of the outer rings could be specified as a location for suburban employment. It would then be easy to specify commuting times from any ring to these suburban rings. Alternatively, employment could be found in local service sectors in each ring, much in the spirit of Solow's model [21]. Unlike the Solow model, there would be no need to restrict employment in the local service sector of a ring to residents of that ring.[27]

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[27] All of the above models have dealt with essentially one-dimensional cities, the key parameter being radial distance. However, in studying multiple workplaces, a more natural context would be a two-dimensional city. Instead of dividing land into concentric rings, one might, for example, consider a checkerboard of land parcels (which Mills [9] has used in a different context). Given the present stage of development of these algorithms, such an approach might present severe computational difficulties.

6. A Numerical Example

In this section a small numerical example of the algorithm is presented. Some of the parameter values were generated from the data underlying Muth's [12] computations, while others were chosen arbitrarily. The example is not meant to be a realistic representation of an urban area.

The city has five circular rings. The first ring has a radius of 1 mile, and the remaining rings are each 1 mile wide. The first and third rings are the only ones which are zoned commercially. All rings permit land to be used for residential purposes. One-third of the land in each ring is assumed to be available for residential or (in rings 1 and 3) commercial use.

There are five basic categories of goods. There is a commercial good, which can be produced in either ring 1 or ring 3 (reflecting our zoning assumptions). The other goods are capital, land, housing, and labor.

The commercial good is produced with inputs of capital, labor, and land. Capital and land inputs are used in the production of housing. Production of the commercial and housing goods is described by CES production functions of the following form:

$$Q = A(\beta_1 x_1^{(\sigma-1)/\sigma} + \beta_2 x_2^{(\sigma-1)/\sigma} + \beta_3 x_3^{(\sigma-1)/\sigma})^{h\sigma/(\sigma-1)},$$

where Q is output, σ is the elasticity of substitution, h is the degree of homogeneity, x_1 denotes the input of capital, x_2 the input of labor, and x_3 the input of land, with $x_i \geq 0$. The production function parameter values used in the numerical example are given in Table I. The β_3 coefficient refers to land input from the particular ring in which the good is being produced. We are assuming that the production function for housing is the same in all rings, but the production function for the commercial good does differ

TABLE I

Production Function Parameters

	Parameter					
Good	β_1	β_2	β_3	σ	h	A
Commercial						
Ring 1	0.10	7.5	0.160	2.00	0.5	100
Ring 3	0.20	10.5	0.160	2.00	0.5	100
Housing[a]	2.42	0.0	0.025	0.75	1.0	1

[a] β_1 and β_3 for housing are derived conditional on the elasticity of substitution, using Muth's [12, p. 314] methodology and data.

between rings 1 and 3 (perhaps reflecting differences in transporting the good to an appropriate market).

The amount of land used in the production of the commercial good in rings 1 and 3 is endogenous. If land in ring 1 is used, we say a CBD forms (whose radius is endogenous but bounded above by the radius of ring 1). If land from ring 3 is used in the production of the commercial good, we say a SBD (suburban business district) forms, whose size is also endogenous, subject to the zoning constraint.

There are two different types of households, with 100,000 households of type 1 and 89,000 households of type 2. All households of a given type have identical tastes and initial endowments. Each household is assumed to make 250 commuting work trips per year, with 8.76 hours available for work and commuting on each work day. There are two potential workplaces. The number of households of a particular type which commute to a particular workplace is endogenous. Commuting distances between the various rings and potential workplaces are given in Table II. Commuting times from the various rings to the two potential workplaces are generated from the commuting distances of Table II under the assumption of an average velocity of 21.3 miles per hour.

TABLE II

Round Trip Commuting Distances (miles)[a]

	Ring				
Workplace	1	2	3	4	5
CBD	2	3	5	7	9
SBD	3	2	1	2	4

[a] The distances between ring j ($j = 2,\ldots, 5$) and ring 1 (CBD) and between ring j ($j = 2$, 4, 5) and ring 3 are measured from the midpoint of ring j to the midpoint of ring 1 or ring 3. The distances in column 1 are measured from the outer edge of ring 1 to the midpoints of rings 1 and 3. The distance in the third column of the second row is measured from the edge of ring 3 to its midpoint. The equilibrium solution is not very sensitive to the conventions used in deriving these distances.

Each household of type 1 has an initial endowment of 15 units of capital, while each type 2 household has an initial endowment of 6 units of capital. Type 1 households own seven-tenths of the land in each ring which is available for residential or commercial use, with type 2 households owning the remaining three-tenths. (Land not available for commercial or residential use is not privately owned.) Profits from production are distributed equally to all households.

The preferences of each household are defined by a CES utility function of the form

$$U = \alpha_1 y_1^{(S-1)/S} + \alpha_2 y_2^{(S-1)/S},$$

where U is the utility level, S is the elasticity of substitution, y_1 is the consumption of the commercial good, and y_2 is the consumption of housing in the ring in which the household resides. y_1 and y_2 are assumed to be nonnegative.[28] (See Table III.)

TABLE III

Preference Parameter Values

Household type	α_1	α_2	S
1	0.75	0.25	1.05
2	0.50	0.50	1.20

Some of the results of the numerical computations are presented in Table IV. In the equilibrium solution of Table IV, the absolute value of excess demand for land in any ring is less than 1 percent of the total supply of land in the ring whose use is endogenously determined. The absolute value of excess demand for any of the other goods is less than 0.03% of its total equilibrium supply or demand.[29]

[28] We have chosen production and utility functions for which closed form solutions for profit-maximizing quantities and utility maximizing quantities as a function of prices can be found. Without such closed form solutions, the computational requirements of the algorithm would be prohibitive. Also note that not all the assumptions of Section 2 are satisfied by our specification of tastes and technology. (For example, (D.1b) is not satisfied by our CES utility functions.) However, these assumptions are sufficient, but not necessary, for the successful convergence of the algorithm. If we did want to modify our description of tastes and technology to ensure these assumptions are satisfied, we could do so in an ad hoc way. For example, in order to satisfy (D.1b), the CES utility functions could simply be truncated at some arbitrarily large levels of consumption of the two goods.

[29] These excess demands can easily be reduced, but at the cost of additional computational time. The approximate equilibrium in the numerical example was found in about 89 seconds of CPU time on an IBM370–148 computer. (On a faster computer such as an IBM370-168, this CPU time would be reduced by a factor of roughly 16.) Excess demand vectors corresponding to 484 different price vectors were calculated before the approximate equilibrium solution was found. No prior information about the equilibrium prices was used (that is, search was initiated at the center of the price simplex). Fixed point methods in general are less efficient the higher the the degree of nonlinearity of the functions involved. The numerical example presented was deliberately chosen to be highly nonlinear (as reflected in the low values chosen for the elasticities of substitution).

TABLE IV

An Approximate Equilibrium Solution

	Ring				
	1	2	3	4	5
Land rental[a]	1115	993	753	649	550
Housing price[b]	1923	1911	1884	1870	1856
Population density[c]	21.59	19.79	16.06	8.98	7.93
Fraction of type 1[d]					
CBD	0.12	0.39	0.11	0.00	0.00
SBD	0.00	0.00	0.38	0.00	0.00
Fraction of type 2					
CBD	0.00	0.00	0.00	0.00	0.00
SBD	0.00	0.00	0.00	0.47	0.53
Capital/land in housing[e]	179	164	133	119	105

Additional prices

Commercial good $ 1561

Wage[f]

CBD	$4732/yr
SBD	$4633/yr
Capital	$108/yr

Size of commercial zones

CBD[g]	0.19
SBD	0.08

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[a] Dollars per year per acre. Prices are normalized by setting the rental on a unit of capital at $108/yr. (See Muth [12] for the derivation of this rental value on capital.)

[b] Dollars per year.

[c] Number of households per acre of residential land in the ring.

[d] Entries in the CBD row denote the fraction of all households of type 1 who reside in the particular ring *and* commute to the CBD. Similarly, entries in the SBD row denote the fraction of all households of type 1 who live in the particular ring *and* commute to the SBD. (Fractions have been rounded to the nearest hundredth.)

[e] Units of structure per acre.

[f] This does not represent annual wage income, which equals the wage multiplied by the time per year spent working.

[g] The amount of land in ring 1 devoted to commercial use divided by the amount of land in ring 1 available for commercial or residential use. An analogous definition holds for the SBD entry. Under the assumptions that land closest to the city center within ring 1 is used commercially, and that the fraction of land in any annulus within ring 1 that is available for residential or commercial use is the same, the CBD entry implies a CBD radius of 0.43 miles.

The computation time rises quadratically or faster with the dimension of the simplex. This can be a serious problem since goods such as housing must be parameterized by their ring of origin. However, there are a number of ways of taking advantage of the special structure of some of these models to reduce the computation time substantially.

First, the version of the Scarf algorithm we used allows one to initiate the search for equilibrium prices anywhere on the simplex. Thus prior information about the nature of the equilibrium prices (e.g., the general shape of the land rent gradient) can be used in deciding where to start the algorithm (rather than simply starting at the center of the simplex.)[30]

The most time-consuming part of the computations is the calculation of the excess demands, in particular the exponentiation involved in finding demands and supplies. If we can describe production by means of a constant returns-to-scale technology (using continuous production functions and/or an activity analysis framework), then we can use an alternative labeling scheme which greatly reduces the number of exponentiations which must be performed. Price vectors in the grid are labeled by either a vector of demands (calculated in the same manner as described in Section 4) or a profit-maximizing activity vector, in a manner analogous to Scarf's procedure in [17].

The most direct way of increasing the algorithm's efficiency is to exploit relationships which must characterize an equilibrium position to reduce the dimension of the simplex over which search is conducted. For example, we know that in equilibrium, constant returns to scale industries with positive production levels must make zero profits. We exploited this property in our numerical example to reduce the dimension of the simplex by 5, confining our search to a unit simplex involving only the prices of the commercial good, the wages in each workplace, the price of capital, and the land prices in the five rings. Given a grid vector p^l of these prices, a generalization of the procedure illustrated in Fig. 1 of Section 4 was used in deriving appropriate excess demand labels. This generalization is illustrated in Fig. 2.

Corresponding to p^l we derive a profit-maximizing production vector s (whose coordinates correspond to those of p^l) for the commercial sector only, and an associated level of profits $p^l s$. (Here we exploit the property that housing does not appear as an input in the commercial good production functions.) Next we impose zero profit conditions on the production of housing (one for each ring), given the land prices and capital price embodied in p^l, to derive a vector of housing prices $\hat{p}^l$ consistent with profit maximization in these industries.[31] Then, conditional on p^l and $\hat{p}^l$ and the profits $p^l s$

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[30] Using different starting points may also be useful in checking on the possibility of multiple equilibria.

[31] It is crucial to the efficiency of the algorithm that the unit cost function for housing can be written as a closed form expression in the factor prices.

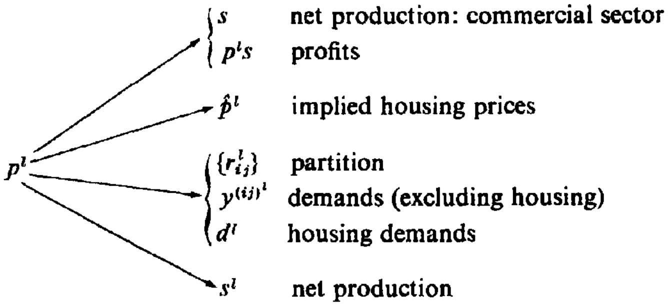
p^l
s net production: commercial sector
$p^l s$ profits
$\hat{p}^l$ implied housing prices
$\{r^l_{ij}\}$ partition
$y^{(ij)^l}$ demands (excluding housing)
d^l housing demands
s^l net production

FIGURE 2

(there are no profits in the housing sector), we derive in a manner analogous to Fig. 1 a partition and utility-maximizing aggregate demands for all the goods. (We have broken the utility-maximizing demand vector into two vectors in Fig. 2.) Next we calculate the cost-minimizing amounts of capital and land from the various rings needed to produce the vector of demands d^l (minus the endowments of housing, which are zero in our example). Then these amounts of capital and land are added to the net commercial production vector s to obtain the net production vector s^l. Then the r^l_{ij}, $y^{(ij)^l}$, and s^l are used as in Section 4 to create excess demand labels. As before, the algorithm will guarantee zero excess demands for all goods whose prices are represented on the simplex. For housing in the various rings, zero excess demand is built into each label as described above.

King [4] and Arnott and MacKinnon [1] exploit zero profit conditions in analogous manners in their papers. However, Arnott and MacKinnon also exploit the property that the equilibrium level of utility of a household of a given type will be the same in any ring in which households of that type reside in the equilibrium partition.[32] By exploiting some additional special structure of their model (in particular that households do not own any of the land in the urban area) they are able to confine their search to a simplex involving a single price. While we cannot expect to obtain such a drastic reduction in the dimension of the simplex for the more general models discussed in this paper, a considerable increase in the efficiency of the algorithm can still be obtained by exploiting certain characteristics of equilibrium positions. Also, if one can eliminate from the simplex prices that are parameterized by ring (such as housing and land prices), then the urban area can be divided into many more rings without increasing computational requirements prohibitively, and hence better approximations to continuous cities can be obtained.

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[32] Given the structure of their model, the maximum utility level U^l attainable by a household of a given type in one specific ring can be derived from a grid price vector p^l on a simplex of reduced dimension. Then, using the indirect utility function, the maximum levels of utility attainable in other rings (which will be functions of other prices in addition to p^l) are set equal to U^l, and those equations are then solved for those additional prices.

7. Concluding Remarks

In this paper we have described a computational technique which we feel will be a useful supplement to purely analytic models in studying resource allocation and distribution in urban areas. We have touched briefly upon just a few of the empirically relevant characteristics and policies which can be simulated using our framework but which are analytically intractable in most of the current urban models. Much work remains to be done, however, especially in analyzing the location of concentrations of production and the associated issue of scale economies. We feel that the techniques developed will be most useful in the study of relatively small scale models, much in the spirit of Muth [12], with parameter value choices based upon empirical evidence (such as econometrically estimated demand and production functions).

References

1. R. J. Arnott and J. MacKinnon, The effects of the property tax: A general equilibrium simulation, *J. Urban Econ.* 4 (1977), 389–407.
2. M. J. Beckmann, On the distribution of urban rent and residential density, *J. Econ. Theory* 1 (1969), 60–67.
3. G. Debreu, "Theory of Value," Wiley, New York, 1959.
4. A. T. King, Computing general equilibrium prices for spatial economies, *Rev. Econ. Statistics* 59 (1977), 340–350.
5. J. MacKinnon, Urban general equilibrium models and simplicial search algorithms, *J. Urban Econ.* 1 (1974), 161–183.
6. J. MacKinnon, An algorithm for the generalized transportation problem, *Reg. Sci. Urban Econ.* 5 (1975), 445–464.
7. O. H. Merrill, "Applications and Extensions of an Algorithm That Computes Fixed Points of Certain Non-Empty Convex Upper Semicontinuous Point to Set Mappings," Technical Report No. 71-7, Department of Industrial Engineering, University of Michigan, Ann Arbor, Mich., 1971.
8. E. S. Mills, An aggregative model of resource allocation in a metropolitan area, *Amer. Econ. Rev.* 57 (1967), 197–210.
9. E. S. Mills, Markets and efficient resource allocation in urban areas, *Swedish J. Econ.* 74 (1972), 100–117.
10. E. S. Mills and J. MacKinnon, Notes on the new urban economics, *Bell J. Econ. Management Sci.* 4 (1973), 593–601.
11. R. Muth, "Cities and Housing," Univ. of Chicago Press, Chicago, 1969.
12. R. Muth, Numerical solution of urban residential land use models, *J. Urban Econ.* 2 (1975), 307–332.

13. Y. Oron, D. Pines, and E. Sheshinski, Optimum vs. equilibrium land use pattern and congestion toll, *Bell J. Econ. Management Sci.* **4** (1973), 619–636.
14. D. Richter, A computational approach to the study of neighborhood effects in general equilibrium urban land use models, *in* "The Economics of Neighborhood" (D. Segal, Ed.), Academic Press, New York, in press.
15. D. Richter, The computation of a general equilibrium in a public goods economy, *Math. Programming* **14** (1978), 186–207.
16. D. Richter, Existence and computation of a Tiebout general equilibrium, *Econometrica* **46** (1978), 779–805.
17. H. Scarf, On the computation of equilibrium prices, *in* "Ten Economic Studies in the Tradition of Irving Fisher," (William Fellner et al., Eds.), pp. 207–30, Wiley, New York, 1967.
18. H. Scarf with the collaboration of T. Hansen, "The Computation of Economic Equilibria," Yale Univ. Press, New Haven, Conn., 1973.
19. R. Solow, Congestion, density and the use of land in transportation, *Swedish J. Econ.* **74** (1972), 161–173.
20. R. Solow, Congestion cost and the use of land for streets, *Bell J. Econ. Management Sci.* **4** (1973), 602–618.
21. R. Solow, On equilibrium models of urban location, *in* "Essays in Modern Economics" (J. M. Parkin, Ed.), Longmans, London, 1973.

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Resource Extraction, Substitute Production, and Monopoly*

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1. INTRODUCTION

The development of price and extraction of a nonrenewable resource owned by a monopolist has been analyzed by Kamien and Schwartz [4], Lewis [5], Stiglitz [8], Sweeney [9], and others. The present paper analyzes how the extraction of a resource owned by a monopolist is affected by the existence of a perfect substitute for the resource.

To focus on the existence of a substitute produced by a "backstop technology" (see [6]), we disregard changes in the demand function over time (see [8]) and costs depending on current extraction (see [9]), or on remaining reserves (see [2, 7]). Section 2 gives the monopolistic extraction paths in three cases; (1) no substitute exists, (2) the resource owning monopolist controls the production of the substitute, and (3) the substitute is supplied competitively. Section 3 compares these three cases for a general class of demand functions. In Section 4 the special case with a constant demand elasticity and zero extraction costs is treated, before some conclusions are given in Section 5.

2. THE THREE MONOPOLY CASES

In all three cases a monopolist is assumed to maximize discounted profits W, where

$$W = \int_0^T e^{-rt}[F(x(t) + y(t)) - bx(t) - cy(t)]\,dt, \qquad (1)$$

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subject to

$$\dot{S}(t) = -x(t), \qquad S(0) = S_0, \tag{2}$$

$$S(t) \geq 0, \qquad x(t) \geq 0, \qquad y(t) \geq 0. \tag{3}$$

Here x and y stand for resource extraction and substitute production by the monopolist, while S is the remaining resource stock (S_0 is given). The discount factor (r) and the unit cost of producing the substitute (c) are both positive, and the unit cost of resource extraction (b) satisfies $0 < b < c$. The function $F(x + y)$ gives the monopolist's total revenue.

In the case with no backstop technology we have the additional constraint $y(t) = 0$, and the revenue function is

$$F(x) = xf(x), \tag{4}$$

where $f^{-1}(p)$ is the demand function and p is the price of the resource. Throughout our analysis we assume that

$$\operatorname{Lim}_{x \to 0} f(x) = \infty \tag{5}$$

so that we can disregard the possibility of zero resource extraction when there is no substitute. We also assume that $f' < 0$ and that $xf(x)$ is strictly concave, so that the marginal revenue function declines with x everywhere. Under these assumptions it is well known [5, 8] that the monopolist's optimal extraction path makes the marginal current profit rise with the rate r, with the initial extraction determined so that

$$\int_0^\infty x(t)\, dt = S_0. \tag{6}$$

Defining

$$q = g(x) = f(x) + xf'(x) - b \tag{7}$$

as the marginal current profit, we therefore have

$$q_1(t) = q_1(0)\, e^{rt}, \tag{8}$$

where the subscript 1 refers to case number 1. The value of $q_1(0)$ is determined so that (6) holds when $x = g^{-1}(q)$ is used.

If substitute production is possible and controlled by the monopolist there are no additional constraints to our problem (1)–(3), and $F(x + y) = (x + y)f(x + y)$ like before. Since $c > b$ and $r > 0$, it is obviously not optimal for the monopolist to start producing the substitute before the resource is exhausted. Furthermore, after the date of exhaustion we must have marginal revenue equal to marginal cost, i.e., $q_2(t) + b = c$ from (7) (the subscript 2 refers to case number 2). Like before, we have $q_2(t)$ rising with the rate r as long as resource extraction takes place, and it is easy to verify that $q_2(t)$ must reach its long-run value $(c - b)$ exactly when exhaustion occurs (see, for instance, Dasgupta and Stiglitz [1] or Hoel [3]). The solution in this case is therefore given by

$$q_2(t) = (c - b)\, e^{-r(T_2 - t)}, \tag{9}$$

where T_2 is the date of exhaustion. The value of T_2 is determined by the fact that (6) must hold with $x(t) = g^{-1}(q_2(t))$ for $t < T_2$ and $x(t) = 0$ for $t > T_2$ inserted.

Our third monopoly case is slightly more complicated than the two previous cases. Here we have the constraint $y(t) = 0$ like in case number 1 (remembering that y is the quantity of the substitute produced by the monopolist). Furthermore, the monopolist cannot sell any resources at a price exceeding the competitive substitute price c. To simplify the mathematics, we assume that the monopolist can sell as much as he wishes, up to the total demand $f^{-1}(p)$, at $p = c$. Since $r > 0$, it will not be profitable for the monopolist to have $0 < x(t) < f^{-1}(c)$ for any period, since he at the price c could obtain a higher discounted profit by letting $x(t) = f^{-1}(c)$ first and $x(t) = 0$ later in such a period. We can therefore solve the optimization problem by introducing the additional constraint

$$x(t) \geq f^{-1}(c) \qquad \text{or} \qquad x(t) = 0. \tag{10}$$

For the same reason as $0 < x(t) < f^{-1}(c)$ never would be optimal it is clear that it cannot be optimal to have $x(t) > 0$ after a period where $x(t) = 0$. The optimization problem (1)-(3) with the additional constraints $y(t) = 0$ and (10) therefore has the same solution as maximizing

$$\int_0^{T_3} e^{-rt}[F(x(t)) - bx(t)]\, dt$$

subject to (2), (3), and $x(t) \geq f^{-1}(c)$, where the date of exhaustion T_3 is a control variable. The Hamiltonian corresponding to this problem is

$$H(x, \lambda, t) = e^{-rt}[F(x) - bx] - \lambda x, \tag{11}$$

where λ is a positive constant since S does not enter the function H. The optimal extraction path $x_3(t)$ must satisfy

$$H_x(x_3(t), \lambda, t) \leq 0,$$

where $<$ implies $x_3(t) = f^{-1}(c)$. Together with (4), (7), and (11) this means that

$$g(x_3(t)) \leqslant e^{rt}\lambda, \tag{12}$$

where $<$ implies $x(t) = f^{-1}(c)$. The horizon T_3 must satisfy $H(x_3(T_3), \lambda, T_3) = 0$, i.e., from (4) and (11)

$$[e^{-rT_3}(f(x_3(T_3)) - b) - \lambda]\, x_3(T_3) = 0. \tag{13}$$

From (12) we see that $x_3(t)$ is nonincreasing. Furthermore, the constraint $x(t) \geq f^{-1}(c)$ must sooner or later be binding, otherwise the present case would be identical case 1, and since (8) implies that $x_1(t) < f^{-1}(c)$ for a sufficiently large value of t, this would give us a contradiction. These two properties of the solution $x_3(t)$ imply that $x_3(T_3) = f^{-1}(c)$, so that (13) gives us

$$\lambda = e^{-rT_3}(c - b). \tag{14}$$

The solution for $q_3(t) = g(x_3(t))$ follows from (12) and (14):

$$q_3(t) = \mathrm{Min}[\bar{q}, (c - b)\, e^{-r(T_3 - t)}], \tag{15}$$

where

$$\bar{q} = g(f^{-1}(c)) < (c - b). \tag{16}$$

Like before, T_3 is determined by the fact that (6) must hold with $x(t) = g^{-1}(q_3(t))$ for $t \leq T_3$ and $x(t) = 0$ for $t > T_3$ inserted. In other words, the monopoly solution in this case has two phases: In the first phase the marginal current profit rises with the rate r until the resource price reaches its upper limit c in the end of this phase. In the second phase the resource price is constant and equal to c, and this phase lasts until the resource is completely exhausted. Notice that the second phase must always occur in a solution (cf. (15) and (16), which imply that $q_3(t_3) = \bar{q}$ for t sufficiently close to T_3). However, the first phase need not exist. This is true no matter how large S_0 is if $\bar{q} \leq 0$. Since $b > 0$, it is clear that a *sufficient* condition for $\bar{q} \leq 0$ is that the demand elasticity $(-f/(xf'))$ is smaller than one for all $p \leq c$. If the second phase does not exist (due to $\bar{q} \leq 0$ or S_0 sufficiently low to make $e^{-rT_3}(c - b) \geq \bar{q}$), the monopolist's solution is simply to set $x(t) = f^{-1}(c)$ until the resource is exhausted.

3. COMPARISON OF THE THREE CASES

We now compare the three monopoly solutions. In particular, we compare the initial price and extraction, remembering that p is strictly increasing in q $(p = f(g^{-1}(q)))$ and that x is strictly decreasing in q $(x = g^{-1}(q))$.

The first difference we notice is that while the natural resource never is completely exhausted when no substitute exists, it is exhausted in finite time in both the cases in which a substitute exists.

Since the first two monopoly cases both are characterized by $q(t)$ increasing with the rate r whenever extraction takes place, $q_2(t) - q_1(t)$ must have the same sign for all $t < T_2$. But since the resource is exhausted in finite time when a substitute exists, and x is higher the lower q is, we must have $q_2(t) < q_1(t)$ for all $t < T_2$. From the relationship between $q(t)$, $p(t)$ and $x(t)$ we therefore can conclude that

$$\begin{aligned} p_2(t) &< p_1(t) \qquad \text{for all } t, \\ x_2(t) &> x_1(t) \qquad \text{for all } t < T_2 . \end{aligned} \tag{17}$$

In other words, the resource is extracted at a faster rate before exhaustion occurs in the case with a substitute than in the case without a substitute. The price of the resource and substitute is therefore always lowest in the case with a substitute.

Let us now compare the date of exhaustion in the two cases with a substitute. Assume first that $T_3 \geq T_2$. From (9) and (15) we see that this implies $q_3(t) \leq q_2(t)$ and $x_3(t) \geq x_2(t)$, with a strict inequality holding for t sufficiently close to T_3. But since the same total stock S_0 is assumed to be extracted by the date of exhaustion in both cases, this inequality must imply $T_3 < T_2$. This contradiction proves that $T_3 \geq T_2$ cannot be true, i.e., we have

$$T_3 < T_2 . \tag{18}$$

In other words, the resource is exhausted at an earlier date when the substitute is supplied competitively than when it is controlled by the resource owning monopoly.

From (9) it is clear that we may well have $p_2(t) > c$ for all t. Since $p_3(t) \leqslant c$, this means that $p_2(t) > p_3(t)$ for all t is possible. However, we see from (9), (15), and (18) that

$$\begin{aligned} & p_3(0) > p_2(0), \\ & x_3(0) < x_2(0) \qquad \text{if} \quad p_3(0) < c. \end{aligned} \tag{19}$$

This result is rather surprising. The case in which the substitute is supplied competitively is in a sense more competitive than the case in which the monopolist controls substitute production as well as resource extraction. Intuitively, one might expect the price alway to be lowest in the most competitive case, at least for some well-behaved demand functions. The result above shows that this is never true unless the constraint $p_3(t) \leqslant c$ is binding for all t: The initial price is *higher*, and initial extraction *lower*, in the case with a competitively supplied substitute than in the case in which the monopolist controls the substitute production, as long as $p_3(0) < c$.

Since $T_3 < T_2$, we obviously must have $x_3(t) > x_2(t)$ for some $t < T_3$, i.e.,

$$\begin{aligned} p_3(t) &< p_2(t), \\ x_3(t) &> x_2(t) \end{aligned} \tag{20}$$

for t smaller than but sufficiently close to T_3. This result helps explain why $p_3(0) > p_2(0)$ is not so surprising after all: The existence of the constraint $p_3(t) \leq c$ when the substitute is supplied competitively prevents the monopolist from charging as high a price sometime in the future as he would have if he controlled the production of the substitute. It then seems reasonable that some of this reduced future price will be compensated for by raising the price early in the period of extraction.

From (17) and (20) we see that

$$\begin{aligned} p_3(t) &< p_1(t), \\ x_3(t) &> x_1(t) \end{aligned} \tag{21}$$

for t smaller than but sufficiently close to T_3. However, with the general assumptions used so far it does not seem possible to sign $p_3(0) - p_1(0)$ or $x_3(0) - x_1(0)$ unambiguously even for the case in which $p_3(0) < c$. In the next section we see that this can be done for the special case with zero extraction costs and a constant demand elasticity.

4. ZERO EXTRACTION COSTS AND A CONSTANT DEMAND ELASTICITY

In this section we assume that $b = 0$ and that

$$f(x) = x^{-a}, \qquad \text{where} \qquad 0 < a < 1. \tag{22}$$

This implies that

$$q = g(x) = (1 - a)\, x^{-a} \qquad \text{and} \qquad \bar{q} = (1 - a)c. \tag{23}$$

From (8) and (23) we get

$$x_1(t) = x_1(0)\, e^{-(r/a)t}, \tag{24}$$

which inserted into (6) gives, after integrating,

$$x_1(t) = \frac{r}{a} S_0 e^{-(r/a)t}. \tag{25}$$

It is well known [5, 8] that the extraction path described by (25) is identical to the extraction path under perfect competition.

From (9) and (23) we get

$$x_2(t) = \left(\frac{1-a}{c}\right)^{1/a} e^{(r/a)(T_2 - t)}. \tag{26}$$

Inserting this equation into (6) and integrating (remembering that $x_2(t) \to 0$ for $t \to T_2$) gives

$$\left(\frac{1-a}{c}\right)^{1/a} e^{(r/a)T_2} = \frac{r}{a} S_0 + \left(\frac{1-a}{c}\right)^{1/a},$$

so that (26) may be rewritten as

$$x_2(t) = \left[\frac{r}{a} S_0 + \left(\frac{1-a}{c}\right)^{1/a}\right] e^{-(r/a)t}, \tag{27}$$

which holds for all $t \leq T_2$. Comparing (27) with (25) of course confirms our general result (17).

Dasgupta and Stiglitz [1], Hoel [3], and Nordhaus [6] have shown that the competitive extraction path when a substitute exists is given by (9), except that q_2 is substituted by $p_2 - b$. In the present case this is the same as substituting c with $c(1-a)^{-1}$ in (9). This means that the competitive solution is given by (27) with $(1-a)/c$ substituted by $1/c$ (see [1, Sect. 2.1]). We therefore have the following result for our present case: While the competitive and monopolistic extraction paths are identical when no substitute exists, the presence of a substitute gives the competitive path a higher rate of resource extraction up till the date of exhaustion than the rate of extraction is when a monopolist controls resource extraction and substitute production.

Let us now turn to the case in which a monopolist extracts the resource and the substitute is supplied competitively. From our result (19) and what we said above, it is clear that unless $p_3(0) = c$, this intermediate market structure does *not* have an initial price and initial extraction rate which lies between the prices and extraction rates of full monopoly and full competition. To study the present market solution in more detail, we find an explicit solution for $x_3(t)$.

From (15) and (23) we get

$$q_3(t) = c \cdot \mathrm{Min}[1 - a, e^{-r(T_3 - t)}],$$

which together with (23) gives

$$x_3(t) = c^{-1/a} \cdot \mathrm{Max}[1, (1-a)^{1/a} e^{(r/a)(T_3 - t)}]. \tag{28}$$

The case in which $p_3(t) = c$ for all t is uninteresting, as it simply gives $x_3(t) = c^{-1/a}$ for all t until exhaustion occurs. Let us therefore assume that

S_0 is sufficiently large to make $p_3(0) < c$. Denoting the time when $p_3(t) = c$ is reached by t_3, we see from (28) that

$$e^{(r/a)(T_3 - t_3)} = (1 - a)^{-1/a} \tag{29}$$

or

$$T_3 - t_3 = -\frac{1}{r}\ln(1 - a). \tag{29'}$$

In other words, the period with constant extraction is independent of c and S_0.

Inserting (28) in (6) and using $x_3(t) = 0$ for $t > T_3$ gives us

$$[(1 - a)^{1/a} e^{(r/a)T_3}] \int_0^{t_3} e^{-(r/a)t}\,dt + (T_3 - t_3) = c^{1/a}S_0,$$

which after integrating, using (29) and (29′) and rearranging terms gives

$$(1 - a)^{1/a} e^{(r/a)T_3} = \frac{r}{a} S_0 c^{1/a} + \frac{a + \ln(1 - a)}{a}.$$

By inserting this expression into (28) we obtain

$$x_3(t) = \operatorname{Max}\left\{c^{-1/a}, \left[\frac{r}{a} S_0 + \frac{a + \ln(1 - a)}{a} c^{-1/a}\right] e^{-(r/a)t}\right\} \tag{30}$$

which is valid for all $t < T_3$. From our assumption $p_3(0) < c$ we have

$$x_3(0) = \frac{r}{a} S_0 + \frac{a + \ln(1 - a)}{a} c^{-1/a}. \tag{31}$$

Since $0 < a < 1$, we have $a + \ln(1 - a) < 0$. Comparing (31) with (25) we therefore can conclude that in our present case

$$\begin{aligned} &p_3(0) > p_1(0), \\ &x_3(0) < x_1(0) \qquad \text{if} \quad p_3(0) < c. \end{aligned} \tag{32}$$

In other words, the initial price is *higher* and the initial extraction is *lower* when a substitute is supplied competitively than when no substitute exists. This rather surprising result can be given an interpretation similar to the one we gave after (20): Introducing a competitively supplied substitute affects the monopolist in the same way as introducing a maximal price c. If such a maximal price depresses the monopolist's future price, some of this price reduction may be compensated for by raising the price early in the period of extraction.

When a substitute exists, one expects the price to be lower the lower this cost is. This is indeed the case when the monopolist controls substitute production: From (27) we get $\partial x_2(t)/\partial c < 0$ for all t, which means that the extraction will be higher, exhaustion will occur earlier and the price will be lower the lower is the cost of producing the substitute. From what we said above about the competitive solution, it is clear that the same results hold under full competition. It is also easy to see that these results are valid also for the more general case treated in Sections 2 and 3.

From (31) it is clear that as long as $p_3(t) < c$ we get $\partial x(t)/\partial c > 0$. In other words, the initial price is *higher* and the initial extraction is *lower* the lower the cost of producing the substitute is. For larger values of t we have $p_3(t) = c$, so that the opposite result is true for this part of the extraction period. Clearly $\partial p_3(0)/\partial c < 0$ must imply $\partial t_3/\partial c > 0$ by the definition of t_3 (i.e., $p_3(t_3) = c$). But since $T_3 - t_3$ is independent of c (cf. (29′)), this means that the resource will be exhausted earlier the lower is the cost of producing the substitute.

We saw above that the initial resource price is higher the lower the cost of substitute production is, as long as this cost is relatively high. For lower costs of producing the substitute (giving $p_3(0) = c$), the initial resource price is lower the lower this cost is. One may be interested in knowing more precisely how low c may be before $p_3(0) = c$. From (30) it is clear that the critical value of c, denoted by $\bar{c}$, must satisfy

$$\bar{c}^{-1/a} = \frac{r}{a} S_0 + \frac{a + \ln(1-a)}{a} \bar{c}^{-1/a},$$

i.e.,

$$\bar{c} = \left[\frac{-\ln(1-a)}{rS_0}\right]^a. \tag{33}$$

From (33) we see that $\bar{c}$ is higher the lower the initial resource stock S_0 is and the lower the demand elasticity $1/a$ is (i.e., the closer to zero $1 - a$ is, remembering our assumption $a < 1$). In the limiting case when $a = 1$ we get $\bar{c} = +\infty$, i.e., $p_3(0) = c$ no matter what values S_0 and c have.

5. CONCLUSIONS

We have seen that for quite general demand functions, the intermediate market structure in which a monopolist controls resource extraction, but a substitute is supplied competitively, gives a higher initial price and a lower initial resource extraction than we get when a single monopolist controls both resource extraction and substitute production. This conclusion holds as long as the cost of producing the substitute exceeds the initial price charged by a single monopolist in control of both resource extraction and production of the substitute.

In the special case treated in Section 4, we found that introducing a competitively supplied substitute would increase the initial price the resource monopolist would charge, as long as the cost of producing the substitute exceeds this price.

A market structure with resource extraction fully monopolized and substitute production perfectly competitive is of course a very special situation. If our results were only valid for this special market structure, their significance would be rather limited. However, there is nothing in the preceding analysis indicating that these results hold only for the market structure studied. There probably also exist more realistic market structures giving similar results, for instance some cases in which the market conditions both of resource extraction and substitute production lie somewhere between perfect competition and full monopoly, with the resource extraction being more monopolized, in some sense, than the substitute production.

REFERENCES

1. P. Dasgupta and J. E. Stiglitz, "Uncertainty and the Rate of Extraction under Alternative Institutional Arrangements," Technical Report No. 179, Institute for Mathematical Studies in the Social Sciences, Stanford University, May 1976.
2. G. Heal, The relationship between price and extraction cost for a resource with a backstop technology, *Bell J. Econ.* 7 (1976), 371–378.
3. M. Hoel, "Resource Extraction under some Alternative Market Structures," Mathematical Systems in Economics 39, Verlag Anton Hain, Meisenheim am Glan, 1978.
4. M. I. Kamien and N. L. Schwartz, A note on resource usage and market structure, *J. Econ. Theory* **15** (1977), 394–397.
5. T. R. Lewis, Monopoly exploitation of an exhaustible resource, *J. Environmental Econ. Management* 3 (1976), 198–204.
6. W. D. Nordhaus, The allocation of energy resources, *Brookings Papers Econ. Activity* 3 (1973), 529–576.
7. R. M. Solow and F. Y. Wan, Extraction costs in the theory of exhaustible resources, *Bell J. Econ.* 44 (1976), 359–370.
8. J. E. Stiglitz, Monopoly and the rate of extraction of exhaustible resources, *Amer. Econ. Rev.* **66** (1976), 655–661.
9. J. L. Sweeney, Economics of depletable resources: Market forces and intertemporal bias, *Rev. Econ. Stud.* **44** (1977), 125–142.

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Probabilities of Election Outcomes for Large Electorates*

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1. INTRODUCTION

This paper applies results for quadrivariate normal orthant probabilities to election situations to derive probabilities of the occurrences of several interesting events as the number of voters approaches infinity. Since large electorates are not uncommon, the derived limit probabilities have obvious relevance for practical situations. This is discussed further in the final section of the paper.

The generic voting situation used throughout the paper consists of m candidates or alternatives $a_1, a_2, \ldots, a_m$, and an odd number n of voters. Each voter is assumed to have one of the $m!$ linear orders on $\{a_1, a_2, \ldots, a_m\}$ as his or her preference order. The distribution of voters' preference orders is presumed to be governed by the so-called impartial culture hypothesis. This means that a randomly chosen voter will have a randomly chosen linear order as his or her preference order with probability $1/m!$, and that this probability is independent of other voters' preferences.

The events whose limit probabilities are derived in ensuing sections involve aspects of the simple majority relation on $\{a_1, \ldots, a_m\}$ along with several point-count methods for determining winners in voting situations. Each derivation is based on four discrete variates, say v_1 through v_4, that involve the linear orders on the alternatives and have zero expected values under the

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impartial culture hypothesis. The variates are defined in each case so that the primary event under consideration occurs if and only if $\bar{v}_i > 0$ for each i, where $\bar{v}_i$ is the average of the values of v_i for the n voters' preference orders. Since the event $\{\bar{v}_i > 0: i = 1, 2, 3, 4\}$ is equivalent to $\{\bar{v}_i n^{1/2} > 0: i = 1, 2, 3, 4\}$, we obtain the limiting quadrivariate normal distribution of $(\bar{v}_1 n^{1/2}, \bar{v}_2 n^{1/2}, \bar{v}_3 n^{1/2}, \bar{v}_4 n^{1/2})$ from the four-variate extension of the central limit theorem [12, 18]. This distribution has zero mean and covariance matrix $[E(v_i v_j)]$ with corresponding correlation matrix ρ. The desired limit probability of the primary event is then obtained as the positive orthant probability of the limiting distribution. Each correlation matrix that we obtain has a form for which the positive orthant probability is known from previous research.

The foregoing approach is applied to five types of voting events. These events and the results obtained for their limit probabilities are outlined following a few introductory definitions. Each event is then examined more closely in succeeding sections.

Let S be the binary relation of strict simple majority for a generic voting situation, so that $a_i S a_j$ if and only if more voters prefer a_i to a_j than prefer a_j to a_i. An odd number of voters with linear preference orders ensures that S is complete. An alternative is a *Condorcet alternative* if and only if it bears the relation S to [has a simple majority over] every other alternative. An alternative is the *Condorcet loser* if all other alternatives bear the relation S to it.

Given a real vector $w = (w_1, w_2, \ldots, w_m)$, the w score of alternative a_i equals $\sum_j w_j n_{ij}$, where n_{ij} is the number of voters who have a_i in the jth position in their preference orders. (First position is most preferred; mth position is least preferred.) An alternative is a *w-alternative* if and only if its w score is as great as the w score of every other alternative. When $w = (m - 1, m - 2, \ldots, 1, 0)$, a w-alternative is called a *Borda alternative*, and, when $w = (1, 0, \ldots, 0)$, a w-alternative is referred to as a *simple plurality alternative*.

The first event of interest occurs if and only if there is a Condorcet alternative. As Condorcet [2] noted, cycles in S can prevent the existence of such an alternative. Let P_m be the limit as $n \to \infty$ of the probability that there is a Condorcet alternative with m alternatives. Guilbaud [11] showed that

$$P_3 = \tfrac{3}{4} + (3/(2\pi)) \sin^{-1} \tfrac{1}{3}. \tag{1}$$

May [14] presented a theorem that was later proved by Fishburn [7] which implies that

$$P_4 = \tfrac{1}{2} + (3/\pi) \sin^{-1} \tfrac{1}{3}. \tag{2}$$

Neimi and Weisberg [15] extended the work of Ruben [17] to obtain values of P_m for all $m \leq 49$. However, since Ruben's approach is a recursive procedure involving evaluation of functions by quadrature, it yields only numerical values of P_m without giving specific forms as in (1) and (2). Specific forms for P_5 and P_6 are derived in the next section of the present paper.

The second event we consider occurs if and only if the simple majority relation S is transitive. Let P_m' be the limit probability that S is transitive when there are m alternatives. Although $P_3' = P_3$, P_4' is not equal to P_4 since there can be a Condorcet alternative when S cycles on the other three alternatives. An expression for P_4' is obtained in Section 3.

Our third event involves the effect of abstentions on election outcomes. It is assumed that each potential voter has the same probability $k \in [0, 1]$ of actually voting. Given m alternatives, we then let $P_m(k)$ be the limit probability that there is a Condorcet alternative for the set of all potential voters, there is also a Condorcet alternative for the set of all nonabstaining voters, and these two Condorcet alternatives are identical. The conditional probability that there is a Condorcet alternative for the nonabstaining voters that coincides with the global Condorcet alternative, given that the latter exists, is therefore $P_m(k)/P_m$. Gehrlein and Fishburn [9] have shown for the two-alternative case that

$$P_2(k) = (1/\pi)\cos^{-1}(-k^{1/2}).$$

In Section 4 we derive a simple expression for $P_3(k)$, thus extending the study of the effects of abstentions on Condorcet alternatives to three-candidate elections.

The fourth event is concerned with coincidence between the Borda and simple plurality point-count winners. Given m alternatives, let Q_m be the limit probability that a Borda alternative will be identical to a simple plurality alternative. (As $n \to \infty$, the probabilities of tied Borda alternatives or tied plurality alternatives goes to zero, so that it suffices to consider cases with unique Borda and simple plurality alternatives.) Since the Borda and simple plurality w's are identical when $m = 2$, $Q_2 = 1$. An exact relation for Q_3 is determined in Section 5.

The fifth event considered is the coincidence between the Condorcet loser and the plurality winner. As before, the probability of tied plurality winners goes to zero as $n \to \infty$, so it is sufficient to consider the case in which the simple plurality winner is unique. For m alternatives, let T_m be the limit probability that a Condorcet loser will be a simple plurality alternative. Section 6 presents an exact relation for T_3.

Of related interest is the likelihood of coincidence between the Condorcet alternative and the w-alternative. Given $m = 3$ and $w = (1, \lambda, 0)$, with $0 \leq \lambda \leq 1$, Gehrlein and Fishburn [10] obtain an expression for the limit probability $P_3^*(\lambda)$ that there is a Condorcet alternative and the w-alternative is identical to the Condorcet alternative. These limiting coincidence probabilities for the Borda and simple plurality cases are given, respectively, by

$$
\begin{aligned}
P_3^*(\tfrac{1}{2}) &= \tfrac{3}{2} - (3/(2\pi))[\cos^{-1}(8^{1/2}/3) + \cos^{-1}(2^{1/2}/3)], \\
P_3^*(0) &= \tfrac{1}{4} + (3/(4\pi))[\sin^{-1}(2^{1/2}/3^{1/2}) + \sin^{-1}(1/6^{1/2}) + \tfrac{1}{2}\sin^{-1}\tfrac{1}{3}] \qquad (3) \\
&\quad + (3/(4\pi^2))\{[\sin^{-1}(2^{1/2}/3^{1/2})]^2 - \tfrac{1}{2}[\sin^{-1}\tfrac{1}{3}]^2\} \\
&\quad + (3/(4\pi^2))\, L,
\end{aligned}
$$

where

$$L = 2Li_2(f, \cos^{-1}\tfrac{1}{3}) - Li_2(f^2, \cos^{-1}\tfrac{1}{3}) + \tfrac{1}{4}Li_2(f^4) - \tfrac{1}{2}Li_2(f^2) \qquad (4)$$

with $f = 2 - 3^{1/2}$, $Li_2(z)$ the dilogarithmic function with real argument z, and $Li_2(z, \theta)$ the real part of the dilogarithmic function with complex argument (z, θ). Lewin [13] provides a detailed analysis of dilogarithmic functions. A simpler expression for L is obtained in the next section.

Numerical values of the foregoing coincidence probabilities that are accurate to six decimals are $P_3^*(\tfrac{1}{2}) = 0.822119$ and $P_3^*(0) = 0.690763$. The latter two numbers should be divided by $P_3 = 0.912260$ to obtain the respective coincidence probabilities given that there is a Condorcet alternative. Numerical values of other events are noted in later sections.

Our main purpose is to develop analytical expressions for the probabilities of the events mentioned above. Section 7 discusses some of the practical implications of the derived results. The references that we have found most useful in this effort are cited as we proceed. In addition to these, we note here that Dutt [5] has developed an efficient method for evaluating quadrivariate normal orthant probabilities based on Gaussian quadrature that is used by Dutt and Lin [6] to compute a table of orthant probabilities for a variety of specific correlation matrices. Although none of their matrices coincide with those in the present paper, their approach provides an alternative method of computing the probabilities of the events that are analyzed in ensuing sections.

2. CONDORCET ALTERNATIVES

To determine P_5, the limit probability for the existence of a Condorcet alternative when $m = 5$, we obtain the limiting probability P_5^1 that a_1 is the Condorcet alternative. Then $P_5 = 5P_5^1$ by the symmetry of impartial culture. The four discrete variates for this case are u_1 through u_4, defined on the set of linear orders on $\{a_1, \ldots, a_5\}$ by

$$
\begin{aligned}
u_i &= 1 \text{ if } a_1 \text{ is ranked ahead of (preferred to) } a_{i+1}, \\
u_i &= -1 \text{ otherwise.}
\end{aligned}
$$

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Since the probability of $\bar{u}_i = 0$ vanishes as $n \to \infty$, P_5^1 is the limit of the probability that $\{\bar{u}_i > 0: i = 1, 2, 3, 4\}$.

For each voter the probability that $u_i = 1$ is $\frac{1}{2}$ and the probability that $u_i = -1$ is $\frac{1}{2}$ under impartial culture. In addition, $E(u_i) = 0$, $E(u_i^2) = 1$, and $E(u_i u_j) = \frac{1}{3}$ for $i \neq j$. Hence, as n gets large, the distribution of $(\bar{u}_1 n^{1/2}, \bar{u}_2 n^{1/2}, \bar{u}_3 n^{1/2}, \bar{u}_4 n^{1/2})$ approaches the quadrivariate normal distribution with correlation matrix

$$\rho_1 = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ - & 1 & \frac{1}{3} & \frac{1}{3} \\ - & - & 1 & \frac{1}{3} \\ - & - & - & 1 \end{bmatrix}.$$

Hence P_5^1 is the positive orthant probability $\Phi_4(\rho_1)$ of the quadrivariate normal distribution with correlation matrix ρ_1. Cheng [3] presents an expression for $\Phi_4(\rho_1^*)$ when

$$\rho_1^* = \begin{bmatrix} 1 & \alpha & \alpha\beta & \alpha\beta \\ - & 1 & \alpha\beta & \alpha\beta \\ - & - & 1 & \alpha \\ - & - & - & 1 \end{bmatrix},$$

which, in our special case of $\alpha = \frac{1}{3}$ and $\beta = 1$, yields

$$P_5 = (5/16) + (15/(4\pi))\sin^{-1}\tfrac{1}{3} - (5/(4\pi^2))[\sin^{-1}\tfrac{1}{3}]^2 + (5/\pi^2)L, \qquad (5)$$

where L is given by (4). Since (3) and (5) both contain L, P_5 can be written in terms of $P_3^*(0)$ and $\sin^{-1}$ values.

However, a simpler relation for P_5 can be obtained from Abrahamson [1] or Posnyakov [16], who examine orthant probabilities when all correlations are equal. Posnyakov's relation implies that

$$P_5 = (5/16) + (15/(4\pi))\sin^{-1}\tfrac{1}{3} + (15/(2\pi^2))\int_0^{1/3} \frac{\sin^{-1}[\gamma/(1+2\gamma)]}{(1-\gamma^2)^{1/2}}\,d\gamma. \qquad (6)$$

Together, (5) and (6) imply that

$$L = \tfrac{1}{4}[\sin^{-1}\tfrac{1}{3}]^2 + \tfrac{3}{2}\int_0^{1/3} \frac{\sin^{-1}[\gamma/(1+2\gamma)]}{(1-\gamma^2)^{1/2}}\,d\gamma,$$

which can be used instead of (4). Substitution of the new expression for L in (3) yields a simpler expression for $P_3^*(0)$.

An expression for P_6 is obtained directly from (1), (5) or (6), and the recursion relation $P_6 = 3 - 5P_3 + 3P_5$ derived by Gehrlein and Fishburn [8].

The result is

$$P_6 = (3/16) + (15/(4\pi))\sin^{-1}\tfrac{1}{3} + (45/(2\pi^2))\int_0^{1/3}\frac{\sin^{-1}[\gamma/(1+2\gamma)]}{(1-\gamma^2)^{1/2}}\,d\gamma.$$

Numerical values of P_3 through P_6 are $P_3 = 0.91226$, $P_4 = 0.82452$, $P_5 = 0.74869$, and $P_6 = 0.68476$.

3. TRANSITIVE SIMPLE MAJORITY

To determine P_4', the limit probability that S is transitive on $\{a_1, a_2, a_3, a_4\}$, we obtain the limit probability R_4' for the event that both of a_1 and a_2 stand in the relation S to both of a_3 and a_4. Since R_4' accounts for four of the 24 linear orders on $\{a_1, a_2, a_3, a_4\}$, it follows from the symmetry of impartial culture that $P_4' = 6R_4'$.

The four discrete variates for the R_4' derivation are x_1 through x_4, defined on the linear orders on $\{a_1, \ldots, a_4\}$ as follows:

for $i = 1, 2$: $x_i = 1$ if a_1 is ranked ahead of a_{i+2},
$x_i = -1$ otherwise;

for $i = 3, 4$: $x_i = 1$ if a_2 is ranked ahead of a_i,
$x_i = -1$ otherwise.

It follows that R_4' is the limit of the probability that $\{\bar{x}_i > 0: i = 1, 2, 3, 4\}$.

Under impartial culture, $E(x_i) = 0$, $E(x_i^2) = 1$, $E(x_1x_4) = E(x_2x_3) = 0$, and $E(x_1x_2) = E(x_1x_3) = E(x_2x_4) = E(x_3x_4) = \frac{1}{3}$. Hence, as $n \to \infty$, $(\bar{x}_1 n^{1/2}, \bar{x}_2 n^{1/2}, \bar{x}_3 n^{1/2}, \bar{x}_4 n^{1/2})$ becomes quadrivariate normally distributed with correlation matrix

$$\rho_2 = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ - & 1 & 0 & \frac{1}{3} \\ - & - & 1 & \frac{1}{3} \\ - & - & - & 1 \end{bmatrix}$$

so that $R_4' = \Phi_4(\rho_2)$. Cheng [3] and David and Mallows [4] derive an expression for $\Phi_4(\rho^*)$ when ρ^* is the same as ρ_2 with $\frac{1}{3}$ replaced by x. In our special case of $x = \frac{1}{3}$, the David and Mallows result yields

$$P_4' = \tfrac{3}{8} + (6/\pi^2)\int_0^{1/3}\frac{\cos^{-1}[-\gamma/(1-2\gamma^2)]}{(1-\gamma^2)^{1/2}}\,d\gamma.$$

Numerical evaluation shows that P_4' is approximately 0.73946. This may be compared to $P_4 = 0.82452$. Hence the probability of getting a Condorcet alternative when S cycles on the other three alternatives is $P_4 = P_4' = 0.08506$.

4. ABSTENTIONS AND CONDORCET ALTERNATIVES

Let $P_3^1(k)$ be the limit probability that a_1 is the global Condorcet alternative from $\{a_1, a_2, a_3\}$ and that, when each potential voter has probability $k > 0$ of actually voting, a_1 is also the Condorcet alternative for the set of non-abstaining voters. Then, by symmetry, $P_3(k) = 3P_3^1(k)$. We shall comment later on $k > 0$.

The discrete variates used for the derivation of $P_3^1(k)$ are defined as follows:

for $i = 1, 2$: $y_i = 1$ if a_1 is ranked ahead of a_{i+1},
$y_i = -1$ otherwise;

for $i = 3, 4$: $y_i = 1$ if a_1 is ranked ahead of a_{i-1} and the voter actually votes,
$y_i = 0$ if the voter abstains,
$y_i = -1$ if a_{i-1} is ranked ahead of a_1 and the voter actually votes.

The probabilities for $y_i = 1, 0$, and -1 for $i \in \{3, 4\}$ are respectively $k/2$, $1 - k$, and $k/2$. The desired probability $P_3^1(k)$ is the limit of the probability that $\{\bar{y}_i > 0: i = 1, 2, 3, 4\}$. Note here that the probability of $\bar{y}_3 = 0$ or $\bar{y}_4 = 0$ vanishes as n gets large, provided that $k > 0$.

In the present case $E(y_i) = 0$, $E(y_1^2) = E(y_2^2) = 1$, $E(y_3^2) = E(y_4^2) = k$, $E(y_1 y_3) = E(y_2 y_4) = k$, $E(y_1 y_2) = \frac{1}{3}$, and $E(y_1 y_4) = E(y_2 y_3) = E(y_3 y_4) = k/3$. Hence the distribution of $(\bar{y}_1 n^{1/2}, \bar{y}_2 n^{1/2}, \bar{y}_3 n^{1/2}, \bar{y}_4 n^{1/2})$ approaches the quadrivariate normal distribution that has correlation matrix

$$\rho_3 = \begin{bmatrix} 1 & \frac{1}{3} & k^{1/2} & k^{1/2}/3 \\ - & 1 & k^{1/2}/3 & k^{1/2} \\ - & - & 1 & \frac{1}{3} \\ - & - & - & 1 \end{bmatrix}$$

so that $P_3^1(k) = \Phi_4(\rho_3)$. Cheng [3] presents a result for $\Phi_4(\rho_3^*)$ when

$$\rho_3^* = \begin{bmatrix} 1 & \alpha & \beta & \alpha\beta \\ - & 1 & \alpha\beta & \beta \\ - & - & 1 & \alpha \\ - & - & - & 1 \end{bmatrix}. \qquad (7$$

which, for our special case of $\alpha = \frac{1}{3}$ and $\beta = k^{1/2}$, yields

$$\begin{aligned} P_3(k) = (3/16) &+ (3/(4\pi))[\sin^{-1}\tfrac{1}{3} + \sin^{-1}(k^{1/2}) + \sin^{-1}(k^{1/2}/3)] \\ &+ (3/(4\pi^2))\{[\sin^{-1}\tfrac{1}{3}]^2 + [\sin^{-1}(k^{1/2})]^2 - [\sin^{-1}(k^{1/2}/3)]^2\}. \end{aligned} \tag{8}$$

The following values of $P_3(k)$ are accurate to six decimal places:

$P_3(0.1) = 0.386449,$

$P_3(0.3) = 0.482587,$

$P_3(0.5) = 0.564283,$

$P_3(0.7) = 0.650084,$

$P_3(0.9) = 0.763094.$

The value of $P_3(0)$ depends on our interpretation of Condorcet alternatives in the set of nonabstaining voters when the latter set is empty. The values $P_3(0) = 0$, $P_3(0) = P_3$, and $P_3(0) = P_3/3$ can be obtained under different interpretations. We get $P_3(0) = 0$ under our original definition since no alternative beats any other alternative by a simple majority within the empty set of voters. The value $P_3(0) = P_3$ arises if we relax the original definition and say that a Condorcet alternative is any alternative that ties or beats every other alternative, so that all alternatives are Condorcet alternatives within the empty set of voters. If within the latter context we are concerned about the probability that there is a global Condorcet alternative that is the same as the winner based on a random selection from the set of Condorcet alternatives determined by the actual voters (when this set is not empty), then $P_3(0) = P_3/3$. When $k \to 0$, the same value of $P_3(k)$ is obtained under all three interpretations.

It may also be noted that none of the three values for $P_3(0)$ in the preceding paragraph equals the limit (0.277406) of $P_3(k)$ as $k \to 0$ since, by (8), this limit is $P_3^2/3$. In other words, the result obtained by setting $k = 0$ and then letting $n \to \infty$ is different from the result obtained by first letting $n \to \infty$ and then letting $k \to 0$ which, as May [14] notes, should come as no surprise in this type of situation.

5. Borda and Simple Plurality

To determine Q_3, the limit probability that the Borda and simple plurality alternatives coincide when $m = 3$, we obtain the limit probability Q_3^1 that a_1 is both the Borda alternative and the simple plurality alternative. Then $Q_3 = 3Q_3^1$. The variates for this case are z_1 through z_4, defined on the linear orders on $\{a_1, a_2, a_3\}$ as follows:

$$z_1 = \begin{cases} 1 & \text{if } a_1 \text{ is first,} \\ -1 & \text{if } a_2 \text{ is first,} \\ 0 & \text{if } a_3 \text{ is first,} \end{cases} \qquad z_2 = \begin{cases} 1 & \text{if } a_1 \text{ is first,} \\ -1 & \text{if } a_3 \text{ is first,} \\ 0 & \text{if } a_2 \text{ is first,} \end{cases}$$

$$\text{for } i = 3, 4: \quad z_i = \begin{cases} 2 & \text{if } a_1 \text{ is first and } a_{i-1} \text{ is last,} \\ 1 & \text{if } a_1 \text{ is ranked immediately before } a_{i-1}, \\ -1 & \text{if } a_{i-1} \text{ is ranked immediately before } a_1, \\ -2 & \text{if } a_{i-1} \text{ is first and } a_1 \text{ is last.} \end{cases}$$

Thus, for n voters, a_1 beats a_2 and a_3 under simple plurality if $\bar{z}_1 > 0$ and $\bar{z}_2 > 0$, and a_1 beats a_2 and a_3 under the Borda point-count system if $\bar{z}_3 > 0$ and $\bar{z}_4 > 0$.

Under impartial culture, $E(z_i) = 0$, $E(z_1^2) = E(z_2^2) = \frac{2}{3}$, $E(z_3^2) = E(z_1^2) = 2$, $E(z_1 z_2) = \frac{1}{3}$, $E(z_1 z_4) = E(z_2 z_3) = \frac{1}{2}$, and $E(z_1 z_3) = E(z_2 z_1) = E(z_3 z_1) = 1$. Hence the limiting distribution of $(\bar{z}_1 n^{1/2}, \bar{z}_2 n^{1/2}, \bar{z}_3 n^{1/2}, \bar{z}_1 n^{1/2})$ is quadrivariate normal with correlation matrix

$$\rho_1 = \begin{bmatrix} 1 & \frac{1}{2} & 3^{1/2}/2 & 3^{1/2}/4 \\ & 1 & 3^{1/2}/4 & 3^{1/2}/2 \\ & \cdot & 1 & \frac{1}{2} \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

so that $Q_3^1 = \Phi_4(\rho_1)$. Since ρ_1 coincides with ρ_3^4 as given by (7) when $\alpha = \frac{1}{2}$ and $\beta = 3^{1/2}/2$, Cheng's result for this special case yields

$$Q_3 = (123/144) + (3/(4\pi^2))[\sin^{-1}(13^{1/2}/4)]^2,$$

which is approximately 0.758338.

6. SIMPLE PLURALITY AND CONDORCET LOSER

Let T_3^1 be the limit probability that alternative a_1 in the set of alternatives $\{a_1, a_2, a_3\}$ is both the simple plurality winner and the Condorcet loser. The symmetry of impartial culture implies that $T_3 = 3T_3^1$.

The four discrete variables that are used to obtain a relation for T_3^1 are as follows:

$$\text{for } i = 1, 2: \quad \begin{aligned} w_i &= 1 \text{ if } a_1 \text{ is ranked first,} \\ w_i &= -1 \text{ if } a_{i+1} \text{ is ranked first,} \\ w_i &= 0 \text{ if } a_{4-i} \text{ is ranked first;} \end{aligned}$$

$$\text{for } i = 3, 4: \quad \begin{aligned} w_i &= -1 \text{ if } a_1 \text{ is ranked ahead of } a_{i-1}, \\ w_i &= 1 \text{ if } a_{i-1} \text{ is ranked ahead of } a_1. \end{aligned}$$

The value of T_3^1 is given by the limit probability of $\{\bar{w}_i > 0: i = 1, 2, 3, 4\}$.

Under impartial culture, $E(w_i) = 0$, $E(w_1^2) = E(w_2^2) = \frac{2}{3}$, $E(w_3^2) = E(w_4^2) = 1$, $E(w_1w_2) = E(w_3w_4) = \frac{1}{3}$, $E(w_1w_3) = E(w_2w_4) = -\frac{2}{3}$, and $E(w_1w_4) = E(w_2w_3) = -\frac{1}{3}$. As $n \to \infty$, $(\bar{w}_1n^{1/2}, \bar{w}_2n^{1/2}, \bar{w}_3n^{1/2}, \bar{w}_4n^{1/2})$ becomes quadrivariate normally distributed with correlation matrix

$$\rho_5 = \begin{bmatrix} 1 & \frac{1}{2} & -2^{1/2}/3^{1/2} & -1/6^{1/2} \\ — & 1 & -1/6^{1/2} & -2^{1/2}/3^{1/2} \\ — & & 1 & \frac{1}{3} \\ — & — & — & 1 \end{bmatrix}$$

so that $T_3^1 = \Phi_4(\rho_5)$. Cheng [3] presents a relation for $\Phi_4(\rho_5^*)$ when

$$\rho_5^* = \begin{bmatrix} 1 & \alpha & \beta & \alpha\beta \\ — & 1 & \alpha\beta & \beta \\ & — & 1 & \alpha\beta^2 \\ — & — & — & 1 \end{bmatrix},$$

which, for our special case of $\alpha = \frac{1}{2}$ and $\beta = -2^{1/2}/3^{1/2}$, yields

$$\begin{aligned} T_3 = {} & \tfrac{1}{4} - (3/(4\pi))[\sin^{-1}(2^{1/2}/3^{1/2}) + \sin^{-1}(1/6^{1/2}) - \tfrac{1}{2}\sin^{-1}\tfrac{1}{3}] \\ & + (3/(4\pi^2))\{[\sin^{-1}(2^{1/2}/3^{1/2})]^2 - \tfrac{1}{2}[\sin^{-1}\tfrac{1}{3}]^2\} \\ & + (3/(4\pi^2))L, \end{aligned}$$

where L is given by (4). Numerical evaluation yields $T_3 = 0.033843$.

7. Discussion

Table I summarizes the numerical results obtained in this study. A major contribution of these results involves the comparison of election procedures for large electorates when the impartial culture hypothesis is applicable. For example, suppose a large electorate wishes to choose the Condorcet alternative as often as possible when there is one, but hesitates to adopt as a procedure that requires direct comparison of each pair of alternatives for the purpose of determining if there is a Condorcet alternative. The electorate might then consider either the plurality or Borda procedure. Table I shows for $m = 3$ that the probability that the Borda winner is the Condorcet alternative, given that the latter exists, is 0.9012. Moreover, the probability that the plurality winner is the Condorcet alternative, given that there is a Condorcet alternative, is 0.7572. Hence a direct comparison of these probabilities favors the Borda procedure.

This advantage, however, must be weighed against the increased complexity of the Borda procedure. For example, the Borda method requires voters to rank alternatives, whereas plurality asks them to state only their first choices.

TABLE I

Limit Probabilities for Election Outcomes under Impartial Culture with m Alternatives

Outcome	m = 3	4	5	6
A Condorcet alternative exists	0.91226	0.82452	0.74869	0.68476
Simple majorities are transitive	0.91226	0.73946		
Borda and plurality winners coincide	0.75834			
Borda and Condorcet alternatives coincide	0.82212			
Plurality and Condorcet alternatives coincide	0.69076			
Plurality winner and Condorcet loser coincide	0.03384			

A potential effect of using Borda instead of plurality could be to increase the proportion of voters who abstain, and this in turn would affect the likelihoods of electing the global Condorcet alternative (based on all potential voters) when there is one.

Other factors also enter into the comparison of plurality and Borda. For example, for $m = 3$ these procedures produce the same winner about 75 percent of the time. Another consideration is that a Condorcet loser cannot be elected by the Borda procedure--assuming no abstentions and sincere voting—whereas the plurality procedure has about a four percent chance of electing a Condorcet loser, given that one exists.

REFERENCES

1. I. G. Abrahamson, Orthant probabilities for the quadrivariate normal distribution, *Ann. Math. Statist.* **35** (1964), 1685–1703.
2. Marquis de Condorcet, "Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité de voix," Paris, 1785; reprinted, Chelsea, New York, 1973.
3. M. C. Cheng, The orthant probability of four Gaussian variates, *Ann. Math. Statist.* **40** (1969), 152–161.
4. F. N. David and C. L. Mallows, The variance of Spearman's rho in normal samples, *Biometrika* **48** (1961), 19–28.

5. J. E. Dutt, A representation of multivariate normal probability integrals by integral transformations, *Biometrika* **60** (1973), 637–645.
6. J. E. Dutt and T. K. Lin, A short table for computing normal orthant probabilities of dimensions 4 and 5, *J. Statist. Comput. Simul.* **4** (1975), 95–120.
7. P. C. Fishburn, A proof of May's theorem $P(m, 4)$ $2P(m, 3)$, *Behav. Sci.* **18** (1973), 212.
8. W. V. Gehrlein and P. C. Fishburn, The probability of the paradox of voting: A computable solution, *J. Econ. Theory* **13** (1976), 14–25.
9. W. V. Gehrlein and P. C. Fishburn, The effects of abstentions on elections outcomes, *Public Choice*, in press.
10. W. V. Gehrlein and P. C. Fishburn, Coincidence probabilities for simple majority and positional voting rules, *Social Sci. Res.*, in press.
11. G. Th. Guilbaud, Les théories de l'intérêt général et le problème logique de l'aggrégation, *Économie Appliquée* **5** (1952), 501-584.
12. N. L. Johnson and S. Kotz, "Distributions in Statistics: Continuous Multivariate Distributions," Wiley, New York, 1972.
13. L. Lewin, "Dilogarithms and Associated Functions," Macdonald Press, London, 1958.
14. R. M. May, Some mathematical remarks on the paradox of voting, *Behav. Sci.* **16** (1971), 143–151.
15. R. G. Niemi and H. F. Weisberg, A mathematical solution to the probability of the paradox of voting, *Behav. Sci.* **13** (1968), 317-323.
16. V. V. Posnyakov, A representation of the multidimensional normal distribution function, *Ukrainian Math. J.* **23** (1971), 473–476.
17. H. Ruben, On the moments of order statistics in samples from normal populations, *Biometrika* **41** (1954), 200–227.
18. S. S. Wilks, "Mathematical Statistics," Wiley, New York, 1962.

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Optimal Control Problems from Second-Order Difference Equations*

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We examine solutions $\hat{x} \in l_\infty(R^n)$ of the equations $F_t(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}) = 0$ and derive conditions for the existence of objective functions G_t so that $\hat{x}$ solves $\max_{x \in l_\infty} \sum G_t(x_t, x_{t+1})$. We specialize the conditions to time autonomous equations and apply them to a competitive industry in temporary equilibrium. The objective function in the example is of the cost benefit form: consumer surplus minus opportunity cost of labor minus industry-wide cost of investment.

1

We examine conditions under which a system of second-order difference equations is the Euler equation for a calculus of variations problem. Such systems can arise in economics from a group of agents individually optimizing their own behavior which generates the system of equations. If this system is also the Euler equation for a single optimization problem then this can both simplify the analysis of such systems and provide economic understanding of them.

Presented here is a discrete time analog of the continuous time problem studied by Brock [2]. The gist of the problem is: Since the solution to an optimal control problem satisfies a second-order difference equation (differential equation in continuous time), can we reverse the procedure and start with a solution to a second-order difference equation and find an optimal control problem that has the same solution? (This is not a new question. See, for example, [1]).

We show that if the second-order system satisfies certain conditions then we can find a corresponding optimal control problem, regardless of the underlying mechanics that gave rise to the system in the first place.

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We present here the general time dependent solution, the special case of the time-autonomous solution, and an application to a temporary equilibrium model of a competitive industry.

2

Mathematically, the problem is: Given an element $x \in l_\infty(R^n)$ and a sequence of C^1 functions, F_t, that satisfy

$$F_t(x(t-1), x(t), x(t+1)) = 0, \qquad t = 1, 2, \ldots, \tag{1}$$

where $x(0) = \alpha$ (a given initial condition), when does there exist a sequence of real valued C^2 functions, G_t, such that x also satisfies

$$D_2G_{t-1}(x(t-1), x(t)) + D_1G_t(x(t), x(t+1)) = 0, \qquad t = 1, 2, \ldots, \tag{2}$$

where the D_i are the partial differential operators?

Note that (2) is a system of n equations, and hence for (1) to be compatible with (2) we shall assume that F_t is a map into R^n.

First, we outline the optimization problem that leads to (2) as a necessary condition for optimality. Define

$$H_\alpha(x) = \sum_{t=0}^{\infty} G_t(x(t), x(t+1)). \tag{3}$$

For a given α we seek to maximize $H_\alpha(x)$ for $\|x\|_\infty \le 1$. If we assmue:

(a) G_t are concave C^1 functions;

(b) $\sup \sum_{t=1}^{n} |G_t(x(t), x(t+1))| < \infty$, $\|x\|_\infty \le 1$;

(c) $\sum_{t=0}^{\infty} |D_2G_{t-1}(x(t-1), x(t)) + D_1G_t(x(t), x(t+1))| < \infty$, $\|x\|_\infty \le 1$;

then a maximizing element, $\hat{x}$, exists. Furthermore, if $\|\hat{x}\|_\infty < 1$, then necessarily $DH_\alpha(\hat{x}) = 0$. (Here, $DH_\alpha(\hat{x})$ is the Frechet derivative of H_α at $\hat{x}$. See [8] for details.) This latter condition, along with the fact that $DH_\alpha \in l_1(R^n)$, implies

$$D_2G_{t-1}(\hat{x}(t-1), \hat{x}(t)) + D_1G_t(\hat{x}(t), \hat{x}(t+1)) = 0, \qquad t = 1, 2, \ldots. \tag{4}$$

This optimal control problem is not as restrictive as it appears. If we parameterize the problem by "$\|x\|_\infty \le r$" then for each $r > 0$, a solution $\hat{x}_r$ exists in that

$$H_\alpha(x) \le H_\alpha(\hat{x}_r), \qquad \|x\|_\infty \le r.$$

Then if for some r, $\|\hat{x}_r\|_\alpha < r$, we have

$$H_\alpha(x) \leq H_\alpha(\hat{x}_r), \qquad x \in l_\alpha(R^n),$$

that is: A solution to $\max\{H_\alpha(x) \mid x \in l_\alpha(R^n)\}$ exists. Note that if $\|\hat{x}_r\|_\alpha = r$ for all r then there is no $\hat{x} \in l_\alpha(R^n)$ such that $H_\alpha(\hat{x}) = \sup\{H_\alpha(x) \mid x \in l_\alpha(R^n)\}$. Since in these types of optimal control problems we seek bounded solutions, either the objective functions G_t have to be of some restricted class or we have to place a bound on $\|x\|_\alpha$.

See Dechert [5] for the development of the existence of a solution and the necessary condition for optimality.

3

First, let us explain the notation that is used. The integrals in Sections 3 and 4 are line integrals in the following sense: Let f: $R^n \to R^n$ be measurable with component functions $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, and let ϕ: $[0, 1] \to R^n$ be continuoulsy differentiable with component functions $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$.

$$\int_{\phi(0)}^{\phi(1)} f(x)\,dx = \int_0^1 \left\{ \sum_{i=1}^{n} f_i(\phi(t)) \frac{d\phi_i}{dt} \right\} dt.$$

$f(x)\,dx$ is said to be exact when this integral does not depend on the (path) function ϕ. The necessary and sufficient condition for a C^1 function f to be exact is $\partial f_i/\partial x_j = \partial f_j/\partial x_i$.

Furthermore, if f: $R^n \to R^{n^2}$ is the matrix of functions

$$\begin{bmatrix} f_{11}(x) & \cdots & f_{1n}(x) \\ \vdots & & \\ f_{n1}(x) & \cdots & f_{nn}(x) \end{bmatrix}.$$

Then the ith component of $\int_{\phi(0)}^{\phi(1)} f(x)\,dx$ is

$$\int_0^1 \left\{ \sum_{j=1}^{n} f_{ij}(\phi(t)) \frac{d\phi_j}{dt} \right\} dt.$$

For the most part we have tried to keep the notation clean so that the material is readable. Unfortunately, this places an additional burden on those who want to work out the proofs in detail since it involves using the more cumbersome notation. Also, [ ]* denotes transpose.

The main result in answer to the question posed in Section 1 is:

THEOREM 1. *Let F_t be a sequence of functions, F_t: $R^n \times R^n \times R^n \to R^n$.*

The necessary and sufficient conditions that there exist a sequence of C^2 *functions,* $G_t\colon R^n \times R^n \to R$ *which satisfy*

$$D_2G_{t-1}(x, y) + D_1G_t(y, z) = F_t(x, y, z), \qquad t = 1, 2, \ldots, \tag{5}$$

are that there exist C^1 *functions* $a_t, b_t\colon R^n \times R^n \to R^n$ *such that*

(i) $F_t(x, y, z) = a_t(x, y) + b_t(y, z)$,
(ii) $D_1a_{t+1}(x, y) = [D_2b_t(x, y)]^*$,
(iii) $a_t(x, y)\, dy$ *and* $b_t(x, y)\, dx$ *are exact,*
(iv) $D_1 \int_{y_0}^{y} a_t(x, \eta)\, d\eta = \int_{y_0}^{y} [D_1a_t(x, \eta)]^*\, d\eta$, *and* $D_2 \int_{x_0}^{x} b_t(\xi, y)\, d\xi = \int_{x_0}^{x} [D_2b_t(\xi, y)]^*\, d\xi$.

Proof (Necessity). Since the G_t are C^2 functions, (5) implies that the F_t are C^1 functions. Differentiate (5) to get

$$D_{21}G_{t-1}(x, y) = D_1F_t(x, y, z), \tag{6a}$$

$$D_{12}G_t(y, z) = D_3F_t(x, y, z). \tag{6b}$$

Since (6a) is independent of z and (6b) is independent of x, (i) follows. Match up the variables and subscripts in (6a) and (6b) to get (ii). Condition (iii) is the statement that $D_2G_{t-1}(x, y)\, dy$ and $D_1G_t(x, y)\, dx$ are both exact. Finally, differentiate (with respect to x):

$$\int_{y_0}^{y} a_t(x, \eta)\, d\eta = G_{t-1}(x, y) - G_{t-1}(x, y_0)$$

to get

$$\begin{aligned} D_1 \int_{y_0}^{y} a_t(x, \eta)\, d\eta &= D_1G_{t-1}(x, y) - D_1G_{t-1}(x, y_0) \\ &= b_{t-1}(x, y) - b_{t-1}(x, y_0) \\ &= \int_{y_0}^{y} D_2b_{t-1}(x, \eta)\, d\eta \\ &= \int_{y_0}^{y} [D_1a_t(x, \eta)]^*\, d\eta. \end{aligned}$$

Similarly for the second condition of (iv).

(Sufficiency). Define the functions G_t:

$$G_0(x, y) = \int_{y_0}^{y} a_1(x, \eta)\, d\eta \tag{7}$$

$$G_t(x, y) = \int_{x_0}^{x} b_t(\xi, y)\, d\xi + \int_{y_0}^{y} a_{t+1}(x_0, \eta)\, d\eta, \qquad t = 1, 2, \ldots.$$

Then

$$D_2G_0(x, y) = a_1(x, y),$$

and

$$\begin{aligned} D_2G_t(x, y) &= D_2 \int_{x_0}^{x} b_t(\xi, y)\, d\xi + a_{t+1}(x_0, y) \\ &= \int_{x_0}^{x} [D_2 b_t(\xi, y)]^*\, d\xi + a_{t+1}(x_0, y) \\ &= \int_{x_0}^{x} D_1 a_{t+1}(\xi, y)\, d\xi + a_{t+1}(x_0, y) \\ &= a_{t+1}(x, y), \qquad t = 1, 2,\ldots \end{aligned}$$

Also

$$D_1G_t(x, y) = b_t(x, y), \qquad t = 1, 2,\ldots .$$

Thus

$$D_2G_{t-1}(x, y) + D_1G_t(y, z) = F_t(x, y, z). \qquad \text{Q.E.D.}$$

The inherent lack of uniqueness in the functions G_t is due to the fact that Eqs. (7) define the G_t up to an additive constant which is determined by the choice of (x_0, y_0) in the line integrals.

COROLLARY. *The functions G_t given by Eqs. (7) are differentiably concave if the matrix*

$$\begin{bmatrix} D_1b_t(x, y) & D_2b_t(x, y) \\ D_1a_{t+1}(x, y) & D_2a_{t+1}(x, y) \end{bmatrix} \tag{8}$$

is negative semidefinite.

Conditions (ii) and (iii) of the theorem imply that (8) is a symmetric matrix. Also, by replacing F_t with $-F_t$, the G_t are convex when (8) is positive semidefinite.

It is worth pointing out that the specification of the functions G_t by Eq. (7) could also have been given by

$$G_0(x, y) = \int_{y_0}^{y} a_1(x, \eta)\, d\eta,$$

$$G_t(x, y) = \int_{y_0}^{y} a_{t+1}(x, \eta)\, d\eta + \int_{x_0}^{x} b_t(\xi, y_0)\, d\xi, \qquad t = 1, 2,\ldots$$

That these two functions agree can be shown by defining

$$K_t(x,y) = \int_{x_0}^{x} b_t(\xi,y)\,d\xi + \int_{y_0}^{y} a_{t+1}(x_0,\eta)\,d\eta - \int_{y_0}^{y} a_{t+1}(x,\eta)\,d\eta - \int_{x_0}^{x} b_t(\xi,y_0)\,d\xi.$$

Then $K_t(x,y)$ is continuous, $K_t(x_0,y_0) = 0$, and $D_1K_t(x,y) = b_t(x,y) - D_1\int_{y_0}^{y} a_{t+1}(x,\eta)\,d\eta - b_t(x,y_0) = 0$ by conditions (ii) and (iv). Similarly, $D_2K_t(x,y) = 0$, and hence $K_t(x,y) = 0$.

Before turning to an application of this theorem we should make some comments about this theorem vis-a-vis the optimal control problem in Section 2 and the question raised in Section 1. Basically we are interested in answering the question: Given an element $\hat{x} \in l_\infty(R^n)$ that satisfies the second-order difference equation $F_t(\hat{x}(t-1), \hat{x}(t), \hat{x}(t+1)) = 0$ with $\hat{x}(0) = \alpha$, when is this the optimal policy function for the optimal control problem: $\max \sum G_t(x(t), x(t+1))$ with $x(0) = \alpha$? The theorem is only a partial answer to this question. It gives conditions for the functions G_t to satisfy Eq. (5) for all (x, y, z), and not just the specific values $(x(t-1), x(t), x(t+1))$ for which $F_t = 0$.

In certain cases the solution to the second-order difference equation is the solution to an optimal control problem. A set of sufficient conditions are given by:

THEOREM 2. *Suppose F_t satisfy conditions* (i)–(iv) *of Theorem* 1 *and matrix* (8) *is negative semidefinite. Further, let $\hat{x} \in l_\infty(R^n)$ be such that $\hat{x}(0) = \alpha$, and*

$$F_t(\hat{x}(t-1), \hat{x}(t), \hat{x}(t+1)) = 0, \qquad t = 1, 2, \ldots . \tag{9}$$

Let G_t be defined by Eqs. (7) *and suppose that*

$$\sum_{t=0}^{\infty} | G_t(\hat{x}(t), \hat{x}(t+1))| < \infty$$

$$\sum_{t=0}^{\infty} \| D_iG_t(\hat{x}(t), \hat{x}(t+1))\| < \infty, \qquad i = 1, 2.$$

Then $\hat{x}$ is a solution to the optimal control problem

$$\max_{x \in l_\infty} \sum_{t=0}^{\infty} G_t(x(t), x(t+1)),$$

$$x(0) = \alpha.$$

Proof. G_t is concave since matrix (8) is negative semidefinite. Hence,

$$G_t(x(t), x(t+1)) - G_t(\hat{x}(t), \hat{x}(t+1)) \\ \leq (D_1G_t)(x(t) - \hat{x}(t)) + (D_2G_t)(x(t+1) - \hat{x}(t+1)),$$

where

$$D_iG_t = D_iG_t(\hat{x}(t), \hat{x}(t+1)).$$

Thus for any $x \in l_\infty(R^n)$,

$$\sum_{t=0}^{\infty} \{G_t(x(t), x(t+1)) - G_t(\hat{x}(t), \hat{x}(t+1))\} \\ \leq \sum_{t=0}^{\infty} \{(D_1G_t)(x(t) - \hat{x}(t)) + (D_2G_t)(x(t+1) - \hat{x}(t+1))\}.$$

The sum on the right converges absolutely, and so we can rearrange the terms to get

$$\sum_{t=0}^{\infty} \{G_t(x(t), x(t+1)) - G_t(\hat{x}(t), \hat{x}(t+1))\} \\ \leq \sum_{t=1}^{\infty} [(D_2G_{t-1}) + (D_1G_t)](x(t) - \hat{x}(t)) = 0,$$

where we have used $x(0) = \alpha = \hat{x}(0)$ and Eqs. (5) and (9). Q.E.D.

Note that the summability conditions are for the single element $\hat{x}$, which is assumed to be known and which satisfies the difference equation (9).

4

The corresponding time autonomous system of Eqs. (1)-(3) are

$$\delta^{t-1}F(x(t-1), x(t), x(t+1)) = 0, \tag{1'}$$

$$\delta^{t-1}D_2G(x(t-1), x(t)) + \delta^t D_1G(x(t), x(t+1)) = 0, \tag{2'}$$

$$H_\alpha(x) = \sum_{t=0}^{\infty} \delta^t G(x(t), x(t+1)), \tag{3'}$$

where $\delta > 0$ is the discount factor.

The conditions of the theorem and corollary can be stated as:

(i′) $F(x, y, z) = a(x, y) + \delta b(y, z)$,

(ii′) $D_1a(x, y) = [D_2b(x, y)]^*$,

(iii′) $a(x, y)\,dy$ and $b(x, y)\,dx$ are exact, (iv′) $D_1 \int_{y_0}^{y} a(x, \eta)\, d\eta = \int_{y_0}^{y} [D_1 a(x, \eta)]^*\, d\eta$, and $D_2 \int_{x_0}^{x} b(\xi, y)\, d\xi = \int_{x_0}^{x} [D_2 b(\xi, y)]^*\, d\xi$,

$$G(x, y) = \int_{x_0}^{x} b(\xi, y)\, d\xi + \int_{y_0}^{y} a(x_0, \eta)\, d\eta, \tag{7'}$$

$$\begin{bmatrix} D_1 b(x, y) & D_2 b(x, y) \\ D_1 a(x, y) & D_2 a(x, y) \end{bmatrix} \tag{8'}$$

is negative semidefinite.

When $\delta < 1$ the solution to the optimal problem specified by the function F becomes simple:

THEOREM 3. *Let F satisfy* (i′)–(iv′) *and* (8′). *Let $x(0) = \alpha$, and $\hat{x} \in l_\infty(R^n)$ satisfy*

$$F(\hat{x}(t-1), \hat{x}(t), \hat{x}(t+1)) = 0, \qquad t = 1, 2, \ldots, \tag{10}$$

and let G be defined by (7′). *Then $\hat{x}$ is a solution to the optimal control problem*

$$\max_{x \in l_\infty} \sum_{t=0}^{\infty} \delta^t G(x(t), x(t+1)), \tag{11}$$

$$x(0) = \alpha.$$

Proof. Since G is continuous, for any $x \in l_\infty(R^n)$,

$$\left| \sum_{t=0}^{\infty} \delta^t G(x(t), x(t+1)) \right| \le (1-\delta)^{-1} \max_{\substack{|\xi| \le \|x\| \\ |\eta| \le \|x\|}} |G(\xi, \eta)|,$$

and so (11) is a well-posed problem. Matrix (8′) implies that G is concave and so

$$\begin{aligned} G(x(t), x(t+1)) &- G(\hat{x}(t), \hat{x}(t+1)) \\ &\le D_1 G(t)(x(t) - \hat{x}(t)) + D_2 G(t)(x(t+1) - \hat{x}(t+1)), \end{aligned}$$

where $D_i G(t) = D_i G(\hat{x}(t), \hat{x}(t+1))$. Thus,

$$\begin{aligned} &\sum_{t=0}^{\infty} \delta^t \{G(x(t), x(t+1)) - G(\hat{x}(t), \hat{x}(t+1))\} \\ &\qquad \le \sum_{t=0}^{\infty} \delta^t \{D_1 G(t)(x(t) - \hat{x}(t)) + D_2 G(t)(x(t+1) - \hat{x}(t+1))\}. \end{aligned}$$

G is continuously differentiable, so an argument similar to the one at the beginning of the proof shows that the sum on the right converges absolutely.

We can rearrange the terms to get

$$\sum_{t=0}^{T} \delta^t\{G(x(t), x(t+1)) - G(\hat{x}(t), \hat{x}(t+1))\}$$
$$\geq \sum_{t=1}^{T} \delta^{t-1}[D_2G(t-1) + \delta D_1G(t)](x(t) - \hat{x}(t)) = 0,$$

where we have used the fact that $x(0) = \alpha = \hat{x}(0)$ and Eq. (10). Q.E.D.

5

As an application of some of these ideas consider an industry which consists of N competitive firms, where the ith firm chooses a sequence of utilization levels of capital and labor, $(\hat{k}_i(t), \hat{l}_i(t)) \in R^m$ for $t = 1, 2,\ldots$, that satisfy the second-order equations:

$$0 = \begin{bmatrix} -c_i'(\hat{k}_i(t) - \hat{k}_i(t-1)) \\ 0 \end{bmatrix} + \delta \begin{bmatrix} p(t)\dfrac{\partial f_i}{\partial k_i}(\hat{k}_i(t), \hat{l}_i(t)) + c_i'(\hat{k}_i(t+1) - \hat{k}_i(t)) \\ p(t)\dfrac{\partial f_i}{\partial l_i}(\hat{k}_i(t), \hat{l}_i(t)) - w(t) \end{bmatrix}, \qquad (12)$$

where $\hat{k}_i(0)$ are given initial stocks and the variables are

$p(t)$, the current price of output,

$w(t)$, the current wages of labor,

f_i, the ith firm's production function,

c_i, the ith firm's cost of investment,

and c_i' denotes marginal cost of investment.

While we might choose to interpret (12) as the necessary condition for $(\hat{k}_i(t), \hat{l}_i(t))$ to solve

$$\max \sum_{t=0}^{\infty} \delta^t\{p(t)f_i(k_i(t), l_i(t)) - w(t)\,l_i(t) - c_i(k_i(t+1) - k_i(t))\},$$

we might also choose to consider (12) as describing $\hat{k}_i(t)$, $\hat{l}_i(t)$ as the behavior rules of the ith firm. In Brock [4] the schemes for expectations give rise to second-order (differential) equations which can be interpreted as the behavior rules for the economic agents.

In a temporary equilibrium framework, Eqs. (12) for $i = 1, \ldots, N$ and the market clearing conditions

$$\hat{p}(t) = D\left(\sum_{i=1}^{N} f_i(\hat{k}_i(t), \hat{l}_i(t))\right), \tag{13}$$

$$\hat{w}(t) = S\left(\sum_{i=1}^{N} \hat{l}_i(t)\right),$$

define $\hat{k}_i(t)$, $\hat{l}_i(t)$ as the Lucas Perfect Foresignt Equilibrium, where D is the market (price) demand function for output and S is the (wage) supply function of labor. In equilibrium the sequence of input decisions made by the ith firm satisfies

$$0 = \begin{bmatrix} -c_i'(\hat{k}_i(t) - \hat{k}_i(t-1)) \\ 0 \end{bmatrix} + \delta \begin{bmatrix} D(\hat{Q}(t)) \dfrac{\partial f_i}{\partial k_i}(\hat{k}_i(t), \hat{l}_i(t)) + c_i'(\hat{k}_i(t+1) - \hat{k}_i(t)) \\ D(\hat{Q}(t)) \dfrac{\partial f_i}{\partial l_i}(\hat{k}_i(t), \hat{l}_i(t)) - S(\hat{L}(t)) \end{bmatrix}, \tag{14}$$

where $\hat{Q}(t) = \sum_{i=1}^{N} f_i(\hat{k}_i(t), \hat{l}_i(t))$ and $\hat{L}(t) = \sum_{i=1}^{N} \hat{l}_i(t)$, and $\hat{k}_i(0)$ is the given initial capital stocks. Equation (14) represents a system of $n = Nm$ equations which, with the following notation:

$$x(t) = \begin{bmatrix} \begin{bmatrix} k_1(t) \\ l_1(t) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} k_N(t) \\ l_N(t) \end{bmatrix} \end{bmatrix} = \left[\begin{bmatrix} k_i(t) \\ l_i(t) \end{bmatrix}\right],$$

we can represent the functions $a(x(t-1), x(t))$ and $b(x(t), x(t+1))$ by

$$a(x(t-1), x(t)) = \left[\begin{bmatrix} -c_i'(k_i(t) - k_i(t-1)) \\ 0 \end{bmatrix}\right],$$

$$b(x(t), x(t+1)) = \left[\begin{bmatrix} D(Q(t)) \dfrac{\partial f_i}{\partial k_i}(k_i(t), l_i(t)) + c_i'(k_i(t+1) - k_i(t)) \\ D(Q(t)) \dfrac{\partial f_i}{\partial k_i}(k_i(t), l_i(t)) - S(L(t)) \end{bmatrix}\right], \tag{15}$$

where $Q(t) = \sum_{i=1}^{N} f_i(k_i(t), l_i(t))$ and $L(t) = \sum_{i=1}^{N} l_i(t)$. Conditions (ii′), (iii′), and (iv′) are satisfied, and a function G defined by Eq. (7′) is:

$$G(x, y) = \int_0^{\sum_{i=1}^{N} f_i(k_i, l_i)} D(q)\, dq - \int_0^{\sum_{i=1}^{N} l_i} S(l)\, dl - \sum_{i=1}^{N} c_i(\kappa_i - k_i), \tag{16}$$

where

$$x = \left[\begin{bmatrix} k_i \\ l_i \end{bmatrix}\right] \quad \text{and} \quad y = \left[\begin{bmatrix} \kappa_i \\ \lambda_i \end{bmatrix}\right],$$

and the lower limits of integration have been arbitrarily chosen. Equation (16) is in fact the result of evaluating the line integral of Eq. (7′), which can be verified by differentiating (16).

The integrals in (16) are Lebesgue integrals, which we assume exist for the given functions D and S. Along an optimal solution, the objective function can be written as

$$\begin{aligned} G(\hat{x}(t), \hat{x}(t+1)) = {} & \hat{p}(t)\hat{Q}(t) - \hat{w}(t)\hat{L}(t) - \sum_{i=1}^{N} c_i(\hat{k}_i(t+1) - \hat{k}_i(t)) \\ & + \int_0^{\hat{Q}(t)} [D(q) - \hat{p}(t)]\, dq \\ & + \int_0^{\hat{L}(t)} [\hat{w}(t) - S(l)]\, dl. \end{aligned}$$

This demonstrates the welfare economics principle that producer surplus, consumer surplus, and labor surplus are maximized in competitive equilibrium.

The matrix of second partials of $G(x, y)$ given by (16) is:

$$D_{11}G = D'(Q) \begin{bmatrix} \partial f_1 \\ \partial f_2 \\ \vdots \\ \partial f_N \end{bmatrix} \begin{bmatrix} \partial f_1 \\ \partial f_2 \\ \vdots \\ \partial f_N \end{bmatrix}^{*} + D(Q) \begin{bmatrix} \partial^2 f_1 & & & & \\ & \ddots & & & \\ & & \partial^2 f_2 & & \\ & & & \ddots & \\ & & & & \partial^2 f_N \end{bmatrix}$$

$$+ \begin{bmatrix} \left[\begin{array}{c|c} \partial^2 c_1 & 0 \\ \hline 0 & S'(L) \end{array}\right] & & \\ & \ddots & \\ & & \left[\begin{array}{c|c} \partial^2 c_N & 0 \\ \hline 0 & S'(L) \end{array}\right] \end{bmatrix},$$

$$D_{12}G = - \begin{bmatrix} \left[\begin{array}{c|c} \partial^2 c_1 & 0 \\ \hline 0 & 0 \end{array}\right] & & \\ & \ddots & \\ & & \left[\begin{array}{c|c} \partial^2 c_N & 0 \\ \hline 0 & 0 \end{array}\right] \end{bmatrix},$$

$D_{21}G = D_{12}G$ and $D_{22}G = -D_{12}G$. The notation here is: ∂f_i is the vector of partial derivatives of f_i, $\partial^2 f_i$ is the matrix of mixed partial derivatives of f_i, similarly for c_i. Under the usual assumptions that $D(Q) \geq 0$, $D'(Q) \leq 0$, $S(L) \geq 0$, $S'(L) \geq 0$, f_i are concave and c_i are convex, this matrix is negative semidefinite.

A more elegant proof[1] that $G(x, y)$ is concave is to note that each term in (16) is concave. For example,

$$F(Q) = \int_0^Q D(q)\,dq$$

has $F'(Q) = D(Q) > 0$ and $F''(Q) = D'(Q) < 0$. Since $Q = \sum f_i(k_i, l_i)$ is concave, then so is $F(Q)$. In applications, the matrix of mixed partial derivatives of G is necessary to analyze the local stability of the solution (as in Dechert [6]), and so we present it here.

The conclusion of this analysis is that the Lucas Perfect Foresight Equilibrium Solution of (13) (14) is also the solution to the optimal control problem

$$\max \sum_{t=1}^{\infty} \delta^t G(x(t), x(t+1)), \qquad \text{given } x(0),$$

where G is given by Eq. (16). This objective function can be thought of as the consumer value of output (the integral under the demand curve) minus the opportunity cost of labor[2] (the integral under the supply curve) minus the total cost of investment in the industry.

This type of cost benefit analysis can be found in a variety of contexts. Samuelson [9] examines a spatial equilibrium with consumer surplus minus transportation costs. In an intertemporal model, Lucas and Prescott [7] investigate an investment equilibrium with an objective function of consumer surplus minus investment costs. Two of the advantages of this type of analysis applied to a system of equations such as (12)-(13) are, first: a basic economic interpretation of the equilibrium solution, especially with regard to welfare economics; and second: a mathematical analysis of the equilibrium solution, especially with regard to stability of steady state solutions.

6

In this paper we have given a partial answer to the question: If $\hat{x} \in l_\infty(R^n)$ and $\alpha \in R^n$, is there a sequences of functions G_t so that $\hat{x}$ is the solution to

$$\max_{x \in l_\infty} \sum_{t=0}^{\infty} G_t(x(t), x(t+1)),$$
$$x(0) = \alpha?$$

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[1] An anonymous referee pointed this out to me.

[2] Barbara White pointed out this interpretation.

We have tried to motivate this problem in economics with our example of Section 5. There is more work to be done here. In Section 3 we pointed out that we have answered the question only when there exist G_t such that $D_2G_{t-1}(x, y) + D_1G_t(y, z) = F_t(x, y, z)$ for all x, y, z, and not just for the particular values of $\hat{x}(t-1), \hat{x}(t), \hat{x}(t+1)$. At the end of Section 5 we referred to the question of stability of solutions, which plays an important role in economic analysis (see Brock's survey article [3]). We need to find sufficient conditions so that an $\bar{x} \in R^n$ that satisfies $F_t(\bar{x}, \bar{x}, \bar{x}) = 0$ has the property that $\lim_{t \to \infty} |\hat{x}(t) - \bar{x}| = 0$ for $\alpha \neq \bar{x}$. Also important is the question of sensitivity of solutions with respect to initial conditions. That is, if $\hat{x}$ satisfies $F_t(\hat{x}(t-1), \hat{x}(t), \hat{x}(t+1)) = 0$ with $\hat{x}(0) = \alpha$ and $\tilde{x}$ satisfies $F_t(\tilde{x}(t-1), \tilde{x}(t), \tilde{x}(t+1)) = 0$ with $\tilde{x}(0) = \beta$, how does $\| \tilde{x} - \hat{x} \|$ behave with respect to $| \alpha - \beta |$? These are important issues because they answer the question, How do present differences affect future differences?

Another area of application of this inverse problem of optimal control is to adaptive control processes.[3] In models where an agent has a rule for behavior, $u(t) = R(x(t-1), x(t))$, and the consequence of this action is $x(t+1) = f(x(t), u(t))$, one can analyze the rule R through the corresponding optimal control problem. This poses the following questions concerning robustness of solutions for rules of thumb: If R is from some class of rules (e.g., linear rules, first-order rules, etc.), how sensitive is the time path $x(t)$ of the state of the system to changes in R? (This is important for modeling purposes: Is the solution well behaved even though the rule of thumb R is only an approximation of the agent's behavior?) If the transition function f is from some paramertized class of functions, how sensitive is the time path $x(t)$ to changes in the parameters? (This is important for the agent's estimation problem: If the transition function is unknown the agent will be attempting to estimate its parameters. We would anticipate that if the solutions were very sensitive to which f prevails, the system would be highly unstable.)

REFERENCES

1. R. Bellman and R. Kalaba, An inverse problem in dynamic programming and automatic control, *J. Math. Anal. Appl.* 7 (1963), 322–325.
2. W. A. Brock, "Some Results in Dynamic Integrability," Department of Economics Working Paper, University of Chicago, December 1975.
3. W. A. Brock, "Applications of Recent Results on the Asymptotic Stability of Optimal Control to the Problem of Comparing Long Run Equilibria," Department of Economics Working Paper No. 129, Cornell University, November 1976.
4. W. A. Brock, On Models of expectations that arise from maximizing behavior of economic agents over time, *J. Econ. Theory* **5** (1972).

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[3] This was suggested to me by Richard H. Day.

5. D. Dechert, "Optimal Control Problems in Economics," Modelling Research Group Working Paper 7706, Department of Economics, University of Southern California, May 1977.
6. D. Dechert, "An Industry on Dynamic Equilibrium: The Incentive to Form a Cartel," Mimeograph, Department of Economics, University of Southern California, June 1978.
7. R. E. Lucas, Jr., and E. C. Prescott, Investment under uncertainty, *Econometrica* **39**, No. 5 (1971).
8. W. Rudin, "Functional Analysis," McGraw-Hill, New York, 1973.
9. P. A. Samuelson, Intertemporal price equilibrium: A prologue to the theory of speculation, *in* "The Collected Scientific Papers of Paul A. Samuelson," Vol. II, Essay 73, MIT Press, Cambridge, Mass., 1966.

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Time Preferences, Conditional Risk Preferences, and Two-Period Cardinal Utility*

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1. INTRODUCTION AND SUMMARY

In recent years there has been extensive research on multiperiod decision problems in which economic agents are assumed to possess preferences over certain current/uncertain future consumption plans. Frequently, these problems are cast in a simplified two-period setting—see for instance, the papers of Drèze and Modigliani [2, 3], Leland [9], Sandmo [16–18] and Mirman [11], which deal with the consumption/savings (portfolio) problem. Almost without exception the agent's preferences are assumed to be representable by an expected TPC (Two-Period Cardinal) utility function, with the reader generally referred to the classic axiomatization of von Neumann and Morgenstern [22] (or Savage [19]).

The purpose of this paper is to investigate relationships among three types of preferences and their associated utility representations in a two-period context. The objects of choice are ordered pairs (x, F) in the product set $S = C_1 \times \mathcal{X}$ where $C_1 = (0, \infty)$ is the set of certain consumption possibilities for the first period, and $\mathcal{X}$ is the set of cumulative distribution functions on $C_2 = (0, \infty)$, the elements of which represent risky consumption

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possibilities for the second period. We consider the following three types of preference utility structures:

(i) *time preferences* over certain consumption pairs in $C_1 \times C_2$ described by the binary relation $\precsim^t$ on $C_1 \times C_2$ and represented by a (continuous, strictly monotone) ordinal index $U: C_1 \times C_2 \to \mathbb{R}$ with $(c_1, c_2) \precsim^t (c_1', c_2')$ iff $U(c_1, c_2) \leq U(c_1', c_2')$;

(ii) a set of *conditional risk preferences* $\{\precsim_x, x \in C_1\}$, each of which is defined over $\underline{X}$ and conditioned on a fixed level of first-period consumption and is representable according to the expected utility principle where $V_x: C_2 \to \mathbb{R}$ is a (continuous and strictly monotone increasing) second-period NM (von Neumann Morgenstern) index such that, for all $F, G \in \underline{X}$,

$$F \precsim_x G \text{ iff } \int_{C_2} V_x(c_2)\, dF(c_2) \leq \int_{C_2} V_x(c_2)\, dG(c_2) \tag{1.1}$$

(each V_x is unique up to a positive affine transformation); and

(iii) preferences over $S = C_1 \times \underline{X}$, described by the complete preordering $\precsim$ and represented by a (continuous, strictly monotone) *TPC* (two-period cardinal) utility function $W: C_1 \times C_2 \to \mathbb{R}$ for which

$$(x, F) \precsim (x', G) \text{ iff } \int_{C_2} W(x, c_2)\, dF(c_2) \leq \int_{C_2} W(x', c_2)\, dG(c_2). \tag{1.2}$$

The two-period NM index W is unique up to a positive affine transformation.

Assuming the existence of a two-period von Neumann–Morgenstern utility W, we consider in Section 2 when and how such a function can be constructed from a time preference index U and a *single conditional* NM utility function W_x. These results formalize a procedure suggested by Hirshleifer [7] and Drèze and Modigliani [3].

Unfortunately, even when the necessary conditions for constructing a TPC utility are satisfied, not every (perfectly standard) U and W_x pair is compatible mathematically and economically. Several interesting examples of incompatibilities are presented in Section 3. For instance, if time preferences are representable by a strictly quasi-concave ordinal utility function (from a subclass of the constant elasticity of substitution, or CES, family) and if the conditional index W_x exhibits risk aversion, then for x' infinitesimally larger than x the constructed conditional NM utility $W_{x'}(c_2)$ need not be risk averse everywhere and, in fact, can exhibit strong risk *affinity*.

Now instead of *assuming* the existence of a TPC utility function, one can ask whether it, in fact, is *implied* by the set of axioms sufficient for the existence of the ordinal time preference function U and a complete set of conditional second-period NM indices $\{V_x\}$. This is seen not to be the case. In Section 4, we identify that additional axiom, referred to as "coherence," which is necessary and sufficient for the existence of a TPC utility function.

The coherence postulate essentially requires a very special kind of meshing together of the consumer's set of conditional risk preferences $\{\precsim_x\}$ and time preferences $\precsim^t$.

Our notion of coherence is formally similar to the concept of "time adjustment calculus" formulated by Prakash [13] for the case of "date-income" pairs. We comment on this paper briefly toward the end of Section 4. Some related issues also are discussed by Kreps and Porteus in their paper [8] on temporal resolution of uncertainty and by Fishburn [4] in the context of multilinear expected utility.

In the last section, the TPC utility model is compared with the OCE ("Ordinal Certainty Equivalent") representation hypothesis developed in Selden [20]. We show that the OCE representation of $\precsim$ is a two-period expected utility representation if and only if coherence is assumed. Since under the OCE theory given time preferences and conditional risk preferences are not incompatible (in the sense discussed above), it is the addition of coherence and hence the desire to have a TPC utility function which produces the possibility of incompatibilities.

2. CONSTRUCTION OF A TPC INDEX

Imagine a decision maker confronting a choice among alternative "certain x uncertain" consumption pairs. Suppose we *know* that a two-period expected utility function *exists*. Then how can it be constructed from the individual's "time" preferences and "risk" preferences?

To begin, we assume the following:

Assumption 1. There exists a complete preference preordering $\precsim$ on $C_1 \times \tilde{X}$.

Assumption 2. The preordering $\precsim$ is NM representable as in Eq. (1.2), where W is the continuous TPC index.

It follows from (A.1) that there will exist a conditional preordering $\precsim_x$ on each subset $\{x\} \times \tilde{X}$, where $x \in C_1$, and from (A.2) that each such ordering possesses an expected utility representation with $W_x: C_2 \rightarrow \mathbb{R}$ being the *conditional* NM utility. The collection of orderings $\{\precsim_x\}$ is referred to as the individual's "risk preferences."

Turning next to "time preferences," a basic assumption required for the results in this section is

Assumption 3. *There exists a complete preordering* $\precsim^t$ *on* $C = C_1 \times C_2$ which is (i) continuous and strictly monotone and (ii) representable by the real-valued continuous time preference index $U: C \rightarrow \mathbb{R}$.

We make extensive use of the following notation to highlight an important characteristic of time preferences.

DEFINITION 1. If $x \in C_1$, then $C\{x\}$ denotes the class of pairs in C (temporally) indifferent to some pair in the set $C[x] =_{\text{def}} \{x\} \times C_2$; i.e.,

$$C\{x\} =_{\text{def}} \{(c_1, c_2) \in C \mid (c_1, c_2) \sim^t (x, c_2'), \text{ where } (x, c_2') \in C[x]\}.$$

As proved in Lemma 1 below, four types of regions may appear as $C\{x\}$ when $\succsim^t$ is continuous and monotone. This is illustrated in Fig. 1, where each type of region except the fourth is exhibited by a member of the CES class of utility functions: (1) $U(c_1, c_2) = c_1c_2$, (2) $U(c_1, c_2) = (c_1^{1/2} + c_2^{1/2})^2$, (3) $U(c_1, c_2) = (c_1^{-1} + c_2^{-1})^{-1}$. (For an interpretation of the elasticity of substitution as a measure of (intuitive) intertemporal complementarity cf., Selden [21].)

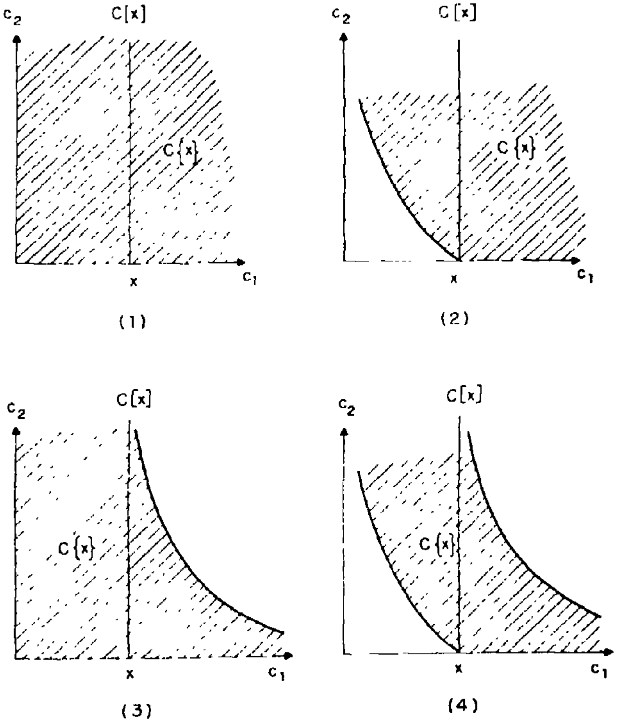

FIGURE 1

LEMMA 1. *If* $\preccurlyeq^t$ *is continuous and strictly monotone, the set* $C\{x\}$, $x \in C_1$, *is one of the following four types*:

(1) $C\{x\} = C_1 \times C_2$,

(2) $C\{x\} = \{c \mid c' \preccurlyeq^t c\}$ *for some* $c' \in C$,

(3) $C\{x\} = \{c \mid c \preccurlyeq^t c''\}$ *for some* $c'' \in C$,

(4) $C\{x\} = \{c \mid c' \preccurlyeq^t c \preccurlyeq^t c''\}$ *for some* $c', c'' \in C$.

The proof follows from the continuity and strict monotonicity of $\preccurlyeq^t$ and the connectedness of C.

Finally, let us explicitly relate the time preference and TPC utilities.

Assumption 4. The preference orderings $\preccurlyeq^*$ and $\preccurlyeq^t$ are related by the condition $W = h \circ U$, where h is some strictly monotone increasing function.

(In Section 4 we invoke an analogous assumption independent of the existence of W. Our present goal is to state a simple but quite useful relationship which must be satisfied by W, U, and a single arbitrary prespecified conditional NM utility W_x. For this purpose, Assumptions 1 through 4 are adequate.)

Given a conditional NM index W_x and a time preference relation on $C\{x\}$, the following formalizes a procedure suggested by way of an example in Hirshleifer [7, pp. 237–239][1]:

THEOREM 1. *Let* $x \in C_1$ *be arbitrarily chosen, and suppose* (A.1)–(A.4) *hold. Let* W_x *be the conditional* NM *utility function* (*corresponding to* $\preccurlyeq_x$ *on* $\{c_1\} \times \underset{\sim}{X}$). *Then for all* $c \in C\{x\}$, *W can be computed according to*

$$W(c_1, c_2) = W_x U_x^{-1}(U(c_1, c_2)), \tag{2.1}$$

where $x \in C_1$ *and* $U_x := U \mid C[x]$.

The proof is given in Appendix A. Thus the composition $W_x \circ U_x^{-1}$ defines the increasing monotonic transform h which distinguishes the TPC index from the time preference function U.

A number of brief comments about Theorem 1 are in order. First, it does *not* prove the existence of an expected utility representation; rather the theorem only provides a *computational formula* which is valid provided a TPC utility exists. Second, as can be seen from the proof, the restriction of the validity of the formula to the domain $C\{x\}$ is absolutely crucial. Third, whereas W_x and U_x^{-1}, in general, depend on x, h does not.

We shall use Theorem 1 to shed some light on interconnections posited by a TPC utility between $\preccurlyeq^t$ and the collection of conditional risk preferences

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[1] Hirshleifer credits Drèze and Modigliani [3] with introduction of this procedure. However, the latter seem more concerned with indirect utility for wealth than with *conditional* (period-two) expected utility for uncertain consumption.

$\{\precsim_x \mid x \in C_1\}$. The first question for us to consider is whether a prespecified $\precsim^t$ and a single conditional risk preference relation $\precsim_x$ on $\{x\} \times \underline{X}$ may be *incompatible* with the *existence* of a two-period von Neumann–Morgenstern utility representation of $\precsim$ on *the whole* of $S = C_1 \times \underline{X}$.

3. INCOMPATIBLE TIME AND CONDITIONAL RISK PREFERENCES

We continue for the moment to suppose that the orderings $\precsim$ and $\precsim^t$ satisfy (A.1)–(A.4). Let $x \in C_1$ be preassigned and suppose that in C, $C\{x\}$ has an upper boundary (i.e., it is of type 3 or 4 in terms of Lemma 1). Any (ordinal) utility function U representing the corresponding time preferences must be *conditionally bounded* which, in turn, forces the boundedness of W_x.[2]

The impact of this boundary condition is illustrated by the following example, in which a specific $(\precsim^t, \precsim_x)$-pair is seen to be incompatible with the existence of a TPC representation of $\precsim$.

EXAMPLE 1. Suppose time preferences are represented by $U(c_1, c_2) = (c_1^{-\delta} + c_2^{-\delta})^{-1/\delta}$, where $\delta > 0$. If $c_1 = x$, U_x is bounded by x. Assume further that for some x, the conditional risk preferences $\precsim_x$ are NM representable where the second-period NM index takes the form (up to a positive affine transform) $\log c_2$, which is *not* bounded above. According to (A.4) these utility functions must satisfy, for $c_1 = x$, the equation

$$\log c_2 = h(U(x, c_2)). \tag{3.1}$$

But since the log function is not bounded above and since U_x is bounded, the required relation (3.1) cannot hold for any increasing function defined on all im $U = (0, \infty)$. Therefore, NM representable preferences $\precsim$ over S cannot be compatible with the given $\precsim^t$ and $\precsim_x$. This conclusion is echoed if we formally compute $W_x \circ U_x^{-1}(U) = \log(U^{-\delta} - x^{-\delta})^{-1/\delta}$, since the transform (1) is not defined for every possible value of U and (2) is not strictly increasing everywhere.[3]

The following result, complementary to Theorem 1, states simple compatibility requirements on $\precsim^t$ and a *single* $\precsim_x$ such that there can exist an ordering $\precsim$ on all of $S = C_1 \times \underline{X}$ satisfying (A.1)–(A.4).

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[2] To see this, note first that sup $U \mid C[x]$ is finite since there is a $c'' \in C$ such that for each $c \in C\{x\}$, $U(c) < U(c'')$. If $\precsim$ is NM representable, the conditional risk preference index W_x must also be bounded above. Assumption 4 guarantees that $W = h \circ U$ on C; and, since sup $U \mid C[x] < U(c'')$ for some $c'' \in C$, the monotone function h is bounded on im $U \mid C[x]$, forcing the boundedness of $W \mid C[x] = W_x$.

[3] This example also illustrates the fact that the conditional NM index W_x may be bounded for reasons quite distinct from a desire to circumvent the St. Petersburg paradox (Arrow [1]).

THEOREM 2. *Assume* $\precsim^t$ *satisfies* (A.3), *and let the initial condition* $c_1 = x$ *be given. Assume that the continuous, strictly monotone function* $W_x: C_2 \to \mathbb{R}$ *is a conditional* NM *utility for* $\precsim_x$ *over* $\{x\} \times \underline{X}$. *Then there exists on* $S = C_1 \times \underline{X}$ *an ordering* $\precsim$ *satisfying* (A.1)–(A.4) *compatible with* $\precsim^t$ *and* $\precsim_x$ *iff one of the following is true*:

(i) $C\{x\}$ *is type* 1;
(ii) $C\{x\}$ *is type* 2, *and* W_x *is bounded below*;
(iii) $C\{x\}$ *is type* 3, *and* W_x *is bounded above*; *or*
(iv) $C\{x\}$ *is type* 4, *and* W_x *is bounded below and above.*

Only in case (i) *must the* $\precsim$ *be unique.*

The proof is given in Appendix B.

Remark. Though many preference relations on S are obtainable from a (U, W_x)-pair under cases (ii)–(iv) of Theorem 2, all of the resulting TPC, NM utilities coincide in $C\{x\}$.

Theorems 1 and 2 together provide the framework for investigating a second question. Given that $\precsim^t$ and $\precsim_x$ satisfy the conditions of Theorem 2 so that there exists a $\precsim$ for which (A.1)–(A.4) hold (i.e., $\precsim$ is "NM representable"), may they nevertheless be incompatible in some behavioral sense? For instance, will the resulting TPC index W exhibit undesirable properties?

We continue to assume CES time preferences

$$U(c_1, c_2) = (c_1^{-\delta} + c_2^{-\delta})^{-1/\delta}, \tag{3.2}$$

where $\delta > 0$ and the elasticity of substitution is given by $\eta = 1/(\delta + 1)$. We denote the two-period analogue of the Arrow–Pratt [1, 14] relative risk aversion function (in the spirit of Sandmo [16, 17]) by

$$\tau_R(c_1, c_2) =_{\text{def}} -c_2 W_{22}(c_1, c_2)/W_2(c_1, c_2), \tag{3.3}$$

and the *conditional* relative risk aversion function by

$$\tau_R^x(c_2) =_{\text{def}} -c_2 W_x''(c_2)/W_x'(c_2), \tag{3.4}$$

where $x \in C_1$. Suppose the conditional NM index assumes the form $W_x(c_2) = -\delta c_2^{\beta}/\beta$, where $\beta > 0$ and $\tau_R^x(c_2) = \beta + 1$ (the constant value of relative risk aversion being denoted simply τ).[4] Formal application of Theorem 1 yields an expression for W in $C\{x\}$ and hence one for $\tau_R(c_1, c_2)$ also in $C\{x\}$.

EXAMPLE 2. Let us assume the following: $\beta = 1$, $\delta = 2$, and $x = 10{,}000$.

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[4] Note that τ_R^x being constant does not imply that $\tau_R^{x'}$ $(x' \neq x)$ will be.

Then the resulting TPC utility function may be described over $C\{x\}$ (but *not* over all of C) by $W(c_1, c_2) = 2(c_1^{-2} + c_2^{-2} - 10^{-8})^{-1/2}$. For this function, direct computation reveals that $\tau_R^c = -2$ and $\eta = -1/3$. If we consider the point $c_1 = x = 10{,}000$, $c_2 = 22{,}400$ in $C\{x\}$, another direct computation using (3.3) yields $\tau_R(10{,}000, 22{,}400) = \tau_R^c(22{,}400) = -2$. However, if current consumption is increased by only 10% to $c_1 = 11{,}000$, then for $(11{,}000, 22{,}400) \in C\{x\}$ we obtain $\tau_R(11{,}000, 22{,}400) = -4.76$! (See Fig. 2.)

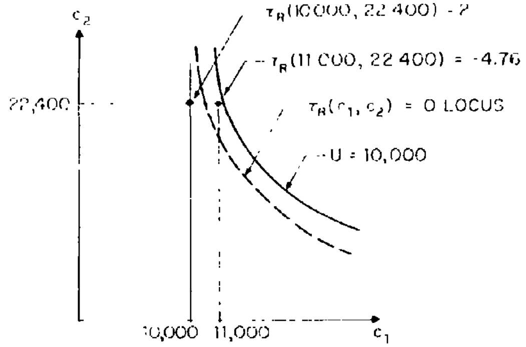

FIGURE 2

More generally, any values of the constant elasticity of substitution and relative risk aversion function satisfying $0 < \tau\eta < 1$ and $\eta < 1$ will produce preferences for which the property of being conditionally risk averse is unstable under small perturbations of first-period consumption. Yet this *seemingly pathological behavior of the TPC representation occurs when U and W_x exhibit perfectly "standard" properties.*

We leave to the interested reader consideration of other properties of the constructed TPC utility functions. Our only concern here is to indicate the possibility of behavioral incompatibilities between the time preferences and conditional risk preferences underlying two-period expected utility functions.

4. EXISTENCE

In this section we consider the existence of a TPC representation of preferences over $S(= C_1 \times \mathcal{X})$. Necessary and sufficient conditions are derived within a framework which directs attention to the interplay of conditional risk preferences and time preferences. By contrast, virtually all previous discussions of existence known to us proceed by applying directly the classical von Neumann–Morgenstern axiomatization. However, such an approach in the present setting confronts an immediate stumbling block in providing the choice space S with a mixture structure. Specifically, if (c_1'', F'') is a mixture of (c_1, F) and (c_1', F'), how should c_1'' be interpreted and/or defined when c_1 and c_1' are distinct? To form general "lotteries" appears to require dropping the distinction between current consumption being "certain" and future consumption being "risky" or "uncertain." On the other hand, without a mixture structure on S, the NM axiomatization cannot be directly applied.

In the present discussion, we overcome this difficulty by relying solely upon mixtures defined only within the "slices" $S[x] = \{x\} \times X$. Further, the classical NM axioms will be applied to *conditional* risk preferences, not to the relation $\precsim$ over S. Specifically, we relax Assumption 2 to read

Assumption 2′. For each $x \in C_1$, the conditional risk preference relation $\precsim_x$ (i.e., $\precsim | S[x]$) is NM representable as in Eq. (1.1), with the continuous NM index V_x strictly monotonically increasing.

This assumption implies that we have a collection of second-period, conditional NM indices $\{V_x : x \in C_1\}$, but no longer have the TPC utility function W. As a result, it is necessary to use a reformulation of (A.4) which does not presume the existence of a W. To do so, first define the natural embedding $\iota: C \to S$ by mapping $(c_1, y) \mapsto (c_1, F_y^*)$ where F_y^* is the one-point c.d.f. with its saltus point at $y \in C_2$.

Assumption 4′. The preordering $\precsim$ satisfies the condition that time preferences are preserved under the natural embedding, i.e., $c \precsim^t c' \Leftrightarrow \iota c \precsim \iota c'$, $\forall c, c' \in C$.

(It is easily verified that (A.1), (A.2), (A.3), (A.4) $\Rightarrow$ (A.1), (A.2), (A.3), (A.4′).)

For the ensuing existence discussion, we use the modified axiom structure (A.1), (A.2′), (A.3), and (A.4′). This represents a substantive weakening of the assumptions used in Sections 2 and 3, since the OCE preferences introduced in [20] and discussed in Section 5 satisfy this new axiom set but need not be NM representable.

The results of Sections 2 and 3 imply that the present modified set of axioms must be supplemented by some property interrelating conditional risk preferences. Theorem 1, in fact, suggests the nature of this interconnection. Let us rewrite Eq. (2.1) as

$$W_x(c_2) = W_{x'}U_{x'}^{-1}(U(x, c_2)) = W_{x'}U_{x'}^{-1}U_x(c_2),$$

where $x, x' \in C_1$. According to this expression, the conditional NM index W_x can be obtained from $W_{x'}$ by using $U_{x'}^{-1}U_x$. The function $U_{x'}^{-1}U_x$ provides a map from points in $C[x]$ to points in $C[x']$, which we call the *transfer* and denote γ.

DEFINITION 2. If $x, x' \in C_1$, the *transfer mapping* $\gamma: C[x] \cap C\{x'\} \to C[x']$ is characterized by the relation $c \sim^t \gamma c$, for each c in the domain of γ (cf., Fig. 3a). Let $S\{x\}$ denote the set of pairs (c_1, F) in S such that $\{c_1\} \times \operatorname{supp} F \subset C\{x\}$, where supp F is the support of F. The *induced transfer mapping* $S[x] \cap S\{x'\} \to S[x']$ associates to a pair (x, F) the pair (x', G), where $F(y) = G(y')$ if $\gamma(x, y) = (x', y')$ (cf., Fig. 3b).

The induced transfer mapping will be denoted by the same symbol γ as that used for the transfer mapping.

If (x, F) is in the domain of the induced transfer γ, then the corresponding pair $(x', G) = \gamma(x, F)$ is obtained by "sliding along the intertemporal

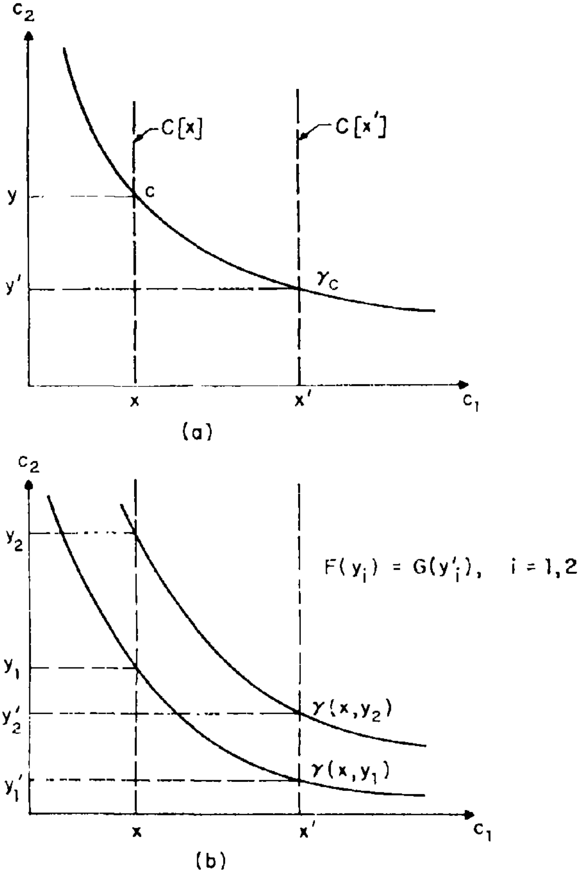

FIGURE 3

indifference curves." Thus for instance, suppose F has n-jump points $\{y_1, \ldots, y_n\}$. Then under the induced transfer one constructs a c.d.f. G which (1) also has n-jump points, $\{y_1', \ldots, y_n'\}$, where each y_i' is obtained by finding that $(x', y_i') \in C[x']$ lying on the same indifference curve as (x, y_i) and (2) has the same "probability structure" as F, i.e., $F(y_i) = G(y_i')$ for each y_i and y_i'. Hence, the transfer is determined completely by the preference relation $\succsim'$. It is clear that the induced transfer is an affine mapping: $\gamma(x, aF + bG) = a\gamma(x, F) + b\gamma(x, G)$ where $a, b \geq 0$ and $a + b = 1$.[5]

The simplest interpretation of $W_x = W_{x'} U_{x'}^{-1} U_x$ involves asserting that the *induced transfer* maps $\succsim_x$ on $S[x]$ onto $\succsim_{x'}$ on $S[x']$. More formally, we provide

DEFINITION 3. Conditional risk preferences $\{\succsim_x \mid x \in C_1\}$ are *coherent* if $\forall s_1, s_2 \in S[x] \cap S[x'], s_1 \sim s_2 \rightarrow \gamma s_1 \sim \gamma s_2$.

Assumption 5. Conditional risk preferences are coherent.

Coherence of the set of conditional risk preferences implies that the induced transfer γ maps a conditional indifference set in $S[x]$ into a conditional indifference set in $S[x']$. Since γ is determined purely by $\succsim'$, coherence is thus a property interrelating time and conditional risk preferences. As should be clear, this integrative property underlies both Theorem 1 and the existence of Examples 1 and 2 in Section 3. Moreover, it plays a crucial role in the proof of our basic two-period NM representation result.

THEOREM 3. (A.1), (A.2′), (A.3), (A.4′), *and* (A.5) *are together necessary and sufficient conditions for the existence of a continuous, strictly monotone increasing* TPC *utility* $W: C \rightarrow \mathbb{R}$.

The proof is to be found in Appendix C.

The relation $s \sim \gamma s$ which plays a crucial role in the proof of Theorem 3 can be given an interesting compensation interpretation. Consider a distribution F with probability mass concentrated at y_1 and y_2 so that $\text{Prob}\{y_1\} = p$, $\text{Prob}\{y_2\} = 1 - p$ (see Fig. 3b). Let $\gamma(x, F) = (x', G)$. We can regard $\gamma(x, F)$ as being obtained from (x, F) by a compensation process: using "time preferences" $\succsim'$, the increase $\Delta x = x' - x$ in current consumption is compensated by adjusting the levels y_1 and y_2 downward to y_1' and y_2' without altering the probability structure (i.e., without changing the value of p). So far we have assumed nothing which would imply that the conditional risk preferences on $S[x']$ are or are not compatible with γ. If the transfer completely compensates, according to the preference relation $\succsim'$, for the alteration

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[5] Note $(aF + bG)(y) = aF(y) + bG(y) = aF'(y') + bG'(y') = (aF' + bG')(y')$ by Definition 2.

Δx, then $(x, F) \sim \gamma(x, F)$. In this case, therefore, the utility of (x, F) determined by any index $S \to \mathbb{R}$ representing $\precsim$ would depend on the time preferences for the possible outcomes of the distribution, and the induced transfer mapping $\gamma: S[x] \cap S[x'] \to S[x']$ would map indifference sets into indifference sets.

Remark. Given the family $\{V_{c_1}\}$ of conditional NM indices, there is a naturally associated function V defined by $V(c_1, c_2) = V_{c_1}(c_2)$. This mapping $V: C \to \mathbb{R}$ must be interpreted with the greatest of care. The temptation is great to view this expression as a TPC utility. However, this would be *totally unwarranted* on the basis of (A.2′). As we have been arguing, (A.1), (A.2′), (A.3), and (A.4′) are *not* sufficient to establish the existence of a TPC representation. One is only justified in using $EV(c_1, \tilde{c}_2)$ for choices between points in S characterized by a *common value of first-period consumption*. The fact that V depends on c_1 reflects a dependence of the second-period risk aversion (as, for instance, in the Arrow–Pratt sense) on the preceding period's level of consumption.

Remark. The thoughtful reader may well note the possibility of an alternative approach to the existence question. Let $\mathscr{S}$ be the space of joint c.d.f.'s defined over $C_1 \times C_2$. Clearly, many elements of $\mathscr{S}$ may not be meaningful economically (because whereas future consumption will in general be uncertain, current consumption will not be). Nevertheless $\mathscr{S}$ is mathematically definable and supports a natural mixture structure which permits application of the traditional von Neumann Morgenstern axioms (e.g., Fishburn [5]). The idea would then be to *embed* the economically meaningful world of $\precsim$, S (where S is not a mixture space) into the mathematically "nice" world of $\mathscr{S}$ and assert the following:

> is NM representable on S iff there is a complete preordering over $\mathscr{S}$ which satisfies the traditional axioms and which agrees with $\precsim$ when restricted to S.

This result is easily verified. However, the approach may not be operationally meaningful since (i) the decision-maker may simply not possess preferences outside of S or (ii) he may have an ordering over all $\mathscr{S}$ which, although NM representable over S, is not consistent with the NM axioms over other regions of $\mathscr{S}$. The embedding approach has still another disadvantage which is more significant economically: the traditional axioms for TPC utility are not known to reveal the interconnection between time and conditional risk preferences developed in this paper. Finally, the result provides no indication of a test which could determine when such an extension is possible; as a consequence, the condition given for existence would be difficult to verify or to disprove in any particular theoretical discussion.

Remark. The connection between time and NM preferences has also been investigated recently by Prakash [13], though in a quite different context. The notion of a "time adjustment calculus" is used to consider preferences over date income pairs $(t, m) \in \mathbb{R}_+ \times \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$. In addition to relatively minor differences in setting occasioned by Prakash's use of the closed half-plane rather than the open positive orthant, more fundamental differences are produced by his use of the standard assumption that uncertainty is present at each time t. Nevertheless, his Theorem 3.5 (restricted to the domain of δ_{st}) should be compared with our Theorem 1 (reformulated in terms of the transfer). Despite the differences in economic context, the results are mathematically analogous. However, Prakash's argument depends upon a claim (his Proposition 3.4) which amounts to asserting that continuous, strictly monotone NM preferences over date-income pairs imply ordinal time preferences whose regions are only of type I (cf., Lemma 1 above). A counterexample in his context is provided by preferences defined by the utility $U(t, m) = (\exp(-2t) + \exp(-2m))^{-1}$, since the date-income pair (log 2, log 2) is then superior to every income at time $t = 0$.[6] Prakash examines neither the question of existence addressed by our Theorem 3 (possibly because in his setting mixtures are always available) nor that of the possible mathematical and/or behavioral incompatibilities considered in Section 3.

5. OCE UTILITY

In this section we show that the coherence postulate represents the conceptual link between TPC utility and the OCE ("Ordinal Certainty Equivalent") representation of $\precsim$ developed in Selden [20].

Let us first introduce some additional notation. Given a first-period consumption of x, the certainty equivalent period-two consumption associated with the c.d.f. F is denoted $\hat{c}_2(x, F)$. Then the "OCE Representation Theorem" can be stated as follows:

THEOREM 4 (Selden [20]).[7] *Under* (A.1), (A.2′), (A.3), *and* (A.4′), *the ordering* $\precsim$ *on* S *is* OCE *representable, in that* $\forall x, x' \in C_1$ *and* $F, G \in \mathcal{X}$,

$$(x, F) \precsim (x', G) \Leftrightarrow U(x, \hat{c}_2(x, F)) \leq U(x', \hat{c}_2(x', G)),$$

where $\hat{c}_2(x, F) = V_x^{-1} \int_{C_2} V_x(c_2)\, dF(c_2)$ *and* $\hat{c}_2(x', G) = V_{x'}^{-1} \int_{C_2} V_{x'}(c_2)\, dG(c_2)$.

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[6] The "discontinuity" argument in the proof of Proposition 3.4 incorrectly asserts that $(s, m') \precsim l$ for all $p \in (0, 1]$.

[7] Although the assumptions employed here are not exactly the same as those introduced in [20], the essential logic of the proof is.

In relating the OCE and TPC representations, we can state the following result as an immediate consequence of Theorems 3 and 4.

COROLLARY 1. *Under* (A.1), (A.2′), (A.3), *and* (A.4′), *any OCE representation can also be expressed as a TPC representation if and only if risk preferences are coherent.*

This result can be expressed in another, related way. If, in addition to (A.1), (A.2′), (A.3), and (A.4′), one invokes the coherence postulate, then corresponding to a given (U, V_x)-pair will be a two-period NM index W which differs from U by the transform $h = V_x \circ U_x^{-1}$. But *this* transform is exactly what is required to undo the nonlinear (in the probabilities) OCE representation:[8]

$$h \circ U\left(x, V_x^{-1} \int_{C_2} V_x(c_2)\, dF(c_2)\right) = V_x \circ U_x^{-1} \circ U_x\left(V_x^{-1} \int_{C_2} V_x(c_2)\, dF(c_2)\right)$$
$$= \int_{C_2} V_x(c_2)\, dF(c_2).$$

We conclude this paper with a brief discussion of the implications of invoking the coherence axiom. Consider once again the case of the pairs (x, F) and (x', G) described in conjunction with Fig. 3b. One can compute the *conditional* certainty equivalent period-two consumption values as follows:

$$\hat{y} = V_x^{-1}(\pi V_x(y_1) + (1 - \pi)\, V_x(y_2)),$$
$$\hat{y}' = V_{x'}^{-1}(\pi V_{x'}(y_1') + (1 - \pi)\, V_{x'}(y_2')).$$

On the basis of conditional risk preferences, $(x, \hat{y})$ will be indifferent to (x, F), and $(x', \hat{y}')$ to (x', G). Now, in terms of Fig. 4 it is straightforward to see that if $(x, y_1) \sim^t (x', y_1')$ and $(x, y_2) \sim^t (x', y_2')$, then coherence *requires* that $(x, \hat{y})$ and $(x', \hat{y}')$ lie on the same time preference indifference curve (and similarly for all such lotteries). In contrast, the OCE representation hypothesis would, in general, allow an indifference curve passing through $(x, \hat{y})$ to lie above or below $(x', \hat{y}')$. As this evidences, the coherence axiom in conjunction with the other assumptions produces a very strong interdependence between time and conditional risk preferences. It is as a consequence of this interdependence that the incompatibilities described in Section 3 arise. Under an OCE representation, a given $(U, \{V_x\})$-pair will not exhibit such incompatibilities as long as coherence (which certainly possesses some intuitive appeal) is not assumed. (This, of course, is not to say that an

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[8] Clearly, no increasing monotonic transform of U, such as h, will affect the ordering $\precsim$ (cf., [20, Corollary 1]).

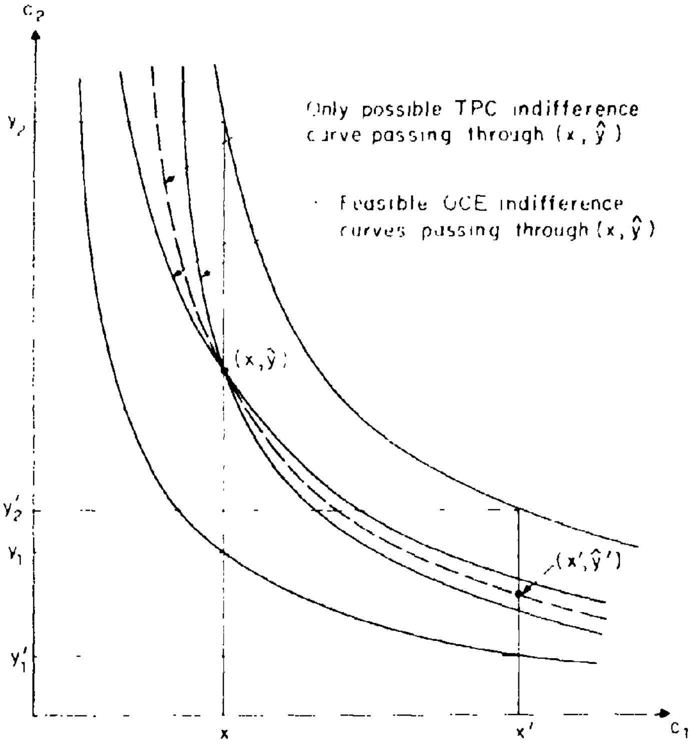

FIGURE 4

arbitrarily specified $(U, \{V_x\})$-collection might not imply unreasonable behavior for instance, in specific applications such as the consumption/savings problem.) As argued in [20], the OCE representation permits one to prescribe U and $\{V_x\}$ and hence model the interrelationship between time and risk preferences. In contrast, under the TPC approach, this interdependence is determined by a desire for the "mathematical simplification" of linearity in the probabilities, or equivalently by adoption of the coherence axiom.

APPENDIX

A. *Proof of Theorem* 1. Let T be a TPC index defined throughout C. Then by (A.4), we have $T = h \circ U$ for a strictly increasing function h: $\operatorname{im} U \to \mathbb{R}$. By restricting consideration to $C\{x\}$ and noting that $\operatorname{im} U \mid C\{x\} = \operatorname{im} U_x$, we obtain $T \mid C\{x\} = h \mid \operatorname{im} U_x \circ U \mid C\{x\}$. On the other hand, by restricting to $C[x]$, we obtain $T_x = h \mid \operatorname{im} U_x \circ U_x$. Therefore, $h \mid \operatorname{im} U_x$

$T_x \circ U_x^{-1}$. Since T_x and W_x are both conditional NM utilities for $\precsim_x$ on $\{x\} \times \underline{X}$, we may write $T_x = A \circ W_x$, where A is affine. Consequently, $T \mid C\{x\} = A \circ W_x \circ U_x^{-1} \circ U \mid C\{x\}$. It follows that $W_x \circ U_x^{-1} \circ U \mid C\{x\}$ is also a TPC, NM index over $C\{x\}$. Q.E.D.

Observe that $h \mid \operatorname{im} U_x$ is affinely equivalent to $W_x \circ U_x^{-1}$, which is not the same as having h affinely equivalent to $W_x \circ U_x^{-1}$ unless $\operatorname{im} U_x = \operatorname{im} U$. This distinction is at the root of all of our labor in proving Theorem 2 (Section 3), where we begin with a candidate for $h \mid \operatorname{im} U_x$ and are required to obtain the extension h.

B. *Proof of Theorem* 2. The general line of argument establishing the necessity of one of the conditions (i)–(iv) is indicated in the discussion of Example 1. To establish sufficiency, i.e., existence of $\precsim$, we shall consider the cases separately.

Case (i). Where $C\{x\} = C_1 \times C_2$, we have $\operatorname{im} U_x = \operatorname{im} U$. But if the monotone transform $h\colon \operatorname{im} U_x \to \mathbb{R}$ is defined by $h = W_x \circ U_x^{-1}$, then $h \circ U$ is well defined. Consequently, the function $W = h \circ U$ defines a $\precsim$ on $S (=_{\text{def}} C_1 \times \underline{X})$ for which it is the TPC index. Moreover, the resulting W has W_x as its conditional NM utility for $\precsim_x$ (on $\{x\} \times \underline{X}$). The uniqueness assertion is clear from Theorem 1.

Case (iii). Again take $h = W_x \circ U_x^{-1}$. In this case the domain of h is equal to $\operatorname{im} U_x$. But $\operatorname{im} U_x$ is just the open interval ($\inf U$, $\sup U_x$), where $\sup U_x$ is finite and less than $\sup U$. Moreover, since W_x is bounded above, h is also bounded above. Therefore, $\lim_{u \to \sup U_x} h(u)$ exists. But this is exactly what is required to extend h to a monotone function $H\colon \operatorname{im} U \to \mathbb{R}$. Now take $\precsim$ to be the relation over S having $W = H \circ U$ as its TPC index. This preordering is not unique because the extension H is not unique. (Note that if W_x and U are continuous, we can choose a continuous extension. Therefore, W is continuous if so desired.)

Cases (ii) and (iv) can be verified by a similar argument. Q.E.D.

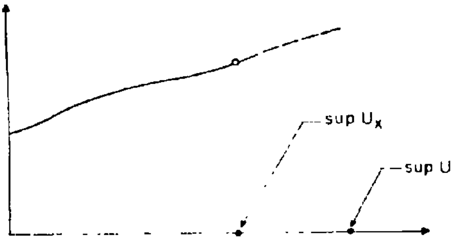

FIGURE 5

C. *Proof of Theorem 3.* The transfer mappings provide [illegible] structure which allows the existence result of classical NM theory to be extended to the certain first-period, uncertain second-period setting considered in this paper. At the level of riskless consumption pairs, the transfer mappings provide no more than a nonstandard, formal specification of [illegible] over $C = C_1 \times C_2$. The induced transfers form a class of extensions of the transfer concept to the level of conditional risk preferences: an extension characterized by the property that the induced mappings be *affine*. This structure can be taken into account provided it is compatible with the structure imposed by the a priori relation $\precsim$ on S. In particular, the transfer allows an affine representation on $S[x]$, where x is an arbitrary current consumption level, to be extended to $S\{x\}$ (which in some circumstances may be all of S). In the general case, the desired representation can be established throughout the entire choice space S by defining the representation over each set $S\{x\}$ of a collection of sets covering S, and by verifying that these definitions are identical where the covering sets $S\{x\}$ overlap.

Let us first define the following property of an induced transfer mapping.

DEFINITION. Risk preferences $\{\precsim_x\}$ are *invariant* under transfer if $s \sim \gamma s$ for any s in $S[x] \cap S\{x'\}$.

In the context of this paper it is easy to show that coherence and invariance are logically equivalent properties.

The following lemmas show that the representation can be extended from $S\{x\}$ to the entire space S, and therefore that the TPC utility can be captured by restricting the extended affine index to the set $\iota(c_1, c_2)$ as in [6].

LEMMA 2. *Given $x' \in C_1$, let λ be a representation of preferences on $S[x']$ which is affine in the probabilities. Then a (slicewise) affine index $\tilde{\lambda}$ on $S\{x'\}$ is defined by $\tilde{\lambda}(s) = \lambda\gamma(s)$ for each $s \in S\{x\}$, where γ is the appropriate transfer.*

Proof. Given that conditional risk preferences are coherent, as suggested above, they will also exhibit invariance, i.e., $\gamma s \sim s$ for each s. Therefore, $s_1 \prec s_2 \Rightarrow \tilde{\lambda}(s_1) < \tilde{\lambda}(s_2)$, and $s_1 \sim s_2 \Rightarrow \tilde{\lambda}(s_1) = \tilde{\lambda}(s_2)$. Therefore, $\tilde{\lambda}$ represents $\precsim$ on $S\{x'\}$. Since both λ and each transfer γ are affine, the extension $\tilde{\lambda}$ must be affine slicewise. ☒

The above lemma establishes the desired representation for arbitrary regions of the form $S\{x\}$, $x \in C_1$. To extend this representation to all of S, a framework of subsets $C^{(N)}$, $N = 0, 1, 2,\ldots$, is chosen such that (1) $C^{(N)} \subset C^{(N+1)} \subset C$, (2) $\bigcup_N C^{(N)} = C$, and (3) an affine representation over choices supported by $C^{(N)}$ can be extended to an affine representation over choices supported by $C^{(N+1)}$. Accordingly, the existence of an affine representation over all S is proved inductively. The following lemma provides the material for constructing $C^{(N)}$ as a union $\bigcup_{n=-N}^{N} C\{x_n\}$ of fundamental equivalence regions determined by the time preference relation $\preccurlyeq^t$. The regions $C\{x_n\}$ are themselves inductively defined.

LEMMA 3. *There is a subset* $\{x_n \mid n = 0, \pm 1, \pm 2,...\}$ *of* C_1 *such that*

(i) $C\{x_n\} \cap C\{x_{n+1}\} \neq \varnothing$ *for each* n,

(ii) $x_n < x_{n+1}$ *for each* n,

(iii) $\bigcup_n C\{x_n\} = C$.

Proof. Proceed inductively by choosing x_0 arbitrarily. It will suffice to demonstrate the extension of the chain $C\{x_{-N}\},..., C\{x_N\}$ to the chain $C\{x_{-N-2}\},..., C\{x_{N+2}\}$. If $C\{x_N\}$ has no right boundary, or if the right boundary approaches no point in $(0, \infty) \times \{0\}$, define x_{N+1} and x_{N+2} by $x_{N+1} = 1 + x_N$, $x_{N+2} = 2 + x_N$; if the right boundary approaches $(x, 0) \in (0, \infty) \times \{0\}$, then set $x_{N+2} = x$ and choose x_{N+1} to satisfy $x_N < x_{N+1} < x_{N+2}$. The choice of x_{-N-1} and x_{-N-2} is made in a similar way based on discrimination among left boundaries.

It is clear that the inductive step preserves properties (i) and (ii). Moreover, from the construction we shall prove that $x_n \to \infty$ as $n \to \infty$, and similarly $x_n \to 0$ as $n \to -\infty$. These limiting behaviors guarantee property (iii) of the claim.

Suppose $\sup x_n < \infty$. Then the left boundary of $C\{x\}$ must lie to the right of $C[x_n]$, where $x = \sup x_n$. Since these verticals approach $C[x]$, the left boundary must lie in $[x, \infty) \times C_2$. But strict monotonicity of $\preccurlyeq^t$ makes this impossible. Therefore $\sup x_n = \infty$.

In a parallel fashion, one may see that $\inf x_n = 0$. ☒

Before proceeding to establish the principal inductive step of the extension proof, we need a preliminary result.

DEFINITION. If $K \subset C_1 \times C_2$, then $S(K)$ denotes the class of all $s \in S$ whose support lies in K.

With this notation, $S[x'] = S(\{x'\} \times C_2)$ and $S\{x'\}$ is $S(C\{x'\})$. Observe that $S(K \cap L) = S(K) \cap S(L)$ if $K, L \subset C$. Moreover, it is easy to show

LEMMA 4. *If* $K, L \subset C_1 \times C_2$ *are two nondisjoint open intervals determined by* $\preccurlyeq^t$, *then each* $s_1 \in S(K \cup L)$ *is indifferent to an element* $s_2 \in S(K) \cup S(L) \subset S(K \cup L)$.

To prove the theorem, it is enough to establish

LEMMA 5. *For* $N = 0, 1, 2,...$ *there is a (slicewise) affine index* λ_N *representing* $\preccurlyeq$ *over* $S(\bigcup_{-N}^{N} C\{x_n\})$ *such that* $\lambda_{N+1} \mid \text{domain } \lambda_N = \lambda_N$.

Proof. Lemma 2 supplies λ_0. Proceed inductively. Let $K = \bigcup_{-N}^{N} C\{x_n\}$,

$L = C\{x_{N+1}\}$. By Lemma 2 there is an affine index λ representing $\precsim$ over L. Since both L and K are intervals determined by $\precsim'$ (use Lemma 1 and propositions (i) and (ii) of Lemma 3), $K \cap L$ is a subinterval of L. We propose to show that some affine modification of λ will coincide with λ_N in $S(K) \cap S(L)$ by using the classical NM uniqueness result. First view $S(K \cap L)$ as $S(\{x_{n+1}\} \times (y, z))$, where $y \in C_2$. Since $S(K \cap L) \subset S(L)$, both λ_N and λ provide affine indices representing $\precsim$ over $S(K \cap L)$; by classical expected utility theory, there is an affine transform τ such that $\tau \circ \lambda \mid S(K \cap L) = \lambda_N \mid S(K \cap L)$. Consequently, λ_{N+1} may be defined by

$$\lambda_{N+1}(s) = \begin{cases} \lambda_N(s), & s \in S(K), \\ \tau \circ \lambda(s), & s \in S(L). \end{cases}$$

According to Lemma 4, the extension to $S(K) \cup S(L)$ determines the extension to $S(K \cup L)$.

Two points must be verified: (1) λ_{N+1} represents $\precsim$ over $S(K \cup L)$, and (2) λ_{N+1} is slicewise affine.

To establish that λ_{N+1} represents $\precsim$, consider s and s'. Suppose $s \precsim s'$. Then we may suppose that $s = \iota c$, $s' = \iota c'$, where $c, c' \in K \cup L$. Since $s \precsim s'$, it must be true that $c \precsim' c'$. Since we need not consider the cases where $\{c, c'\} \subset K$ or where $\{c, c'\} \subset L$, we may assume $c \in K$ and $c' \in L$. Now if $c' \notin K$, then $c \sim' c''$ and $c' \sim' c'''$, where $\{c'', c'''\} \subset C[x_{N+1}]$ and $c'' \precsim c'''$. But the same may be said for any $c \in K$ such that $c \notin L$ and $c \precsim' c'$. Therefore, $\lambda_{N+1}(s) = \lambda_N(s)$, $\lambda_{N+1}(s') = \tau \circ \lambda(s')$, and $\tau \circ \lambda(s') \geq \sup \lambda_{N+1} \geq \lambda_{N+1}(s)$. Thus, $s \precsim s'$ implies $\lambda_{N+1}(s) \leq \lambda_{N+1}(s')$. If $s \sim s'$, then $c \sim' c'$. Thus $\{c, c'\} \subset K \cap L$, and clearly $\lambda_{N+1}(s) = \lambda_{N+1}(s')$. Now it follows immediately that $s \precsim s'$ iff $\lambda_{N+1}(s) \leq \lambda_{N+1}(s')$.

Finally, to establish that an *affine* representation is possible, we use the classical uniqueness result once again to show that λ_{N+1} coincides with an affine transform over $S[x]$ for every x such that $S[x] \subset S(K \cup L)$. This clearly implies that the inductively defined representation over all of S is (slicewise) affine. First, note that there is some affine index φ over $S[x]$, for $x \in C_1$. But then φ and λ_{N+1} give affine indices over $S[x] \cap S(K)$. So $\lambda_{N+1} \mid S(K) \cap S[x] = \tau_1 \circ \varphi \mid S(K) \cap S[x]$. Moreover, $\lambda_{N+1} \mid S(L) \cap S[x] = \tau_2 \circ \varphi \mid S(L) \cap S[x]$. Consequently, $\tau_1 \circ \varphi \mid S(K \cap L) \cap S[x] = \tau_2 \circ \varphi \mid S(K \cap L) \cap S[x]$. Since $S(K \cap L) \cap S[x]$ contains at least two elements which are not indifferent, the affine transforms τ_1 and τ_2 must be identical. Therefore, $\lambda_{N+1} \mid [S(K) \cup S(L)] \cap S[x] = \tau \circ \varphi \mid [S(K) \cup S(L)] \cap S[x]$. Therefore, $\lambda_{N+1} \mid S[x] = \tau \circ \varphi$ when $S[x] \subset S(K \cup L)$. As φ is affine, λ_{N+1} must also be affine on each slice lying in $S(K \cup L)$. ☒ Q.E.D.

References

1. K. J. Arrow, "Essays in the Theory of Risk Bearing," Markham, Chicago, 1971.
2. J. H. Drèze and F. Modigliani, Consumption decisions under uncertainty, *J. Econ. Theory* **5** (1972), 308–335.
3. J. H. Drèze and F. Modigliani, Épargne et consommation en avenir aléatoire, *Cahiers Sem. Econ.* **9** (1966), 7–33.
4. P. C. Fishburn, Axioms for expected utility in *n*-person games, *Internat. J. Game Theory* **5** (1977), 137–149.
5. P. C. Fishburn, "Utility for Decision Making," Wiley, New York, 1970.
6. Jean-Michel Grandmont, Continuity properties of a von Neumann Morgenstern utility, *J. Econ. Theory* **4** (1972), 45–57.
7. J. Hirshleifer, "Investment, Interest and Capital," Prentice–Hall, Englewood Cliffs, N.J., 1970.
8. D. M. Kreps and E. L. Porteus, Temporal resolution of uncertainty and dynamic choice theory, *Econometrica* **46** (1978), 185–200.
9. H. E. Leland, Saving and uncertainty: the precautionary demand for saving, *Quart. J. Econ.* **82** (1968), 465–473.
10. H. Markowitz, "Portfolio Selection: Efficient Diversification of Investments," Wiley, New York, 1959.
11. L. J. Mirman, Uncertainty and optimal consumption decisions, *Econometrica* **39** (1971), 179–185.
12. R. A. Pollak, Additive von Neumann–Morgenstern utility functions, *Econometrica* **35** (1967), 484–494.
13. P. Prakash, On the consistency of a gambler having time preference, *J. Econ. Theory* **15** (1977), 92–98.
14. J. W. Pratt, Risk aversion in the small and in the large, *Econometrica* **32** (1964), 122–136.
15. M. Rossman and L. Selden, The existence of two attribute von Neumann–Morgenstern utility functions given preferences on a "certain × uncertain" product space, unpublished manuscript, University of Pennsylvania, 1976.
16. A. Sandmo, Capital risk, consumption and portfolio choice, *Econometrica* **37** (1969), 586–599.
17. A. Sandmo, The effect of uncertainty on saving decisions, *Rev. Econ. Studies* **37** (1970), 353–360.
18. A. Sandmo, Two-period models of consumption decisions under uncertainty: a survey, *in* "Allocation under Uncertainty: Equilibrium and Optimality," (J. H. Drèze, Ed.), pp. 24–35, Wiley, New York, 1974.
19. L. W. Savage, "The Foundations of Statistics," 2nd ed. Dover, New York, 1972.
20. L. Selden, A new representation of preferences over "certain × uncertain" consumption pairs: the "ordinal certainty equivalent" hypothesis, *Econometrica* **46** (1978), 1045–1060.
21. L. Selden, An OCE analysis of the effect of uncertainty on saving under risk preference independence, *Rev. Econ. Studies*, in press.
22. J. von Neumann and O. Morgenstern, "The Theory of Games and Economic Behavior," Wiley, New York, 1967.

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Equilibrium with Quantity Rationing and Recontracting[*]

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1. INTRODUCTION

A number of recent works rely on a description of the allocation of resources in an economy where prices are fixed at a value which may not achieve equilibrium of supply and demand in the classical sense. They all rest upon a concept of equilibrium at fixed prices which appears to be related to a number of important issues in economic theory. At the microeconomic level, this approach stems from the study of non-tâtonnement processes where transactions take place in each period at the prevailing prices while prices adjust from period to period [18]. In an economy with public goods, an analysis carried out in a similar framework leads to a characterization of the set of Lindhal equilibria [10]. At the macroeconomic level, the concept of equilibrium at fixed prices seems especially fruitful: It makes it possible to obtain within the same model, by varying the characteristic data of the economy, situations of Keynesian unemployment, stagflation, and repressed inflation [4, 5, 6, 7, 8, 14, 21, 29]. It also provides a tool to describe the functioning of centrally planned economies where prices are administered by the planning agency [26]. And, finally, it paves the way to a reconciliation between macro and micro theory, since, when transactions are not restricted any more to a single price system but may take place at any price, the model gives back the traditional competitive equilibrium (see particularly [22, 23]).

Many of these studies use the Hicksian fixprice method. That is, it is assumed that prices are temporary fixed and that the market clears during the period under consideration by means of quantity signals perceived by the agents (quantity rationing). Moreover, it is often postulated that money is the only medium of exchange so that one can imagine that the economy is composed of separate trading posts, where the traders exchange each com-

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modity against money at the ruling prices. Finally, the rules obeyed by a rationing scheme in these studies are simple. If at the ruling prices, the demand and sypply currently expressed by the agents on a given market are compatible, then everybody should realize his plan. If there is an excess demand, then all sellers should realize their plans, while some buyers must be rationed, i.e., compelled to buy less than they intended. One speaks often in this case of a "sellers' market."

The basic features of this equilibrium concept are by now well established, although there may be variations among the authors in the way to present it [7, 8, 13, 28, 30].

Yet its exact status does not seem to be clearly perceived by all workers in the field. The ambiguity comes from the fact that these models are intended to describe "disequilibrium" states where there may be an excess supply or demand on some market, by contrast with the Walrasian model where all markets clear. In any given period, however, the final allocation is clearly the result of a *tâtonnement* on quantities. On reason comes from the assumption that the agents on the "short side" of a market always realize their plans (e.g., the sellers when there is an excess demand). When there are several buyers and sellers this assumption appears to involve a sizeable and costless exchange of information among traders, so that eventually no seller is left with an unsold commodity in the case of an excess demand. Another reason is that there is a simultaneous adjustment on the various markets. Consequently, the allocation reached on some market is influenced by the existence and the extent of a disequilibrium on others (spillover effects). In any case, the final allocation defined by the axioms postulated in the fixprice method appears to be result of an equilibrating process on quantities within the period under consideration.

The purpose of this study is to get a deeper understanding of the exact nature of this equilibrating process by using the theory of games. A simple example may help to justify the need for such a study. Consider an exchange economy with two commodities $h = 0, 1$ (commodity 0 being money), and two consumers i and j. If we look at the Edgeworth box drawn in Fig. 1, w is the initial allocation, and the line AB represents all admissible exchanges at the ruling price system p. In the picture, there is an excess demand for commodity 1, consumer i being the supplier and consumer j the demander. The rules described in this section lead to a final allocation represented by D_i : there, consumer j is constrained by the supply of consumer i. Yet the process leading to such an allocation is far from clear. It is indeed easy to understand why allocations described by points of AB outside the segment D_iD_j are unlikely to be observed. For instance, at any allocation represented by a point of AB on the left of D_i, both traders can improve their situation by moving along AB towards D_i. But any allocation represented by a point of D_iD_j in Fig. 1a, of D_iC in Fig. 1a, would appear to be "stable" in the sense that any

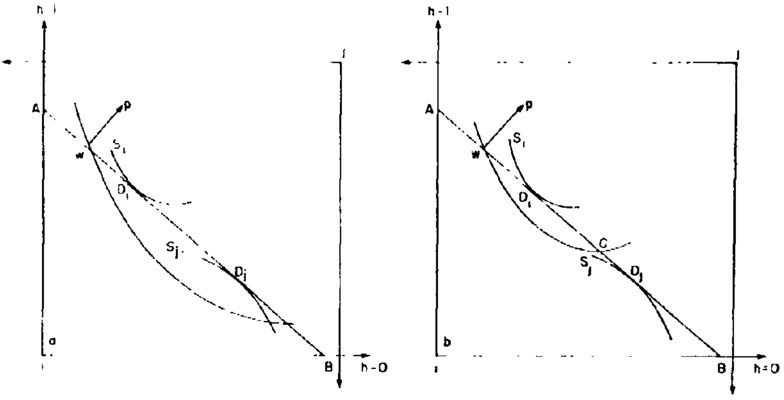

FIGURE 1

move from it while staying on AB, would hurt one of the traders and therefore would be opposed by him.

This heuristic discussion shows that the equilibrium concept leading to a final allocation like D_i, involves some kind of bargaining among traders which makes the sellers somewhat more powerful than the buyers when there is an excess demand on some market. The purpose of this paper is indeed to make this bargaining process clearer. A first result in that direction was already obtained by Younès [28, 30] and later on by Malinvaud and Younès [22, 23]. Their analysis, applied to the case of the fixprice method, shows that allocations like D_i are obtained if one postulates a recontracting process obeying the following axioms:

1. When recontracting, the traders take the ruling price system as given.
2. When recontracting, the traders can only exchange a single commodity against money.
3. The final allocation must be efficient commodity by commodity given the price system.
4. Nobody can be forced to exchange more than he wishes.

In the present study, we keep the first three axioms, and derive the fourth one from more basic considerations. Our starting point is that bargaining takes place among *coalitions* of traders. We assume that, when trying to improve upon a proposed allocation, a coalition will take into account, in a way that we make precise, the possible reactions of the traders who are left out of the coalition (compare with the definition of a bargaining set by Aumann and Maschler [3]). Our basic result is that allocations where the agents on the short side of a market realize their plans are the only ones which are stable—in the sense that no coalition can improve upon it—in large economies.

It is perhaps worthwhile to end this informal description of our analysis by emphasizing an important limitation of our study. In all our paper, the price system is fixed exogeneously and we consider only bargaining processes which take place at the ruling prices. However, since a fixprice allocation is not in general a full Pareto optimum, it would in general be profitable to the agents to change prices during the bargaining process. A more satisfactory theory would be to design bargaining processes taking place without any price limitation which would yield endogeneously determined price systems and corresponding fixprice allocations as stable outcomes. It is hoped that the present study will be a useful first step in this direction.

The paper is organized as follows. In Section 2 we describe our assumptions and the concept of equilibrium which is used in this work. We study in Section 3 the efficiency properties of this concept, and investigate in Section 4 the connection in large economies between equilibrium allocations and the set of allocations which cannot be improved upon by some coalition. All proofs, as well as some auxiliary results which are of independent interest, are gathered in Section 5.

2. DISEQUILIBRIUM ALLOCATIONS

We consider an exchange economy with $l + 1$ commodities indicated by $h = 0, 1,..., l$. Commodity 0 plays a particular role in the analysis and is called money. There are m consumers indicated by $i = 1,..., m$. Each consumer owns an endowment $w_i \in R_+^{l+1}$ which he wishes to trade with the others. The set of feasible net trades z for the ith consumer is a subset Z_i of R^{l+1}. Consumer i has preferences among feasible net trades given by a complete preordering $\succsim_i$ on Z_i. The following assumptions are made throughout the paper:

ASSUMPTION 1. *Z_i is convex.*

ASSUMPTION 2. *The preferences $\succsim_i$ are strictly convex.*

The price system is fixed and equal to a vector $p = (p_0, p_1, ..., p_l)$ in R^{l+1}, all components of which are positive and such that $p_0 = 1$. A trade z is said to be *admissible* if $z_i \in Z_i$ and $p \cdot z = 0$. The result of the trading process is an *admissible allocation* which is described by a collection or admissible net trades (z_i) such that $\sum_{i=1}^{m} z_i = 0$.

Since adjustments on the markets do not take place by price movements but by quantity movements, clearing of the markets is brought about by *quantity constraints* $(\underline{z}_i, \bar{z}_i)$ perceived by the agents that set lower and upper bounds on the net trades that they are allowed to make. An important assumption of the model is the fact that the traders do not perceive constraints in the case of commodity 0. Thus $\underline{z}_i$ and $\bar{z}_i$ are vectors of R^l that satisfy $\underline{z}_i \le 0 \le \bar{z}_i$. A particular *interpretation* of the model is that there are separate markets, or trading posts, one for each commodity $h \ne 0$, where the traders exchange commodity h for money at the ruling prices. The traders however trade simultaneously on all markets. For every $h \ne 0$, let $t(h)$ be a vector (in R^{l+1}) representing an exchange of one unit of commodity h for p_h units of money: $t_0(h) = -p_h$, $t_h(h) = 1$ and $t_k(h) = 0$ for $k \ne 0, h$. Then any trade z_i such that $p \cdot z_i = 0$ can be written $z_i = \sum_{h=1}^{l} z_{ih} t(h)$, and conversely. Then z_{ih} can be interpreted as the intensity of the ith consumer's transaction of commodity h against money, and the constraints $\underline{z}_{ih} \le z_{ih} \le \bar{z}_{ih}$, $h \ne 0$, can be interpreted as constraints on these intensities. Given such a signal $(\underline{z}_i, \bar{z}_i)$, the set of feasible trades satisfying the budget constraint $p \cdot z = 0$ and the quantitative constraints $\underline{z}_{ih} \le z_h \le \bar{z}_{ih}$, $h \ne 0$, is denoted γ_i.

Let a *disequilibrium allocation* be described by an admissible allocation (z_i^*) and by quantity signals $(\underline{z}_i, \bar{z}_i)$ such that for every i, z_i^* maximizes the ith consumer's preferences on the set γ_i. Clearly, such a definition does not involve any restriction at all, since every admissible allocation (z_i^*) can be described in that way by choosing $\underline{z}_{ih} = z_{ih}^* = \bar{z}_{ih}$ for every $h \ne 0$. By contrast, the concept of a Keynesian allocation involves much more stringent conditions which we review now.

To describe whether, at a disequilibrium allocation, the ith consumer perceives binding constraints on the market for commodity h, we introduce the set γ_i^h which is obtained from γ_i by dropping the constraints associated with that commodity. That is, γ_i^h is the set of net trades $z_i \in Z_i$ which satisfy $p \cdot z_i = 0$ and $\underline{z}_{ik} \le z_{ik} \le \bar{z}_{ik}$, $k \ne 0, h$. The consumer is actually constrained in commodity h if there is a net trade z_i in γ_i^h such that $z_i \succ_i z_i^*$.

DEFINITION 1. A *K*-allocation is a disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ such that for every $h \ne 0$, either $0 \le z_{ih}^* < z_{ih}$ for all i and $z_i \in \gamma_i^h$ such that $z_i \succ_i z_i^*$, or $z_{ih} < z_{ih}^* \le 0$ for all i and $z_i \in \gamma_i^h$ such that $z_i \succ_i z_i^*$.

A *K*-allocation displays the following properties. If z_i^* maximizes the ith consumer's preferences on γ_i^h for all i, there is no excess demand or supply for commodity h and everybody realize his plan on that market. On the other hand if, for instance, $0 \le z_{ih}^* < z_{ih}$ whenever $z_i \in \gamma_i^h$ and $z_i \succ_i z_i^*$, there is an excess demand for commodity h and all sellers realize their plans, while some buyers are compelled to buy less of commodity h than they would like.

We study in Section 3 disequilibrium allocations which satisfy les stringent requirements than a *K*-allocation. Specifically, we consider the following condition:

$$(z_{ih} - z_{ih}^*)(z_{jh} - z_{jh}^*) > 0 \qquad \textit{whenever } z_i \in \gamma_i^h,\ z_j \in \gamma_j^h \textit{ and } z_i \succ_i z_i^*,\ z_j \succ_j z_j^*. \tag{H}$$

This condition means that the traders who perceive binding constraints for some commodity h should all be constrained in the same direction. We say that a disequilibrium allocation satisfies the *Hahn–Negishi* condition if (H) holds for every h, because such a requirement was first discussed in the context of *non tâtonnement* models.

An example may clarify the meaning of these conditions. Consider the exchange economy with two commodities ($l = 1$), and two consumers i and j which was pictured in Fig. 1. There, w is the initial allocation, and the set of admissible allocations is represented by the segment AB. The set of admissible allocations which satisfy the Hahn-Negishi condition (H) is represented by the closed segment $D_i D_j$. There exists a unique K-allocation which is represented by the point D_i.

Remark. The foregoing definition of a K-allocation does not specify how shortages are distributed among the agents. Benassy [7, 8] has proved the existence of a K-allocation corresponding to an arbitrary rationing scheme, using a concept of effective demand as in [11, 15].

3. EFFICIENCY

It is natural to ask whether a K-allocation displays any efficiency property. It is first clear that, if we wish a K-allocation to be efficient in the Pareto sense, we have to restrict ourselves to trades that satisfy the budget constraint $p \cdot z = 0$, i.e., to admissible trades. The following typical example [8, 23, 29] shows, however, that a K-allocation need not be Pareto optimal among the class of admissible allocations if we allow the consumers, when recontracting, to trade simultaneously on all markets.

Consider an exchange economy with three commodities ($l = 2$), and two consumers i and j. Assume that there exists a K-allocation (z_i^*, z_j^*), where there is an excess effective supply of commodity 1 as well as of commodity 2, and where consumer i is a net demander of commodity 1 and a net supplier of commodity 2, while the situation is reversed for consumer j. The situation so described is similar, although the analogy must not be pushed too far, to Keynesian unemployment, where there is an excess supply both on the labor market and on the market for output. At the K-allocation, the only *binding* constraint that consumer i perceives is a constraint $\underline{z}_{i2} < 0$ on his supply of commodity 2, where $\underline{z}_{i2} = z_{i2}^* = -z_{j2}^*$. Then z_i^* is the result of the maximization of the preferences $\succsim_i$ subject to $p \cdot z = 0$ and $z_2 \geq \underline{z}_{i2}$. If the preferences $\succsim_i$ are representable by a continuously differentiable utility function U_i, and if z_i^* is interior to $Z_i = -\{w_i\} + R_+^{l+1}$, a straightforward manipulation of the Kuhn–Tucker conditions yields the two relations $U'_{i1} - p_1 U'_{i0} = 0$ and $U'_{i2} - p_2 U'_{i0} < 0$, where the partial derivatives are taken at z_i^*. Similarly, one gets for consumer j, $U'_{j1} - p_1U'_{j0} < 0$ and $U'_{j2} - p_2U'_{j0} > 0$. It is now clear that the two agents can both improve their position by trading simultaneously on all markets. Indeed, consider the new admissible allocation (z_i, z_j), where $z_i = z_i^+ + a_1t^1 + a_2t^2$, with $t^1 = (-p_1, 1, 0)$, $t^2 = (-p_2, 0, 1)$, $a_1 > 0$ and $a_2 < 0$ being "small" numbers. Then

$$U_i(z_i) - U_i(z_i^+) \simeq a_1(U'_{i1} - p_1U'_{i0}) + a_2(U'_{i2} - p_2U'_{i0}) > 0,$$

$$U_j(z_j) - U_j(z_j^+) \sim -a_1(U'_{j1} - p_1U'_{j0}) - a_2(U'_{j2} - p_2U'_{j0}) > 0.$$

When a_1 and a_2 are different from zero, the trade $z_i - z_i^+ = -(z_j - z_j^+)$ involves in general an exchange of all commodities. Suppose now that the traders must exchange only a pair of commodities when trying to improve their position, starting from the given K-allocation. If we allow them to exchange directly commodity 1 for commodity 2, this corresponds to the case $a_1p_1 + a_2p_2 = 0$, and then both traders can be made better off. On the other hand, if the traders are restricted to monetary exchanges (either $a_1 = 0$ or $a_2 = 0$), then a trader cannot improve his position without making the other worse off.

This example suggests that we have to restrict the recontracting process to take place on a single market at a time.

DEFINITION 2. A disequilibrium allocation $(z_i^+, \underline{z}_i, \bar{z}_i)$ is efficient market by market if there is no commodity $h \neq 0$, and no collection of admissible trades $z_i \in \gamma_i^h$, $i = 1,\ldots, m$, such that $\sum_{i=1}^m z_{ih} = 0$ and $z_i \gtrsim_i z_i^+$ for all i, with strict preference for some i.

Note that the consumers, when they recontract by exchanging commodity h among themselves, ignore the quantitative constraints associated with that commodity, but plan to modify their transactions in the other commodities $k \neq h$, subject to the constraints perceived for these commodities.

If we go back to the example described in Fig. 1, we see that the set of efficient allocations (in our restricted sense) coincide to the closed segment D_iD_j. Indeed, for any point between A and D_i, for instance, it is possible to improve the situation of both consumers by moving along the line AB towards D_i. A similar statement is true for any point between D_j and B. But this is no longer possible if we start from points in the segment D_iD_j, for if we wanted to improve the situation of the traders, we would have then to leave the set of admissible allocations which is described by the segment AB.

We saw in Section 2 that the set of admissible allocations which satisfy the Hahn–Negishi condition was represented too by the closed segment D_iD_j. The following proposition states that this is a general result.

PROPOSITION 1. *A disequilibrium allocation is efficient market by market if and only if it satisfies the Hahn–Negishi condition.*

4. RECONTRACTING

We showed in the previous section that the Hahn Negishi condition (H) characterized disequilibrium allocations that were efficient market by market. We now investigate the connection between K-allocations, and the set of disequilibrium allocations that cannot be improved upon, or "blocked," by coalitions. Our goal is to find a notion of blocking such that these two sets tend to coincide when the number of agents is large, and when each of them is negligible compared to the size of the economy.

By analogy with the traditional study of the core, and in view of the restrictions that we imposed on the recontracting process in the previous section, one might be tempted to say that a disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ is stable if there is no coalition S of consumers, no commodity $h \neq 0$, and no collection of admissible net trades $z_i \in \gamma_i^h$, $i \in S$ such that (i) $\sum_{i \in S} z_{ih} = 0$, and (ii) $z_i \gtrsim_i z_i^*$ for all i in S, with strict preference for some member of S. One would then study the relationships between the set of stable disequilibrium allocations, and the set of K-allocations. The following example shows however that, with such a definition, a K-allocation would not in general be stable. Indeed, in many cases, there would be no stable disequilibrium allocation.

Consider an economy with two commodities ($l = 1$), three consumers. Assume that in the absence of any perceived quantitative signal, consumer 1 is willing to supply, say, 20 units of commodity 1, while each other consumer wishes to demand 30 units of commodity 1. There is an excess demand of commodity 1. A K-allocation is thus described by an allocation where consumer 1 delivers 20 units of commodity 1 to the other agents, while consumers 2 and 3 receive, respectively, z_2 and z_3 units of commodity 1, where $(z_2, z_3) \geq 0$ and $z_2 + z_3 = 20$. There are many such allocations, but no one of them is stable if we adopt the definition presented so far. For if, say, $z_2 < 20$, then consumer 1 and 2 can redistribute their endowments among themselves so as to make consumer 2 better off without making consumer 1 worse off. The origin of this phenomenon is the fact that we have allowed so far to much competition among the traders who are actually rationed at a K-allocation (here consumers 2 and 3). The phenomenon can be avoided by requiring that a coalition, when recontracting, must improve the position of *every* member of the coalition. We would not need, however, this additional requirement for the universal coalition (the set of all consumers) since the phenomenon cannot occur in such a case. One thus might be tempted to say that a disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ is stable if it is efficient market by market, and if there is no coalition S, no commodity $h \neq 0$, and no collection of admissible trades $z_i \in \gamma_i^h$, $i \in S$, such that $\sum_{i \in S} z_{ih} = 0$ and $z_i \succ_i z_i^*$ for all members of S. We see later on that with such a definition, a K-allocation is always stable. But we also prove that the set of stable allocations can then be much larger than the set of K-allocations even in large economies (see Section 5). At this stage of the argument we restrict ourselves to showing these points by looking at an example.

Let us go back to the example described in Fig. 1, and assume for the sake of simplicity that there is a continuum of agents of types i and j. We already saw in Section 3 that the allocations represented by points of AB outside the segment D_iD_j were not efficient. Consider now an allocation described by a point P of the segment D_iD_j which differs from D_i and D_j. Let $\tilde{z}_{i1} < 0$ and $\tilde{z}_{j1} > 0$ be the Walrasian supply and demand of the consumers of types i and j, respectively. There exist $\alpha_i > 0$, $\alpha_j > 0$ with $\alpha_i + \alpha_j = 1$ such that $\alpha_i\tilde{z}_{i1} + \alpha_j\tilde{z}_{j1} = 0$. If we consider a coalition S composed of a proportion α_i of consumers of type i and a proportion α_j of consumers of type j, it is possible for every member of S to achieve his most preferred trade and therefore to improve his position compared to the allocation represented by P. On the other hand, if we consider the K-allocation represented by D_i, then no coalition can improve the position of all its members for such a coalition should be exclusively composed of consumers of type j. A similar reasoning shows that the allocation described by D_j would be stable with the definition presented so far, as long as the trade wD_j does not make the traders of type j worse off compared to their initial endowment. Therefore, with the definition presented so far, the set of stable allocations is described by the points D_i and D_j in the case of Figure 1a, and reduces to the K-allocation D_i in Figure 1b.

If we wish to eliminate the allocations like D_j where, on some market h, all the buyers do realize their plans although there is an excess demand, we clearly have to allow more coalitions to recontract out than were permitted in the last definition. In doing so, we have to present a concept which captures the idea that, when there is an excess demand on some market, the sellers are somewhat more powerful than the buyers. It turns out that it is possible to achieve this goal by borrowing some features of the definition of a bargaining set.

Consider a disequilibrium allocation. Let A be the set of agents and let S be a coalition of consumers who wish to recontract out on some market h. The coalition S may need the help of some disjoint subset of traders T in order to achieve its goal. That is, the coalition S wishes to propose a collection of net trades z_i which belong to γ_i^h for all i in $S \cup T$, such that $\sum_{i \in S \cup T} z_{ih} = 0$, $z_i \succ_i z_i^*$ for all i in S and $z_i \sim_i z_i^*$ for all i in T. We saw at the beginning of this section that a K-allocation is not in general stable if we allow such coalitions to always block. It seems however natural to assume that this coalition will indeed propose the new allocation (z_i) and block if no member of T is likely to face an equivalent, or better proposal made by the complementary coalition $U = A\backslash(S \cup T)$. This will be the case if U is empty, which means that an allocation which is not efficient market by market is blocked, of if T is empty, which corresponds to the second definition of blocking that we considered. In all other cases, the allocation will be blocked by S if for every $V \subset U \cup T'$, where T' is an arbitrary nonempty subset of T, there is no collection of trades z_i' which belong to γ_i^h for all i in V such that $\sum_{i \in V} z_{ih}' = 0$ and $z_i' \succsim_i z_i^*$ for all members of V.

If we go back to the example described in Fig. 1, we find that this concept of blocking seems to succeed in capturing the asymmetry between sellers and buyers when there is an excess demand. Let us assume again for simplicity that there is a continuum of players of each type i and j. At the allocation represented by D_j, all sellers of commodity 1 are compelled to sell more than they intended. But they can all achieve their most preferred trade wD_i by exchanging with only a part of the set of the buyers, say T. The buyers who are left out cannot make an equivalent proposal to the members of T. The allocation represented by D_j is blocked. On the other hand, at the K-allocation described by D_i, the buyers are forced to buy less than they intended. It is true that a part S of the buyers can achieve the planned trade wD_j by exchanging with all the sellers. But in that case, the buyers who are left out can make an equivalent proposal to part of the sellers.

DEFINITION 3. The coalition S blocks the disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ if, for some $h \neq 0$,

(α) there exists a (possibly empty) set of traders T, disjoint from S, and a collection of net trades $z_i \in \gamma_i^h$, $i \in S \cup T$, such that $\sum_{i \in S \cup T} z_{ih} = 0$, $z_i \succ_i z_i^*$ for all i in S, and $z_i \sim_i z_i^*$ for all i in T;

(β) whenever $U = A \setminus (S \cup T)$ is nonempty, for every $V \subset U \cup T'$, where T' is some nonempty subset of T, there is no collection of net trades $z_i' \in \gamma_i^h$, $i \in V$, such that $\sum_{i \in V} z_{ih}' = 0$ and $z_i' \succsim_i z_i^*$ for all members of V.

A disequilibrium allocation is stable if there is no blocking coalition.

We wish to show now that with this definition the set of stable allocations and the set of K-allocation tend to coincide when the number of agents is large and when each of them is negligible compared to the market. To prove this claim, we use the procedure of replicating the economy [12]. The economy $\mathfrak{E}^r$ is composed of mr consumers indicated by $a = (i, q)$, with $i = 1, \ldots, m$ and $q = 1, \ldots, r$. Two consumers corresponding to the same index i have the same preferences $\succsim_i$ and the same endowments w_i. We say then that they belong to the same type i. The rth replica of a disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ of the original economy is clearly a disequilibrium allocation of the replica economy $\mathfrak{E}^r$. The same is true for replicas of K-allocations. These replicated allocations are denoted by the same symbol $(z_i^*, \underline{z}_i, \bar{z}_i)$. Then,

THEOREM 1. *A disequilibrium allocation which is stable for all r is a K-allocation.*

Is it true that a K-allocation is stable for every replicated economy $\mathfrak{E}^r$, or at least when r is large enough? The following example shows that this conjecture is false. Consider an exchange economy with two goods $(l = 1)$ and three types of consumers. Assume that, at the ruling prices, consumers of type 1 wish to sell 20 units of commodity 1, while consumers of types 2 and 3 would like to buy 10 and 15 units of that commodity, respectively. At any K-allocation, consumer 1 actually delivers 20 units of commodity 1 to the others, while consumers 2 and 3 receive z_2^* and z_3^* units of that commodity, with $z_2^* + z_3^* = 20$, $5 \leq z_2^* \leq 10$ and $10 \leq z_3^* \leq 15$. Take any such allocation in the replicated economy $\mathfrak{E}^r$, where r is "large," and assume that $5 < z_2^* < 10$, $10 < z_3^* < 15$. We claim that this allocation can be blocked. Indeed, consider the coalition S in $\mathfrak{E}^r$ composed of all buyers of type 3, and of all buyers of type 2 except one, and let T be the set of all sellers. By exchanging with the set T, all members of S can get what they bought at the K-allocation, plus a bonus equal to $z_2^*/(2r - 1)$, which is small enough to make all of them better off when r is large. To prevent the formation of the coalition, the buyer of type 2 who was left out must propose to a member of T to buy 20 units of commodity 1. If such a purchase makes him worse off compared to what he got at the K-allocation, (β) of Definition 3 is satisfied, and S blocks. The phenomenon will occur for any large but finite r.

The example suggests, however, that the phenomenon cannot occur if we have a *continuum* of consumers of each type. For in that case, any coalition S of the type that we described would have to exclude a *positive proportion* of consumers of type 2. These consumers can always make a counterproposal to a small enough proportion of sellers. Therefore, we may conjecture from the example that, if we reason in an abstract economy with a continuum of consumers, a K-allocation is stable. We see that this conjecture is true (see the remark below at the end of Section 5). Such a result, however, would not be very comforting if we were unable to prove some stability property of a K-allocation in finite economies. The example and the remark about economies with a continuum of consumers suggest that we might succeed if we postulate that, in a replicated economy $\mathfrak{E}^r$, a blocking coalition S must contain at most δr consumers of each type, where $\delta < 1$. This is the motivation for the following definition.

DEFINITION 4. Given $0 < \delta < 1$, a disequilibrium allocation is δ-stable in the replicated economy $\mathfrak{E}^r$ if there does not exist a blocking coalition S which contains at most δr consumers of each type.

Then,

THEOREM 2. *Given $0 < \delta < 1$, and a K-allocation, there exists $\bar{r}$ such that it is δ-stable for all $r \geq \bar{r}$.*

As a matter of fact, Theorem 1 is still valid when one replaces the word "stable" by "δ-stable." Therefore we have the following characterization of K-allocations in finite economies:

COROLLARY. *Given $0 < \delta < 1$, a disequilibrium allocation is a K-allocation if and only if there exists $\bar{r}$ such that it is δ-stable for all $r \geq \bar{r}$.*

Remark. Definition 3 can be restated in the following equivalent way. S is said to block a disequilibrium allocation if for some $h \neq 0$, (α') there exists a possibly empty set of traders T disjoint from S and a collection of net trades z_i in γ_i^h such that $\sum_{i \in S \cup T} z_{ih} = 0$, $z_i \succ_i z_i^*$ in S and $z_i \succsim_i z_i^*$ in T, and (β') whenever $U = A \setminus (S \cup T)$ is nonempty for every $V = U \cup T'$, where T' is a nonempty subset of T, there is no collection of z_i' in γ_i^h such that $\sum_{i \in V} z_{ih}' = 0$, $z_i' \succsim_i z_i^*$ in U and $z_i' \succsim_i z_i$ in T'. The reader will immediately verify that a disequilibrium allocation is blocked in the sense of this definition if and only if it is blocked in the sense of Definition 3.

Either definition can be reintrepreted by using the language of the definition of a Bargaining Set [3]. Whenever there exist S and T such that (α) or (α') is satisfied, we can say that $S \cup T$ has an objection against the complementary coalition U. Then (β) or (β') can be interpreted as saying that U has no justified counterobjection. Our concept, however, departs from the usual definition of a bargaining set in one very important respect: *The complementary coalition U is not allowed to make a counterobjection to any member of S.* The equivalent of the definition of the bargaining set in our context would be obtained by replacing in (β'), "T' is a subset of T" by "T' is a subset of $S \cup T$". This obviously would give more power to the complementary coalition U to make a counterobjection, and one would expect that the set of stable allocations would then be much larger than the set of K-allocations, even in large economies. As a matter of fact, in the example described in Fig. 1a, when there is a continuum of traders of each type, it can be shown that, with this new definition, the set of stable allocation is described by the closed segment D_1X, where X is the point of D_iD_j satisfying $U(wX) = \mathrm{Max}(U(wD_i), U(wD_i) - U(wD_j))$. The concept of a bargaining set was therefore not suited to our purpose.

5. PROOFS

PROPOSITION 1. *A disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ is efficient market by market if and only if it satisfies, for all $h \neq 0$:*

$$(z_{ih} - z_{ih}^*)(z_{jh} - z_{jh}^*) \geq 0 \qquad \textit{whenever } z_i \in \gamma_i^h,\ z_j \in \gamma_j^h,$$

$$\textit{and } z_i \succ_i z_i^*,\ z_j \succ_j z_j^*. \tag{H}$$

Proof. Note that if $z_i \in \gamma_i^h$, $z_i \succ_i z_i^*$, then $z_{ih} \neq z_{ih}^*$, otherwise, z_i^* would not maximize the preferences $\succsim_i$ on γ_i.

Let us first show that if there exist $z_i \in \gamma_i^h$, $z_j \in \gamma_j^h$ such that $z_i \succ_i z_i^*$, $z_j \succ_j z_j^*$ and $(z_{ih} - z_{ih}^*)(z_{jh} - z_{jh}^*) < 0$, for some i, j, h, then $(z_i^*, \underline{z}_i, \bar{z}_i)$ cannot be efficient market by market. Indeed, there exist λ_i and λ_j in $(0, 1)$ such that $\lambda_i(z_{ih} - z_{ih}^*) + \lambda_j(z_{jh} - z_{jh}^*) = 0$. Let $z_a' = z_a^*$ for all $a \neq i, j$, $z_i' = \lambda_i z_i + (1 - \lambda_i) z_i^*$ and $z_j' = \lambda_j z_j + (1 - \lambda_j) z_j^*$. We have $z_a' \in \gamma_a^h$ for all a, $\sum_{a=1}^{m} z_{ah}' = 0$, $z_i' \succ_i z_i^*$ and $z_j' \succ_j z_j^*$ because the preferences are convex. Thus $(z_i^*, \underline{z}_i, \bar{z}_i)$ cannot be efficient market by market.

Conversely, let us prove that if a disequilibrium allocation is not efficient market by market, then $(z_{ih} - z_{ih}^*)(z_{jh} - z_{jh}^*) < 0$ for some h and some $z_i \in \gamma_i^h$, $z_j \in \gamma_j^h$ with $z_i \succ_i z_i^*$ and $z_j \succ_j z_j^*$. Now, there exist h and $z_a \in \gamma_a^h$ for all a such that $\sum_{a=1}^{m} z_{ah} = 0$, $z_a \succsim_a z_a^*$ for all a and $z_i \succ_i z_i^*$ for some i. One has surely $z_{ih} \neq z_{ih}^*$. Thus, since $\sum_{a=1}^{m} (z_{ah} - z_{ah}^*) = 0$, there exists j such that $(z_{ih} - z_{ih}^*)(z_{jh} - z_{jh}^*) < 0$. Because the preferences are strictly convex, any $z_j' \in (z_j, z_j^*)$ satisfies $z_j' \in \gamma_j^h$, $z_j' \succ_j z_j^*$ and $(z_{ih} - z_{ih}^*)(z_{jh}' - z_{jh}^*) < 0$. Q.E.D.

We introduce now a set of disequilibrium allocations which is larger than the set of *K*-allocations. At a *K*-allocation, if there is an excess demand on some market *h*, all the sellers realize their plans, while some buyers are compelled to buy less than they would like. The following definition keeps open the possibility that, in such a case, all buyers realize their plans, while some sellers are compelled to sell more than they wish.

DEFINITION 5. A disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ is an *R*-allocation if, for all $h \neq 0$:

(1) $(z_{ih} - z_{ih}^*)(z_{jh} - z_{jh}^*) \geq 0$ whenever $z_i \in \gamma_i^h$, $z_j \in \gamma_j^h$ and $z_i \succ_i z_i^*$, $z_j \succ_j z_j^*$;

(2) either $z_{ih} \geq 0$ for all i and $z_i \in \gamma_i^h$ such that $z_i \succ_i z_i^*$, or $z_{ih} \leq 0$ for all i and $z_i \in \gamma_i^h$ such that $z_i \succ_i z_i^*$.

Condition (1) is the Hahn Negishi condition. Condition (2) states that only side of the market for commodity *h* may be constrained. If we look again at the example described in Fig. 1, we see that the set of allocations satisfying (1) and (2) is represented by the points D_1 and D_2 in Fig. 1a, and reduces to the point D_1 in Fig. 1b. In the general case, a *K*-allocation is always an *R*-allocation, while the converse is not necessarily true.

We wish also to work temporarily with a weaker concept of stability than was given in Definition 4.

DEFINITION 6. A disequilibrium allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$ is weakly stable if it is efficient market by market and if there is no coalition *S*, no commodity

h, and no collection of net trades $z_i \in \gamma_i^h$, $i \in S$ such that $\sum_{i \in S} z_{ih} = 0$ and $z_i \succ_i z_i^*$ for all members of S.

We indicated in Section 4 that the set of *R*-allocations and the set of weakly stable allocations tended to coincide in large economies. The following result makes this statement precise.

PROPOSITION 2. *A disequilibrium allocation is an R-allocation if and only if it is weakly stable in all replicated economies.*

Proof. Let us first show that an *R*-allocation is weakly stable for all r. It suffices to do so for the original economy.

Proposition 1 shows that this *R*-allocation is efficient market by market. Let us suppose now that there exist a coalition S, a commodity h, and a collection of net trades $z_i \in \gamma_i^h$, $i \in S$, such that $\sum_{i \in S} z_{ih} = 0$ and $z_i \succ_i z_i^*$ for every member of S. Condition (2) implies that, say, $z_{ih} > 0$ for all i in S. But this would imply $\sum_{i \in S} z_{ih} > 0$. We get a contradiction, which proves that an *R*-allocation is weakly stable.

Conversely, consider an allocation which is weakly stable for all r. Proposition 1 tells us that (1) holds. On the other hand, since the disequilibrium allocation cannot be improved upon by one-member coalitions, $z_i \succ_i z_i^*$, $z_i \in \gamma_i^h$ implies $z_{ih} \neq 0$. It remains to show that (2) holds. Suppose that it does not. Then, for some h, there exist $z_i \in \gamma_i^h$, $z_i \succ_i z_i^*$ and $z_j \in \gamma_j^h$, $z_j \succ_j z_j^*$ such that $z_{ih} > 0$ and $z_{jh} < 0$. Hence, there are $\alpha_i > 0$, $\alpha_j > 0$ with $\alpha_i + \alpha_j = 1$ such that $\alpha_i z_{ih} + \alpha_j z_{jh} = 0$. From (1), we know that $(z_{ih} - z_{ih}^*)$ and $(z_{jh} - z_{jh}^*)$ have the same sign. Suppose that they are both negative (the other case is treated in a similar way). Define n_i^r as the largest integer less than or equal to $r\alpha_i$ and n_j^r as the smallest integer greater than or equal to $r\alpha_j$. Let

$$z_{ih}^r = \left(\frac{r\alpha_i}{n_i^r}\right) z_{ih} \qquad \text{and} \qquad z_{jh}^r = \left(\frac{r\alpha_j}{n_j^r}\right) z_{jh}.$$

For r large enough, one has $z_{ih} \leq z_{ih}^r < z_{ih}^*$ and $z_{jh} \leq z_{jh}^r < z_{jh}^*$. In addition,

$$n_i^r z_{ih}^r + n_j^r z_{jh}^r = r(\alpha_i z_{ih} + \alpha_j z_{jh}) = 0.$$

Fix such an r. Let z_i' (resp. z_j') be the convex combination of z_i (resp. z_j) and of z_i^* (resp. z_j^*) such that $z_{ih}' = z_{ih}^r$ (resp. $z_{jh}' = z_{jh}^r$). We have $z_i' \in \gamma_i^h$ and $z_i' \succ_i z_i^*$, and the same thing for j. Therefore, in the rth replica, the coalition composed of n_i^r consumers of type i and of n_j^r consumers of type j can do better for all of its members. This leads to a contradiction. Therefore (2) holds, which completes the proof of the proposition. Q.E.D.

We proceed now to the proofs of Theorems 1 and 2. They are independent of Proposition 2.

THEOREM 1. *A disequilibrium allocation which is stable for all r is a K-allocation.*

Proof. Consider a disequilibrium allocation which is stable for all r. It must be efficient market by market. Consider an arbitrary $h \neq 0$, and pick $\tilde{z}_i$ in γ_i^h such that $\tilde{z}_i \succ_i z_i^*$ for all types i who are constrained in commodity h. We know that $\tilde{z}_{ih} - z_{ih}^*$ has the same sign for all these types. Suppose that $\tilde{z}_{ih} - z_{ih}^* > 0$ for all types who are constrained (the other case where $\tilde{z}_{ih} - z_{ih}^* < 0$ is treated in a similar fashion). If the disequilibrium allocation fails to satisfy the definition of a K-allocation on market h, there must exist a nonempty set S^* of types of consumers who are constrained with $z_{ih}^* < 0$ for all i in S^*. Since $\sum_{i=1}^{m} z_{ih}^* = 0$, this implies:

$$\sum_{z_{ih}^* > 0} z_{ih}^* = -\sum_{z_{ih}^* < 0} z_{ih}^* \geq -\sum_{i \in S^*} z_{ih}^* > -\sum_{i \in S^*} \tilde{z}_{ih}.$$

Therefore, if r is large enough, one can find an integer $n_r < r$ such that

$$r \sum_{i \in S^*} z_{ih}^* < -n_r \sum_{z_{ih}^* > 0} z_{ih}^* < r \sum_{i \in S^*} \tilde{z}_{ih}. \tag{5.1}$$

Fix such an r and n_r. Let S be the coalition of all members $a = (i, q)$ of the rth replica such that $i \in S^*$. We claim that S can block the disequilibrium allocation. Let T be an arbitrary set of consumers $a = (i, q)$ composed of n_r agents of each type i such that $z_{ih}^* > 0$. For every $a = (i, q)$ in T let $z_a = z_i^*$. For every $a = (i, q)$ in S, let $z_a = \lambda z_i^* + (1 - \lambda) \tilde{z}_i$, where λ is chosen so that $\sum_{a \in S \cup T} z_{ah} = 0$. It is readily checked that $0 < \lambda < 1$, because of (5.1). Thus $z_a \in \gamma_i^h$ for every $a = (i, q)$ in $S \cup T$, $\sum_{a \in S \cup T} z_{ah} = 0$, $z_a \succ_i z_i^*$ for all $a = (i, q)$ in S and $z_a = z_i^*$ for all $a = (i, q)$ in T.

Let A^r be the set of agents in the rth replica. Then $U = A^r \setminus (S \cup T)$ is certainly nonempty. Consider an arbitrary coalition V which is the union of U and of some nonempty subset T' of T.

Consider an arbitrary collection of net trades z_a', $a \in V$ such that $z_a' \in \gamma_a^h$ and $z_a' \succsim_a z_a^*$ for all a in V (with the convention $z_a^* = z_i^*$ for all $a = (i, q)$ in V). We have

$$\sum_{a \in V} z_{ah}' \geq \sum_{a \in V} z_{ah}^* \geq \sum_{a \in U} z_{ah}^*.$$

But $\sum_{a \in U} z_a^* + \sum_{a \in S \cup T} z_a^* = 0$, and the first part of (5.1) means that $\sum_{a \in S \cup T} z_h^* < 0$. Therefore, $\sum_{a \in V} z_{ah}' > 0$. This means that S blocks, a contradiction to the assumption that the disequilibrium allocation is stable for all r. Therefore, the disequilibrium allocation is a K-allocation. Q.E.D.

Remark that the foregoing proof shows that a disequilibrium allocation which is stable for infinitely many r is a K-allocation. Also, it is obviously possible to choose the blocking coalition S in such a way that it contains at most δr members of each type, where $0 < \delta \leq 1$.

THEOREM 2. *Given* $0 < \delta < 1$ *and a K-allocation, there exists* $\bar{r}$ *such that it is δ-stable for all* $r \geq \bar{r}$.

Proof. Let $0 < \delta < 1$, and consider a K-allocation $(z_i^*, \underline{z}_i, \bar{z}_i)$. For every $h \neq 0$, let S_h^* be the set of types i of consumers who are actually constrained in commodity h. If S_h^* is empty, let $\bar{r}_h = 1$. If S_h^* is not empty, choose for each i in S^* an arbitrary vector $\tilde{\zeta}_i^h$ in γ_i^h such that $\tilde{\zeta}_i^h \succ_i z_i^*$, and let $\tilde{z}_{ih}$ be the hth component of $\tilde{\zeta}_i^h$. Define $B_h = |\sum_{i \in S_h^*} (\tilde{z}_{ih} - z_{ih}^*)| > 0$. Let A_h be the maximum of $|z_{ih}^*|$ over all i such that $i \notin S_h^*$ whenever there are such i, and 0 otherwise. Then define $\bar{r}_h$ as the smallest integer greater than or equal to $A_h/(B_h(1 - \delta))$. Let $\bar{r}$ be the maximum of all $\bar{r}_h$, $h \neq 0$. We claim that the K-allocation is δ-stable for all $r \geq \bar{r}$.

Choose such an r, and let A^r be the set of all consumers in the rth replica. Consider an $h \neq 0$, let S be a nonempty subset of A^r which contains at most δr consumers of each type, and let T be another (possibly empty) subset of A^r which is disjoint from S. Assume that there exist $z_a \in \gamma_i^h$ for every $a = (i, q)$ in $S \cup T$ such that $\sum_{a \in S \cup T} z_{ah} = 0$, $z_a \succ_i z_i^*$ for all $a = (i, q)$ in S and $z_a \sim_i z_i^*$ for all $a = (i, q)$ in T. Since a K-allocation is efficient market by market, the complementary coalition $U = A^r \backslash (S \cup T)$ is nonempty. We wish to show that U can make an equivalent or better proposal to a nonempty subset T' of T.

Since h is fixed, we write from now on S^* for S_h^*. First it is clear that $i \in S^*$ for every member $a = (i, q)$ of S. For every i in S^*, let s_i be the number of consumers of type i who belong to S. We have $0 \leq s_i \leq \delta r$. To fix the ideas, assume that $z'_{ih} > z_{ih}^* \geq 0$ for all i in S^* and all $z'_i \in \gamma_i^h$ with $z'_i \succ_i z_i^*$ (the other case where $0 \geq z_{ih}^* > z'_{ih}$ for all such i and z'_i can be treated in a similar way). Let us partition T into $T_1 = \{a = (i, q) \in T \mid i \in S^*\}$ and $T_2 = \{a = (i, q) \in T \mid i \notin S^*\}$. We claim that U can make an equivalent or better proposal to a subset T' of T which is the union of T_1 and of a nonempty subset T'_2 of T_2.

Indeed, let us suppose for a moment that we have found a nonempty subset T'_2 of T_2 such that, if $T' = T_1 \cup T'_2$ and $V = U \cup T'$,

$$\sum_{a \in V} z_{ah}^* \leq 0 \leq \sum_{a \in V} z_{ah}^* + \sum_{i \in S^*} (r - s_i)(\tilde{z}_{ih} - z_{ih}^*) \tag{5.2}$$

with the convention $z_a^* = z_i^*$ whenever $a = (i, q)$. Then, there exists λ in [0, 1] such that

$$\sum_{a \in V} z_{ah}^* + \lambda \sum_{i \in S^*} (r - s_i)(\tilde{z}_{ih} - z_{ih}^*) = 0. \tag{5.3}$$

But define z'_a for every a in V as follows. For all $a = (i, q)$ in V, let $z'_a = z_i^*$

whenever $i \notin S^*$, and $z'_a = (1-\lambda) z_i^\dagger + \lambda \xi_i^h$ whenever $i \in S^*$. We have $z'_a \succsim_i z_i^\dagger$ for all $a = (i, q)$ in V, and (5.3) is equivalent to $\sum_{a\in V} z'_{ah} = 0$. Therefore S could not block.

To complete the proof, we have to show that there exists a nonempty subset T'_2 of T_2 such that (5.2) holds. To simplify the notations, let $V_1 = U \cup T_1$. First, remark that $z^*_{ah} = 0$ for all a in S implies

$$\sum_{a\in V_1} z^*_{ah} + \sum_{a\in T_2} z^*_{ah} = \sum_{a\in A'} z^*_{ah} = 0.$$

On the other hand, $z^*_{ah} = 0$ for all a in T_1, $z_{ah} = z^\dagger_{ah}$ for all a in T and $z_{ah} = z^\dagger_{ah}$ for all a in S imply

$$\sum_{a\in V_1} z^\dagger_{ah} \geq \sum_{a\in S\cup T} z^\dagger_{ah} = \sum_{a\in S\cup U} z_{ah} > 0.$$

We have therefore shown that

$$\sum_{a\in V_1} z^*_{ah} + \sum_{a\in T_2} z^*_{ah} = 0 < \sum_{a\in V_1} z^\dagger_{ah} \tag{5.4}$$

which proves in particular that $\sum_{a\in T_2} z^*_{ah} < 0$.

The argument is virtually complete now. From (5.4), there must exist a nonempty subset T'_2 of T_2 and an agent $\alpha \in T'_2$ with $z^*_{\alpha h} < 0$ such that, if $V = V_1 \cup T'_2$,

$$\sum_{a\in V} z^\dagger_{ah} \geq 0 < \left(\sum_{a\in V} z^*_{ah}\right) - z^*_{\alpha h}.$$

But from the choice of r, and the fact that $s_i \leq \delta r$ for all $i \in S^*$, we have

$$|z^*_{\alpha h}| \leq A_h \leqslant r(1-\delta) B_h \leqslant \sum_{i\in S^*} (r - s_i)(\hat{z}_{ih} - z^*_{ih})$$

which yields (5.2), as announced. The proof is complete. Q.E.D.

Remark. Consider an economy with a continuum of agents, as in [2]. Let the set of consumers A be the unit interval $[0, 1]$. The class $\mathfrak{a}$ of possible coalitions is represented by the Borel subsets of A. If μ is the Lebesgue measure on A, then for every $S \in \mathfrak{a}$, $\mu(S)$ is the proportion of consumers who belong to S. Assume that every agent a in A has preferences $\succsim_a$ defined on a set of net trades Z_a which satisfy assumptions (1) and (2) of Section 1. In addition, assume that the set $\{(a, x, y) \mid x \succsim_a y\}$ is measurable.

All definitions can be easily reworded in this context. Then it is straightforward to translate the proofs of the propositions to get the following fact.

A disequilibrium allocation is an R-allocation if and only if it is weakly stable.

The nice thing is the fact that Theorems 1 and 2 become:

A disequilibrium allocation is K-allocation if and only if it is stable.

The "if" part is easy to prove by a simple rewording of the proof of Theorem 1. The proof of the fact that a K-allocation is stable goes along the same lines as that of Theorem 2, and is indeed much simpler. We sketch the argument. Suppose that there exist $h \neq 0$, a coalition S with $\mu(S) > 0$, another coalition T disjoint from S, and a measurable function $z(\cdot)$ such that $z(a) \in \gamma^h(a)$ a.e. in $S \cup T$, $\int_{S \cup T} z_h(a) = 0$, $z(a) \succ_a z^*(a)$ a.e. in S, and $z(a) \sim_a z^*(a)$ a.e. in T. One can focus the attention on the case where $z_h(a) \geq z_h^*(a) \geq 0$ a.e. in S. If $T_2 = \{a \in T \mid z_h^*(a) < 0\}$ and $U = A \setminus (S \cup T)$, one can show by the same reasoning used in the proof of Theorem 2 to prove (5.4) that

$$0 \leq \int_U z_h^*(a) \leq -\int_{T_2} z_h^*(a).$$

Now, since μ is atomless, there exists a Borel subset T' of T_2 such that

$$0 \leq \int_U z_h^*(a) = -\int_{T'} z_h^*(a).$$

Obviously $\mu(T') > 0$. It suffices to take $V = U \cup T'$ and $z'(a) = z^*(a)$ for all a in V to show that S cannot block.

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REFERENCES

1. K. J. ARROW, Towards a theory of price adjustment, *in* "The Allocation of Economic Resources" (A. Abramovitz, Ed.), Stanford Univ. Press, Stanford, Calif., 1959.
2. R. J. AUMANN, Markets with a continuum of traders, *Econometrica* **32** (1964), 39–50.
3. R. J. AUMANN AND M. MASCHLER, The bargaining set for cooperative games, *in* "Advances in Game Theory," (M. Dresher, L. S. Shapley, A. W. Tucker, Eds.), Annals of Mathematical Studies, Vol. 52, Princeton Univ. Press, Princeton, N. J., 1964.
4. R. J. BARRO AND H. I. GROSSMAN, A general disequilibrium model of income and employment, *Amer. Econ. Rev.* **61** (1971), 82–93.
5. R. J. BARRO AND H. I. GROSSMAN, Suppressed inflation and the supply multiplier, *Rev. Econ. Stud.* **41** (1974), 87–104.
6. R. J. BARRO AND H. I. GROSSMAN, "Money, Employment and Inflation," Cambridge Univ. Press, London, 1976.

7. J. P. BENASSY, Disequilibrium theory, unpublished Ph.D. thesis, CRMS Working Paper No. 185, University of California at Berkeley, 1973.
8. J. P. BENASSY, Neokeynesian disequilibrium theory in a monetary economy, *Rev. Econ. Stud.* **42** (1975), 503–523.
9. J. P. BENASSY, The disequilibrium approach to monopolistic price setting and general monopolistic equilibrium, *Rev. Econ. Stud.* **43** (1976), 69–81.
10. P. CHAMPSAUR, "Rationing and Lindhal Equilibria," ENSAE, Paris, 1977.
11. R. W. CLOWER, The Keynesian counterrevolution: A theoretical appraisal, *in* "The Theory of Interest Rates" (F. H. Hahn and F. P. Brechling, Eds.), Macmillan, New York, 1965.
12. G. DEBREU AND H. SCARF, A limit theorem on the core of an economy, *Int. Econ. Rev.* **4** (1963), 235–246.
13. J. DREZE, Existence of an exchange equilibrium under price rigidities, *Int. Econ. Rev.* **16** (1975), 301–320.
14. J. M. GRANDMONT AND G. LAROQUE, On Keynesian temporary equilibria, *Rev. Econ. Stud.* **43** (1976), 53–67.
15. H. I. GROSSMAN, Money, interest and prices in market disequilibrium, *J. Polit. Econ.* **79** (1971), 943–961.
16. H. I. GROSSMAN, A choice-theoretic model of an income investment accelerator, *Amer. Econ. Rev.* **62** (1972), 630–641.
17. F. H. HAHN, On non Walrasian equilibria, *Rev. Econ. Stud.* **45** (1978), 1–17.
18. F. H. HAHN AND T. NEGISHI, A theorem on non tatonnement stability, *Econometrica* **30** (1962), 463–469.
19. J. HICKS, "Value and Capital," Oxford Univ. Press, London, 1939.
20. A. LEIJONHUFVUD, "On Keynesian Economics and the Economics of Keynes," Oxford Univ. Press, London, 1968.
21. E. MALINVAUD, "The Theory of Unemployment Reconsidered," Blackwell, Oxford, 1977.
22. E. MALINVAUD AND Y. YOUNES, "Une Nouvelle Formulation Générale pour l'Étude des Fondements Microéconomiques de la Macroéconomie," INSEE and CEPREMAP, Paris, 1974.
23. E. MALINVAUD AND Y. YOUNES, A new formulation for the microeconomic foundations of macroeconomics, Paper presented at an IEA Conference on the Microeconomic Foundations of Macroeconomics, S'Agaro, Spain, July 1975.
24. T. NEGISHI, Existence of an Underemployment Equilibrium *in* "Equilibrium and Disequilibrium in Economic Theory" (G. Schwödiauer, Ed.), proceedings of a conference held in Vienna, Reidel, Boston, 1978.
25. D. PATINKIN, "Money, Interest and Prices," 2nd ed., Harper & Row, New York, 1965.
26. R. PORTES, Macroeconomic equilibrium and disequilibrium in centrally planned economies, D. P. No. 45, Birbeck College, London, 1976.
27. R. M. SOLOW AND J. E. STIGLITZ, Output, employment and wages in the short run, *Quart. J. Econ.* (1968).
28. Y. YOUNES, "Sur une Notion d'Équilibre Utilisable dans le Cas où les Agents Économiques ne sont pas Assurés de la Compatibilité de leurs Plans," CEPREMAP, Paris, 1970.
29. Y. YOUNES, "Sur les Notions d'Équilibre et de Déséquilibre Utilisées dans les Modèles Décrivant l'Évolution d'une Économie Capitaliste," CEPREMAP, Paris, 1970.
30. Y. YOUNES, On the role of money in the process of exchange and the existence of a non Walrasian equilibrium, *Rev. Econ. Stud.* **42** (1975), 489–501.

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The Le Chatelier Principle in Optimal Control Problems*

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The Samuelson-Le Chatelier Principle [11] states the following: let an agent's actions (demands say) be determined as solutions of a static optimization problem. Fixing some demands consider the response of the remaining variable demands to changes in "conjugate" or "dual" parameters. Then the fewer the number of auxiliary constraints inposed on the agent, the larger the absolute response of the variable demands, i.e., the more constrained demand functions are less elastic.

This proposition is often given a dynamic flavoring and is interpreted as stating that long-run elasticities are greater (in absolute terms) than short-run elasticities. Though this interpretation has rarely been examined in a truly dynamic setting, when it has, the interpretation has not always been found to be warranted. (See [6], [13], and [7]).

The objective of this paper is to describe an alternative extension of the static Principle to dynamic behavior. Specifically, we adopt Samuelson's framework except that the static optimization problem is replaced by a problem of optimal intertemporal planning. The solutions of a constrained and unconstrained optimal control problem depend on a vector of exogenous parameters and we compare the responses of each solution to changes in parameters. Two classes of responses are investigated. In I we derive some pseudo-static Le Chatelier results that pertain to behavior at a point in time or to cumulative or average behavior over the planning horizon. A less general but more truly dynamic analysis is carried out in II where the sensitivities of the entire planning profiles of the constrained and unconstrained control problems are compared. The profile of the unconstrained problem is shown to be more sensitive to changes in some parameters.

The Le Chatelier results derived in II are not universally valid; they are established only for certain types of constraints and parameters the specification of which is a major contribution of the analysis. As is the case with the static Principle, our results are local; the constraints imposed are assumed

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to be ineffective at the initial parameter values. However, the analysis is not confined to pairs of "conjugate" variables and parameters. The scope of the analysis is illustrated in III where it is applied to two standard economic problems the investigations of the process of investment by a firm and the optimal path of capital accumulation for an economy.

The paper concludes in IV with a brief discussion of possible extensions of the analysis. Some technical details are treated in appendices.

The method of analysis in II is comparative dynamics, (see [7], [8] and [9]), which may be described briefly as follows: given a differential equation system $\dot{z} = h(z, t, \theta)$ with specified boundary conditions, write its solution as $z(t; \theta)$, θ being a vector of parameters.[1] Substituting the solution into the differential equation yields the identity $\dot{z}(t; \theta) = h(z(t; \theta), t, \theta)$. If the function h is sufficiently smooth (see [10]), we can differentiate the identity to obtain the following variational differential system:

$$\dot{z}_\theta(t; \theta) = h_z(z(t; \theta), t, \theta)\, z_\theta(t; \theta) + h_\theta(z(t; \theta), t, \theta).$$

The solution $z_\theta(t; \theta)$ to the system composed of these equations and the associated boundary conditions describes the effect on the path $z(t; \theta)$ of a change in θ. $z_\theta(0, \theta)$ is called the impact effect. After imposing sufficient regularity conditions on the optimal control problems, we apply this method repeatedly below to the corresponding systems of differential equations defined by Pontryagin's optimum conditions (composed of the Maximum principle, the canonical differential equations and boundary conditions).

I

We deal with control problems that have finite and fixed time horizons and fixed endpoints.

Consider the following "unconstrained" problem:

$$\max_{u,v} \int_0^T F(x(t), u(t), v(t), \theta)\, e^{-rt}\, dt, \tag{1}$$

$$\text{subject to } \dot{x}(t) = f(x(t), u(t), v(t), \theta)$$
$$(u(t), v(t), x(t)) \in C$$
$$x(0) = x^0,\ x(T) = x^T.$$

---

[1] The following notation is adopted throughout. Let $h(x, t) = (h^1(x, t), \dots, h^m(x, t))$ be a vector valued function of time t and the real vector $x = (z, y)$, $z = (z_1, \dots, z_n)$, $y = (y_1, \dots, y_k)$. Then $h_t = \partial h^i/\partial t$, $h^i_{x_j} = \partial h^i/\partial x_j$, $h^i_{x_j x_l} = \partial^2 h^i/\partial x_j \partial x_l$, $h^i_{t x_j} = \partial^2 h^i/\partial t \partial x_j$, h_x^i is the vector with components $h^i_{x_j}$, h^i_{zy} is the $(n \times k)$ matrix whose $(j, l)^{th}$ element is $h^i_{z_j y_l}$, $h = (h^1, \dots, h^m)$ h_x is the $m \times (n + k)$ matrix with rows h_x^i and so on. All vectors are column vectors unless transposed to row vectors by a superscript τ. Finally, given any two square matrices A and B of the same dimension, $A \geq B$ if and only if $A - B$ is positive semidefinite.

x represents the vector of state variables (called capital stocks for concreteness), u and v are each vectors of controls, r is a rate of discount, T is the length of the planning horizon, x^0 and x^T are the (nonnegative) initial and terminal capital stocks and C is a convex constraint set. θ represents an exogenous parameter. Among the parameters considered are r, x^0, x^T and T. When focussing on any one of these, the rate of discount for example, we identify θ with r but it is understood that F and f do not depend on θ. For other parameters θ, of course, the dependence of the control problem and its solutions on θ is due to the fact that F and f depend on θ. F and f are scalar and vector valued functions respectively, assumed to be twice continuously differentiable. We will generally also require that they each be (strictly) concave in (x, u, v). All solutions to (1) and to (1′) below are assumed to lie in the interior of C so that C is ignored in the sequel. Note that time enters the control problem only through the exponential discounting. This specification is adopted purely in order to simplify the notation.

We assume that for each θ in an open set S there exists a solution to (1), which because of the strict concavity of F and f, is unique. The solution may be described in the following way: Let λ denote the current value shadow price vector corresponding to x and define the current value Hamiltonian

$$H(x, u, v, \lambda, \theta) = F(x, u, v, \theta) + \lambda^T f(x, u, v, \theta). \tag{2}$$

The maximized Hamiltonian is the function M,

$$M(x, \lambda, \theta) = \max_{u,v} H(x, u, v, \lambda, \theta). \tag{3}$$

By the Maximum Principle, the unique optimal controls for (1) are given by the solutions $u(x, \lambda, \theta)$ and $v(x, \lambda, \theta)$ of this maximization problem. Correspondingly, there exist unique optimal time profiles for the capital stocks and shadow prices. For each θ we denote them by $x(t; \theta)$ and $\lambda(t; \theta)$ respectively. It is well-known that they are the unique (by the concavity of F and f and the convexity of C), solutions of the following system:

$$\begin{aligned} \dot{\lambda} &= r\lambda - M_x(x, \lambda, \theta) \\ \dot{x} &= M_\lambda(x, \lambda, \theta) \\ x(0; \theta) &= x^0, \; x(T; \theta) = x^T. \end{aligned} \tag{4}$$

Let $\theta_0 \in S$ be a specified value of θ at which we will calculate the derivatives with respect to θ of the optimal plans, and let $\tilde{v}(t; \theta_0) = v(x(t; \theta_0), \lambda(t; \theta_0), \theta_0$. We define the following "constrained" planning problem corresponding to (1):

$$\max_u \int_0^T F(x(t), u(t), \tilde{v}(t; \theta_0), \theta)\, e^{-rt}\, dt \tag{1′}$$

subject to
$$\dot{x}(t) = f(x(t), u(t), \hat{v}(t; \theta_0), \theta),$$
$$x(0) = x^0, \; x(T) = x^T.$$

In (1′) the control v is no longer a choice variable for the planner. Its time profile is prescribed at the level that is optimal for (1) when $\theta = \theta_0$. We assume that for each θ in S (1′) has a solution, which is necessarily unique.

We adopt the obvious notation and define

$$\begin{aligned} H'(x, u, \lambda, t, \theta) &= F(x, u, \hat{v}(t; \theta_0), \theta) + \lambda^T f(x, u, \hat{v}(t; \theta_0), \theta) \\ &= H(x, u, \hat{v}(t; \theta_0), \lambda, \theta), \end{aligned} \tag{5}$$

and

$$M'(x, \lambda, t, \theta) = \max_u H'(x, u, \lambda, t, \theta). \tag{6}$$

The (assumed) unique solution to the maximization problem in (6) is denoted $u'(x, \lambda, t, \theta)$. The optimal profiles $x'(t; \theta)$ and $\lambda'(t; \theta)$ corresponding to (1′) are the unique solutions of the following system:

$$\begin{aligned} \dot{\lambda}' &= r\lambda' - M'_x(x', \lambda', t, \theta) \\ \dot{x}' &= M_\lambda(x', \lambda', t, \theta) \\ x'(0; \theta) &= x^0, \; x'(T; \theta) = x^T. \end{aligned} \tag{7}$$

Our objective is to compare the response to changes in θ of a planner solving (1) with that of a planner solving (1′). The pseudo-static results of this section are based on the following Lemma:

LEMMA. *Denote by $J(\theta)$ and $J'(\theta)$ the maximum value of the objective function in* (1) *and* (1′) *respectively. Then $J_{\theta\theta}(\theta_0) - J'_{\theta\theta}(\theta_0)$ is positive semidefinite.*

Proof[2]. From the definition of (1′), $J(\theta) - J'(\theta) \geq 0$ for all θ, with equality when $\theta = \theta_0$. By the regularity conditions we have assumed, $J - J'$ is twice continuously differentiable. (See Appendix A). The asserted positive semidefiniteness now follows as a necessary condition for the existence of a local minimum for $J - J'$ at θ_0. ▮

The relevance of the Lemma to the Le Chatelier Principle is demonstrated by taking $\theta = x^0$, $\theta_0 = x_0^0$. It is well-known that $J(x^0)$ and $J'(x^0)$ are each concave functions and that

$$J_{x^0}(x^0) = \lambda(0; x^0), \; J'_{x^0}(x^0) = \lambda'(0, x^0). \tag{8}$$

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[2] The same method of proof was applied in [3] to variable profit functions to establish the Le Chatelier Principle in a static model of producer behaviour.

Therefore, the Lemma implies that

$$\lambda'_{x^0}(0; x_0^0) \leq \lambda_{x^0}(0; x_0^0) \leq 0. \tag{9}$$

The interpretation of these inequalities is simpler when x and λ are scalars. In that case $\lambda(0; x^0)$ and $\lambda'(0; x^0)$ can be viewed as inverse demand functions for initial stock, which intersect when $x^0 = x_0^0$. Coupled with the inelastic initial supply of capital x_0^0, they "determine the initial equilibrium market price" $\lambda(0; x_0^0) = \lambda'(0; x_0^0)$. Consider now the effects of a (marginal) increase in supply described in (9). An increase in the supply of capital induces the price to fall given either the unconstrained or the constrained demand. But the price drop is greater in the latter case because the constrained demand function is less price elastic. When x is a vector, an increase in the supply of any single capital good induces a greater reduction in the corresponding price in the constrained problem. However, off-diagonal elements of the matrix $\lambda'_{x^0}(0; x_0^0) - \lambda_{x^0}(0; x_0^0)$ are restricted only by the asserted negative semi-definiteness.

The corresponding application of the Lemma to the parameter $\theta = x^T$ is clear and is omitted. When $\theta = T$, $\theta_0 = T_0$, we make use of the relations

$$J_T(T) = e^{-rT}M(T), J'_T(T) = e^{-rT}M'(T),$$

where $M(T)$ and $M'(T)$ are the values at $t = T$ of M and M' respectively along optimal paths. Since $M(T_0) = M'(T_0)$, we deduce that

$$M_T(T_0) \geq M'_T(T_0), \tag{10}$$

illustrating another sense in which the unconstrained problem exhibits greater flexibility. The Hamiltonian may be interpreted as the total rate of acquisition of benefits, both current (as measured by F) and future (as measured by $\lambda^T\dot{x} = \lambda^T f$). The inequality (10) states that along an optimal path the value of the Hamiltonian at the endpoint of the program undergoes a larger algebraic change in the unconstrained program.

The absolute sizes of the responses cannot be compared in general. However, suppose that T_0 is actually the optimal planning horizon for the unconstrained planner. That is, suppose that T is a choice variable in (1) so that the planner solves $\max_{T \geq 0} J(T)$. If $T_0 > 0$ is optimal, the necessary first and second order conditions for a local maximum imply that $J_T(T_0) = 0$ and $J_{TT}(T_0) \leq 0$. Moreover, it is easy to see that if T_0 is optimal for (1) it is also optimal for (1′) so that $J'_T(T_0) = 0$ and $J'_{TT}(T_0) \leq 0$. We conclude that $0 \geq M_T(T_0) \geq M'_T(T_0)$. The terminal value of the Hamiltonian is depressed in both planning problems by an increase in T, but more so when the flexibility of the planner in reacting to the change is restricted.

In spite of the above, the limitations of the Lemma as a basis for deriving Le Chatelier results should be evident. First, for many parameters θ, J_θ and J'_θ and hence the assertion of the Lemma are not very interesting economically. (A further interesting application of the Lemma is provided in section III, however). Second, when θ is a vector the Lemma places clear qualitative restrictions only on the diagonal elements of the matrix $J_{\theta\theta} - J'_{\theta\theta}$. Thus, as is the case with the Samuelson-Le Chatelier Principle, applications of the Lemma are confined to conjugate or dual variables and parameters where, of course, conjugacy refers to a relationship such as that described in (8) between x^0 and the initial shadow price. Finally and most importantly given our framework of intertemporal optimization, the Lemma is relevant only to impact effects or to cumulative behavior over time, (see III). Therefore, in the next section we consider the flexibility of the entire planning profiles of the unconstrained and constrained optimal programs and we extend the analysis to include nonconjugate variables and parameters.

II

The extensions described in this section require some additional assumptions. First, we assume that x, f and λ are scalars. The comparative dynamic analysis of multistock models is much more complicated and the qualitative results much fewer. (See [7]). Second, we generally require that the optimization problem (1) satisfy one of the following sets of separability conditions:

(A) $\qquad F_v = 0,\ f_{xv} = f_{v\theta} = 0$, or

(B) $\qquad f_v = 0,\ F_{xv} = F_{v\theta} = 0$.

Though these would appear to be severe restrictions, many of the standard models of dynamic optimization dealt with in economics exhibit such separability. (See the examples below). It is easy to show that these conditions imply that the optimal control $v(x, \lambda, \theta)$ that solves (3) is independent of λ and θ. Thus the Le Chatelier results below will be largely (but not completely) confined to optimization problems that are constrained by prescribing the time profile of a control that is optimally dependent only on the state variable.

Finally, we will occasionally require that the stock x be "desirable" in the sense that its shadow price $\lambda(t; \theta)$ be nonnegative. The arguments and results are readily modified to account for a "bad" stock (such as a stock of pollutants).

Our purpose is to compare $x_\theta(t; \theta_0)$ with $x'_\theta(t; \theta_0)$ and $\lambda_\theta(t; \theta_0)$ with $\lambda'_\theta(t; \theta_0)$. This is accomplished in the following theorem by analysing the variational systems corresponding to (4) and (7).

THEOREM. *Let θ be a scalar parameter and suppose that* (A) *or* (B) *holds.*

(a) *If* $x_\theta(t;\theta_0) \geq 0$ *for all* t, *then* $x_\theta(t;\theta_0) \geq x'_\theta(t;\theta_0)$ *for all* t, $\lambda'_\theta(0;\theta_0) \geq \lambda_\theta(0;\theta_0)$ *and* $\lambda'_\theta(T;\theta_0) \geq \lambda_\theta(T;\theta_0)$.

(b) *If* $x_\theta(t;\theta_0) \leq 0$ *for all* t, *then* $x_\theta(t;\theta_0) \leq x'_\theta(t;\theta_0)$ *for all* t, $\lambda'_\theta(0;\theta_0) \leq \lambda_\theta(0;\theta_0)$ *and* $\lambda'_\theta(T;\theta_0) \leq \lambda_\theta(T;\theta_0)$.

The Theorem is more readily interpreted in the context of the specific parameters discussed in the following corollary:

COROLLARY. *Assume either* $(A')\ F_r = 0$ *and* $f_{ur} = 0$, *or* $(B')\ f_r = 0$ *and* $F_{ur} = 0$.

(a) *Let* $\theta = -x^0$, $\theta_0 = -x_0^0$. *Then* $x_{x^0}(t;x_0^0) \leq x'_{x^0}(t;x_0^0) \leq 0$ *for all* t, $\lambda'_{x^0}(0;x_0^0) \geq \lambda_{x^0}(0;x_0^0) \geq 0$ *and* $0 \geq \lambda'_{x^0}(T;x_0^0) \geq \lambda_{x^0}(T;x_0^0)$.

(b) *Let* $\theta = r$, $\theta_0 = r_0$. *Then* $x_r(t;r_0) \leq x'_r(t;r_0) \leq 0$ *for all* t, $\lambda_r(0;r_0) \geq \lambda'_r(0;r_0) \geq 0$ *and* $\lambda_r(T;r_0) \leq \lambda'_r(T;r_0) \leq 0$.

Part of the Corollary, applied to small finite changes in x^0 and r, is illustrated in Figure 1. In both (a) and (b) a larger absolute response is induced in the entire capital profile of the unconstrained as opposed to the constrained

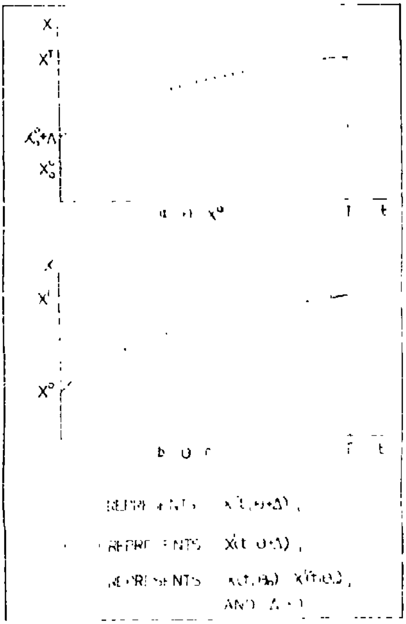

FIG. 1. Shifts of the constrained and unconstrained capital profiles.

planning problem. This is a perfect analogue of the static Le Chatelier Principle; the constrained profile $x'(t; \theta)$ is less elastic than the unconstrained profile $x(t; \theta)$. Note, however, that the relative magnitudes of λ_θ and λ'_θ are determinate only at $t = 0$ and $t = T$. A uniform sign for $\lambda'_\theta - \lambda_\theta$ would apparently be inconsistent with the common initial and terminal stocks in the constrained and unconstrained problems.

The interpretation of the shadow price effects described in the Corollary is analogous to our earlier interpretation of (9). (Indeed the impact effect in (a) is a special case of (9)). For example, view $\lambda(0; r)$ and $\lambda'(0; r)$ as inverse demand functions for initial capital stock (evaluated at the fixed level of initial stock x^0 and) parametrized by r. An increase in r causes a larger downward shift of the unconstrained demand curve, thereby inducing a larger price drop in the unconstrained problem. The shadow price changes at $t = T$ may be similarly interpreted in terms of the "market" for terminal capital stock. The "demand" for terminal capital is reduced by an increase in x^0 and increased by a rise in r, and in both cases the unconstrained demand curve undergoes a larger shift. Given the inelastic supply of terminal capital x^T, the equilibrium prices undergo the changes indicated in the Corollary.

Corresponding results may also be derived with respect to changes in the level of terminal stock, but not quite for arbitrary parameters θ consistent with (A) or (B) and $x_\theta(t; \theta_0) \geq 0$, for example. In the latter case, it does not appear to follow that $x'_\theta(t; \theta_0) \geq 0$ for all t; the constrained and unconstrained profiles *do not necessarily* undergo the same qualitative shifts.[3] Our Le Chatelier result thus states that the unconstrained path experiences a larger algebraic, but not necessarily absolute, shift. Note, however, that the systems (11) and (12) below have identical qualitative properties. Therefore, if the uniform nonnegativity of $x_\theta(t; \theta_0)$ may be deduced from the qualitative properties of (11), it follows that $x'_\theta(t; \theta_0) \geq 0$ for all t. This is the case for the parameter x^0 so that the stronger result stated in the Corollary could be established. Similar comments apply to the shadow price effects in the Theorem and to part (b).

There is a difficulty in defining $x_T(t; T)$ since $x(t; T)$ is defined only for $t \leq T$. However, as described by Nagatani [8], a change in T may be translated into an "equivalent" qualitative change in x^T. A marginal increase in T from T_0 would correspond to an increase in x^T if $\dot{x}(T_0; T_0)$ is negative and to a decrease in x^T if $\dot{x}(T_0; T_0)$ is positive. In this way profile Le Chatelier results may be derived for changes in T.

It is not immediately apparent in most specific optimization models which parameters θ, other than x^0, r and x^T, imply that x_θ is uniformly signed through time. A class of such parameters is described in the following proposition:

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[3] In the Samuelson-Le Chatelier Principle there are only quantitative differences between the reactions of the constrained and unconstrained systems. However, qualitative differences have been found in the recent extension of the Principle by Kusumoto [4]. Note that in the Theorem if we assume the uniform signing of x'_θ rather that x_0, qualitative differences vanish, i.e., we can prove that (a) $x_\theta \geqslant x'_\theta \geqslant 0$, or (b) $x_\theta \leqslant x'_\theta \leqslant 0$, for all t.

PROPOSITION. *Suppose that* $\theta \neq x^0, x^T$, *and that at* $\theta = \theta_0$, $\lambda\, dr/d\theta - M_{x\theta}$ *is uniformly nonnegative (nonpositive) and* $M_{\lambda\theta}$ *vanishes identically in* x *and* λ. *Then* $x_\theta(t; \theta_0) \geq (\leq)\ 0$ *for all* t, $\lambda_\theta(0; \theta_0) \geq (\leq)\ 0$ *and* $\lambda_\theta(T; \theta_0) \leq (\geq)\ 0$. *The corresponding inequalities for* x'_θ *and* λ'_θ *are also valid if* v *satisfies* (A′) *or* (B′).

The proof of the Proposition is similar to that of part (b) of the Corollary and is omitted. For all parameters θ and controls v satisfying the hypotheses of the Proposition, the constrained and unconstrained optimal plans undergo identical qualitative changes. Therefore, if θ is also consistent with (A) or (B) the analogue of the more complete Le Chatelier results described in the Corollary are valid for θ. The discount rate is an example of such a parameter. Other examples are provided in III below. First, however, we prove the Theorem and Corollary.

Proof of Theorem. $x(t; \theta)$ and $\lambda(t; \theta)$ are defined by (4). The corresponding variational system, which defines $x_\theta(t; \theta)$ and $\lambda_\theta(t; \theta)$, is

$$\begin{bmatrix} \dot\lambda_\theta \\ \dot x_\theta \end{bmatrix} = \begin{bmatrix} r - M_{\lambda x} & -M_{xx} \\ M_{\lambda\lambda} & M_{\lambda x} \end{bmatrix} \begin{bmatrix} \lambda_\theta \\ x_\theta \end{bmatrix} + \begin{bmatrix} \lambda \dfrac{dr}{d\theta} - M_{x\theta} \\ M_{\lambda\theta} \end{bmatrix}, \tag{11}$$

$$x_\theta(0; \theta) = \frac{dx^0}{d\theta}, \quad x_\theta(T; \theta) = \frac{dx^T}{d\theta}.$$

Similarly the variational system corresponding to (7) is

$$\begin{bmatrix} \dot\lambda'_\theta \\ \dot x'_\theta \end{bmatrix} = \begin{bmatrix} r - M'_{\lambda x} & -M'_{xx} \\ M'_{\lambda\lambda} & M'_{\lambda x} \end{bmatrix} \begin{bmatrix} \lambda'_\theta \\ x'_\theta \end{bmatrix} + \begin{bmatrix} \lambda' \dfrac{dr}{d\theta} - M'_{x\theta} \\ M'_{\lambda\theta} \end{bmatrix}, \tag{12}$$

$$x'_\theta(0; \theta) = \frac{dx^0}{d\theta}, \quad x'_\theta(T; \theta) = \frac{dx^T}{d\theta}.$$

When the systems are evaluated at $\theta = \theta_0$ they are almost identical. Recall the notation introduced in I and that

$$\begin{aligned} M(x, \lambda, \theta) &= H(x, u(x, \lambda, \theta), v(x), \lambda, \theta) \\ M'(x, \lambda, t, \theta) &= H(x, u'(x, \lambda, t, \theta), \bar v(t; \theta_0), \lambda, \theta). \end{aligned} \tag{13}$$

Because of the strong separability of u and v given either (A) or (B), $u(x, \lambda, \theta)$

$u'(x, \lambda, \theta)$ for all x, λ, t and θ. Differentiation of (13) shows that $M_{x\lambda} = H_{ux}^T u_\lambda + H_{x\lambda} + u_x^T[H_{uu}u_\lambda + H_{u\lambda}] + r_x^T[H_{ru}u_\lambda + H_{r\lambda}]$. Evaluate the derivatives at θ_0. Making use of (A) or (B) we see that $H_{ru} = 0$. Given (A), $H_{r\lambda} = f_r = 0$ because the latter equality is precisely the first order condition that defines $r(x)$. Given (B), $H_{r\lambda} = 0$ trivially. A similar differentiation of M' now shows that at θ_0, $M_{x\lambda} = M'_{x\lambda}$. Similarly it can be shown that $M_{\lambda\lambda} = M'_{\lambda\lambda}$, $M_{\lambda\theta} = M'_{\lambda\theta}$ when $\theta = \theta_0$. $M_{x\theta} - M'_{x\theta} = H_{r\theta}^T r_x = 0$ when $\theta = \theta_0$ because of the strong separability of r and θ. Finally, $M_{xx} - M'_{xx} = r_x^T H_{rr} r_x + 2H_{xr} r_x$. Since $r(x)$ is defined by $H_r = 0$, $r_x = -(H_{rr})^{-1} H_{rx}$. Therefore, $M_{xx} - M'_{xx} = -H_{xr}(H_{rr})^{-1} H_{rx} \geq 0$ by the concavity of H in r.

Two further preliminary results are required: (i) For all x and θ, M is convex in λ. (From (5) and (6) M is the maximum of a family of functions each of which is convex in λ, and hence M is also convex in λ). (ii) For all λ and θ, M is concave in x. (Let $0 < a < 1$ and suppose that $M(x_1, \lambda, \theta) = H(x_1, u_1, r_1, \lambda, \theta)$, $M(x_2, \lambda, \theta) = H(x_2, u_2, r_2, \lambda, \theta)$. Then $M(ax_1 + (1 - a)x_2, \lambda, \theta) \geq H(ax_1 + (1 - a)x_2, au_1 + (1 - a)u_2, ar_1 + (1 - a)r_2, \lambda, \theta) \geq aH(x_1, u_1, r_1, \lambda, \theta) + (1 - a)H(x_2, u_2, r_2, \lambda, \theta) = aM(x_1, \lambda, \theta) + (1 - a)M(x_2, \lambda, \theta)$, by the concavity of H in (x, u, r)). The corresponding statements with respect to M' are similarly true.

Now, subtracting (11) from (12) and evaluating the difference at θ_0, we obtain the following variational system with the indicated sign pattern:

$$\begin{bmatrix} \dot\lambda'_\theta - \dot\lambda_\theta \\ \dot x'_\theta - \dot x_\theta \end{bmatrix} = \begin{bmatrix} r - M_{\lambda x} & -M'_{xx} \\ (?) & (+) \\ M_{\lambda\lambda} & M_{\lambda x} \\ (+) & (?) \end{bmatrix} \begin{bmatrix} \lambda'_\theta - \lambda_\theta \\ x'_\theta - x_\theta \end{bmatrix} + \begin{bmatrix} 0 & M_{xx} - M'_{xx} \\ & (+) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_\theta \\ x_\theta \end{bmatrix}, \tag{14}$$

$$x'_\theta(0; \theta_0) - x_\theta(0; \theta_0) = 0, \quad x'_\theta(T; \theta_0) - x_\theta(T; \theta_0) = 0.$$

Suppose that $x_\theta(t; \theta_0) \geq 0$ for all t. The analysis of the motion described by (14) is facilitated by Figure 2 which indicates the directions of motion on the axes and the initial (z^0) and terminal (z^T) points. Both z^0 and z^T must lie on the vertical axis. It is clear from the diagram that if motion were to enter the first quadrant, it could never return to the vertical axis. Therefore, z^0 must lie on the lower half of the vertical axis ($\lambda'_\theta(0; \theta_0) - \lambda_\theta(0; \theta_0) \leq 0$) and z^T on the nonnegative half ($\lambda'_\theta(T; \theta_0) - \lambda_\theta(T; \theta_0) \geq 0$), as shown in the diagram. Motion is confined to quadrants 2 and 3 so that $x'_\theta - x_\theta$ is nonpositive throughout. Note, however, that the path may cross the nonnegative horizontal axis more than once.

A similar analysis applies to part (b). The directions of motion are indicated in Figure 3. Motion is confined to quadrants 1 and 4 so that $x'_\theta - x_\theta$ is nonnegative throughout. ▮

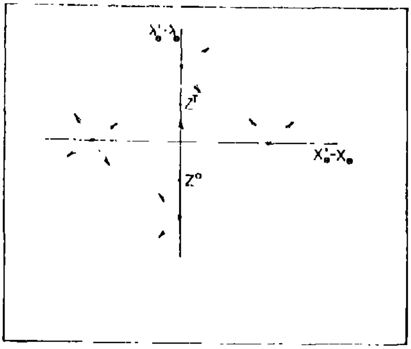

FIG. 2. Phase diagram of $(x'_\theta - x_\theta, \lambda'_\theta - \lambda_\theta)$ when $x_\theta(t:\theta_0) > 0$ for all t.

Proof of Corollary. (a) When $\theta = x^0$ the variational system (11) becomes

$$\begin{bmatrix}\dot\lambda_{x0}\\ \dot x_{x0}\end{bmatrix} = \begin{bmatrix} r - M_{\lambda x} & -\underset{(+)}{M_{xx}} \\ \underset{(+)}{M_{\lambda\lambda}} & M_{\lambda x} \end{bmatrix}\begin{bmatrix}\lambda_{x0}\\ x_{x0}\end{bmatrix}, \quad \begin{matrix} x_{x0}(0; x_0^0) = 1, \\ x_{x0}(T; x_0^0) = 0. \end{matrix} \tag{15}$$

The directions of motion on the axes are shown in Figure 4. Motion begins on the dashed vertical line and terminates on the vertical axis. Therefore, the path must begin at a point z^0 in quadrant 4 and terminate on the nonpositive vertical axis, i.e., $x_{x0} \geq 0$ and $\lambda_{x0} \leq 0$ for all t. Since (12) and (11) have identical qualitative properties, this argument can be repeated to prove that $x'_{x0} \geq 0$ and $\lambda'_{x0} \leq 0$ for all t. The remaining inequalities in (a) now follow from part (a) of the Theorem. (Note that when $\theta = x^0$, (A′) implies (A) and (B′) implies (B)).

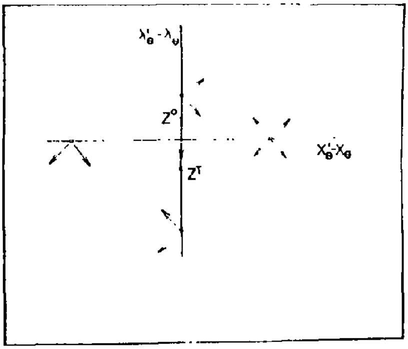

FIG. 3. Phase diagram of $(x'_\theta - x_\theta, \lambda'_\theta - \lambda_\theta)$ when $x_\theta(t; \theta_0) < 0$ for all t.

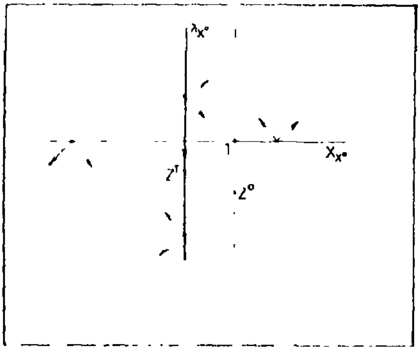

FIG. 4. Phase diagram of (x_{x0}, λ_{x0}).

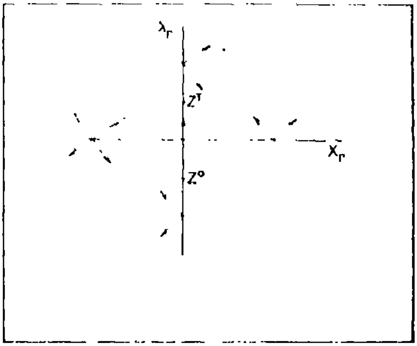

FIG. 5. Phase diagram of (x_r, λ_r).

(b) When $\theta = r$, (11) becomes

$$\begin{bmatrix} \dot{\lambda}_r \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} r - M_{\lambda r} & -M_{rx} \\ & (+) \\ M_{\lambda\lambda} & M_{\lambda x} \\ (+) & \end{bmatrix} \begin{bmatrix} \lambda_r \\ x_r \end{bmatrix} + \begin{bmatrix} \lambda \\ (+) \\ 0 \end{bmatrix}, \tag{16}$$

$$x_r(0; r_0) = x_r(T; r_0) = 0.$$

With the aid of Figure 5 we see that $x_r \leqslant 0$ for all t, $\lambda_r(0; r_0) < 0$ and $\lambda_r(T; r_0) \geqslant 0$. Similarly we can show that $x_r' \leqslant 0$ for all t, $\lambda_r'(0; r_0) < 0$ and $\lambda_r'(T; r_0) \geqslant 0$. The proof is completed by applying part (b) of the Theorem. (When $\theta = r$, (A') and (B') imply (A) and (B) respectively). ▌

Inspection of the proof of the Theorem reveals that the assumptions in (A) and (B) regarding the separability between v and θ are unnecessarily strong. In fact the statement of the Theorem remains valid if they are replaced in part (a) by "$M'_{x\theta} \quad M_{x\theta}$", and in (b) by "$M'_{r\theta} \quad M_{x\theta}$". The corresponding stronger version of the Theorem is applied in one of the examples below.

When r is a scalar, some insight into the sign of $M_{x\theta} - M'_{x\theta}$ is provided by the following observation: from the proof of the Theorem we see that $M_{x\theta} - M'_{x\theta} = H_{r\theta} r_x$. Since only the assumption of strong separability of r and θ has been deleted from (A) and (B), the optimal control r is a function $r(x, \theta)$, defined by the first order condition $H_r(x, u, r, \lambda, \theta) = 0$, where $H_{rr} < 0$. Therefore, r_θ has the sign of $H_{r\theta}$ and we conclude that $M_{x\theta} - M'_{x\theta}$ has the sign of $r_x r_\theta$. An assumption, such as the sign of $r_x r_\theta$, about the relationship between the constrained control and the parameter of interest is necessary apparently because of the analysis of nonconjugate variables and parameters. The discussion of the second example in III below suggests, and indeed it can be established more generally, that a similar assumption is also necessary in order to extend the (static) Samuelson-Le Chatelier Principle to pairs of nonconjugate variables and parameters.

III

In this section we apply our results to two standard problems of intertemporal optimization to illustrate the implications for the sensitivity of optimal programs of certain types of constraints imposed on the planner.

1. *Optimal Economic Growth*

We consider a variation of the standard neoclassical model of optimal growth. Suppose that the aggregate technology may be represented by $y = g(x \cdot s)$, where g is a strictly concave production function satisfying $g_x(0) = \infty$ and $g_x(\infty) = 0$, y is the flow of output of the single homogeneous good in the economy, x is the stock of physical capital and s is the rate of capital utilization. (The stock of labour is assumed constant. If it is variable but growing at a constant rate interpret y, x and c below in per capita terms and the same analysis applies). A higher rate of utilization, whether in the form of the length of time that the capital is operated or the intensity of operation per unit of time, increases the flow of output that can be produced with a given stock x but also increases the rate of depreciation of the capital stock, thereby reducing the potential for future production, i.e., we assume that capital depreciates at the geometric rate $\delta(s)$ where $\delta_s > 0$, $\delta_{ss} > 0$.

The unconstrained problem facing the planner is the following:[1]

$$\max_{c,s} \int_0^T e^{-\rho t} U(c)\,dt, \text{ subject to } \dot{x} = g(sx) - \delta(s)\,x - c,$$
$$x(t) \geq 0,\ c(t) \geq 0, \tag{17}$$
$$x(0) = x^0,\ x(T) = x^T,$$

---

[1] Calvo [2] analyses a similar problem.

where c is the level of consumption, $U(c)$ is a utility function satisfying $U_c > 0$, $U_{cc} < 0$ and $\lim_{c\to 0} U_c(c) = \infty$, and where r, T, x^0 and x^T are as before. We wish to compare the response to parameter changes of a planner that solves (17) with that of a planner that must determine the optimal consumption plan subject to an exogenously prescribed time profile for the rate of utilization of capital. Therefore, we apply the Corollary to (17) identifying u with c and v with s and noting that u and v are separable as in (A′).[5] We see that the planner that is free to vary the time path of capital utilization optimally will exhibit greater sensitivity to changes in $x^0(x^T)$ and r as described in the Corollary. Note that since the shadow price is equal to the marginal utility of consumption along an optimal program of the constrained or unconstrained problem, the shadow price effects indicated are readily translated in terms of changes in the optimal levels of consumption at $t = 0$ and $t = T$.

2. *The Demand for Investment by a Firm*

A firm produces a single output y by means of a concave production function $G(x, L)$, $(G_L(x, 0) = \infty, G_L(x, \infty) = 0)$, where x denotes the stock of capital and L the stock of labour. In both cases service flows are proportional to stocks. The prices p and w of y and L respectively are parametric to the firm, but the firm encounters costs of adjustment (see Treadway [12] and [13] for example) when it changes the level of capital stock. These costs are represented by the function $C(I) = qI + \frac{1}{2}aI^2$ where I is the level of gross investment and q and a are positive parameters. The imperfect variability of x is due precisely to the positivity of a.

Taking the maximization of the expected discounted value of profits as its objective the firm solves

$$\max_{I,L} \int_0^T e^{-rt}[pG(x, L) - wL - C(I)]\,dt,$$

subject to $\dot{x} = I - \delta x$,

$$\begin{aligned} &x(t) \geqslant 0,\ L(t) \geqslant 0, \\ &x(0) = x^0,\ x(T) = x^T, \end{aligned} \tag{18}$$

where r is the rate of interest prevailing in a competitive capital market, δ is a constant geometric rate of depreciation and T, x^0 and x^T are as above. For simplicity the firm's expectations regarding prices and costs of adjustment have been assumed to be static. Below we consider uniform shifts in these expectations though we could equally well have assumed time varying expectations and analysed nonuniform changes in expectations.

---

[5] The assumptions we have made are not sufficient to imply the concavity of the Hamiltonian in (x, c, s). However, we can establish the following weaker conditions which are nevertheless sufficient for the arguments in section II: (i) $M_{xx} < 0$, $M'_{xx} > 0$, and (iii) the Pontryagin optimum conditions are suflicient to define the solutions of (17) and the related constrained problem.

Identify L with r and I with u, noting that they satisfy (B′). We apply the preceding analysis to compare the optimal program of the firm that solves (18) with that of a firm that is locked into a long-term contract that fixes its employment of labour over the planning horizon.

First take p as a parameter (with initial value p_0) and apply the Lemma of section I. Denote by $y(t; p)$ and $y'(t; p)$ the optimal output programs for the unconstrained and constrained firms respectively. It is readily verified that $J_p = \int_0^T e^{-rt} y(t; p)\, dt$, $J'_p = \int_0^T e^{-rt} y'(t; p)\, dt$ and that $J(p)$ and $J'(p)$ are each convex in p. Therefore, we deduce that

$$\frac{d}{dp}\int_0^T e^{-rt} y(t; p_0)\, dt \geq \frac{d}{dp}\int_0^T e^{-rt} y'(t; p_0)\, dt \geq 0. \tag{19}$$

The discounted cumulative production of each firm is increased by a rise in the output price but less so for the firm that cannot adjust its employment of labour.

Profile effects with respect to changes in x^0, r (and x^T) are derived as above by applying the Corollary. The parameter q of the cost function is consistent with (B) and x_q and x'_q are nonpositive over time. Therefore, by part (b) of the Theorem, $x_q(t; q_0) \leq x'_q(t; q_0) \leq 0$ for all t. Similar arguments apply to the parameter a.

Finally, suppose that $G_{Lr} \leq 0$ and consider as parameters the prices of output and labour. Using the Proposition we can show that $x_p(t; p) \geq 0$ and $x_w(t; w) \leq 0$ for all t and similarly for the constrained profile. The parameters p and w do not satisfy the separability of (B) so that the Theorem cannot be applied. But it is readily verified (see Appendix B) that the hypotheses of the stronger version of the Theorem, described at the end of II, are equivalent to the property $G_{rL} \leq 0$. Therefore, we can conclude that

$$x_p(t; p_0) \geq x'_p(t; p_0) \geq 0,$$

and (20)

$$x_w(t; w_0) \leq x'_w(t; w_0) \leq 0 \text{ for all } t.$$

Using (20) it can be shown that

$$y_p(t; p_0) \geq y'_p(t; p_0) \geq 0 \text{ for all } t, \tag{21}$$

strengthening (19), at least for the above production functions. (However, it is a straightforward but tedious exercise to demonstrate that for general technologies the unconstrained firm does not exhibit uniformly greater flexibility through time as in (20) and (21).)

Note that the assumption $G_{L,x} \geq 0$ was made by Kusumoto [4] in his extension of the Samuelson-LeChatelier Principle. It can be shown that the Principle is not valid in general if $G_{L,x} < 0$. It can also be shown, though the proof is too lengthy to be include in this paper, that $G_{L,x} \geq 0$ is necessary for the unconstrained capital profile to be more price elastic, in general, in the sense of (20).

The shadow price effects associated with all of the above results may be translated into statements about changes in gross investment, because for both firms an optimal plan requires that the shadow price equal the marginal cost of investment.

IV

To conclude we comment briefly on some possible modifications and extensions of the results of II.

The assumption of a prescribed terminal stock x^t may be deleted. Suppose, for example, that x^t may be chosen by the planner subject to a nonnegativity constraint. It for each θ in a neighborhood of θ_0 it is the case that both the unconstrained and constrained planners choose positive levels of terminal stock, then $\lambda(T; \theta) = \lambda'(T; \theta) = 0$. Now replace the terminal condition for the variational system (14) by $\lambda_\theta(T; \theta_0) = \lambda'_\theta(T; \theta_0) = 0$ and use Figures 2 and 3 to prove the Theorem. The proof is only slightly more complicated when zero terminal stocks are optimal.

It is more difficult to extend the analysis to an infinite horizon planning problem.[6] Assuming (i) that the infinite horizon unconstrained problem possesses a solution that converges to a unique saddle point equilibrium, the equations of motion (7) that correspond to the constrained problem are of the form $\dot{z} = h(z, t, \theta)$, $z = (\lambda', x')$, where $\lim_{t \to \infty} h(z, t, \theta) = h^*(z, \theta)$, i.e., the system (7) is nonautonomous but it is asymptotically autonomous. The Theorem may be established for $T = \infty$ if we assume (i), (ii) the constrained problem has a solution, and (iii) that the solution converges to the steady state $z^*(\theta)$ defined by $h^*(z^*(\theta), \theta) = 0$. (The proof proceeds in two stages. First perform a comparative statics analysis of the steady state equilibria defined by (4) and (7) respectively and derive the steady state Le Chatelier results. Then use the latter to define the terminal conditions for the variational system (14)). However, even assuming (i), neither of (ii) or (iii) is true in general. For example, the validity of (iii) depends on the speed of convergence of $h(z, t, \theta)$ to $h^*(z, \theta)$. Further discussion is beyond the scope of this paper.

[6] Recent papers related to comparative dynamics in infinite horizon (and multistock) models include [1] and [5].

An extension in another direction is more immediate. Though we have constrained the planner by prescribing the time profiles of a subset of his controls, other forms of constraints are conceivable. For example, suppose that the unconstrained planner is free to choose the level of terminal stock and constrain the other planner by fixing terminal stock at the level that is optimal given θ_0. An analogue of the Theorem is readily established. Of course the separability conditions (A) and (B) are no longer relevant because even the constrained planner may vary all controls.

Finally, we mention two additional extensions that constitute subjects for future research. First it would be desirable to determine the extent to which the Theorem may be generalized to multistock control problems. Second, the regularity conditions we have imposed imply that the optimal controls are smooth functions of the state and costate variables. Since many of the planning problems that appear in economics involve controls that have a finite number of discontinuities or switch points, it would be of interest to extend the analysis to such problems. Such an extension would make use of the results of [9].

APPENDIX A

We wish to show that $J(\theta)$ and $J'(\theta)$ are twice continuously differentiable. Suppose $\theta \neq r, x^0, x^T, T$. Consider J, (the argument for J' is similar), and suppose for simplicity that x and λ are scalars. By definition, $J(\theta) = \int F(x(t;\theta), u(x(t;\theta), \lambda(t;\theta), \theta), v(x(t;\theta), \lambda(t;\theta), \theta), \theta)\, e^{-rt} dt$, where we have suppressed the limits integration. The twice continuous differentiability of F and f implies that $u(x, \lambda, \theta)$ and $v(x, \lambda, \theta)$ are continuously differentiable and that (by [8; pp. 170-177, 194, 198]) x_θ, λ_θ, $\dot{x}_\theta$ and $\dot{\lambda}_\theta$ exist and are continuous in (t, θ). By a standard theorem in basic calculus, J_θ exists and is given by

$$J_\theta(\theta) = \int \frac{d}{d\theta} F(x, u, v, \theta)\, e^{-rt}\, dt = \int \frac{d}{d\theta} M(x, \lambda, \theta)\, e^{-rt}\, dt - \int \frac{d}{d\theta} (\lambda \dot{x})\, e^{-rt}\, dt.$$

Making use of the canonical equations (4), we see that

$$\begin{aligned}\int \frac{d}{d\theta} M(x, \lambda, \theta)\, e^{-rt}\, dt \\ = \int (M_x x_\theta + M_\lambda \lambda_\theta + M_\theta)\, e^{-rt}\, dt &= \int M_\theta e^{-rt}\, dt + \int ((r\lambda - \dot{\lambda})\, x_\theta \\ + \dot{x}\lambda_\theta)\, e^{-rt}\, dt &= \int \frac{d}{dt} (e^{-rt} \lambda x_\theta)\, dt + \int \frac{d}{d\theta} (\lambda \dot{x})\, e^{-rt}\, dt + \int M_\theta e^{-rt}\, dt.\end{aligned}$$

It follows that

$$J_\theta(\theta) = -e^{-rt}\lambda x_\theta \big|_0^T + \int_0^T M_\theta e^{-rt}\, dt = \int_0^T M_\theta(x(t;\theta), \lambda(t;\theta), \theta)\, e^{-rt}\, dt.$$

But now the continuous differentiability of M_θ (inherited from F and f) implies, by a repeat of the above arguments, that J_θ is continuously differentiable.

For the parameters r, x^0, x^T and T, the desired differentiability may be proved in a similar fashion. For example, from above, $J_{x^0}(x^0)$ exists and equals $\lambda(0; x^0)$, which is continuously differentiable in x^0.

APPENDIX B

We verify some of the assertions in the second example.

It is clearly optimal for the firm that attempts to solve (18) to choose L at each instant to maximize short-run profits. Therefore the firm solves

$$\max_I \int_0^T e^{-rt}[\pi(p, w; x) - C(I)]\, dt, \text{ subject to } \dot{x} = I - \delta x$$

$$x(0) = x^0,\ x(T) = x^T, \tag{22}$$

where

$$\pi(p, w; x) = \max_{L \geq 0} pG(x, L) - wL. \tag{23}$$

π is the variable profit function dual to G (see Diewert [3]) and is increasing in p, decreasing in w and convex and linear homogeneous in (p, w). Denote by $L(x, p, w)$ and $y(x, p, w)$ the profit maximizing labour input and output, respectively, in (23). By Hotelling's Lemma ([2]), $L(x, p, w) = -\pi_w(p, w; x)$ and $y(x, p, w) = \pi_p(p, w; x)$.

The constrained firm solves

$$\max_I \int_0^T e^{-rt}[pG(L^\wedge, x) - wL^\wedge - C(I)]\, dt \tag{24}$$

subject to $\dot{x} = I - \delta x$

$$x(0) = x^0,\ x(T) = x^T,$$

where the profile $L^\wedge(t; p_0, w_0)$ is fixed at the level that is optimal given the initial prices p_0 and w_0.

Consider the signs of L_x, L_p and L_w. L_x has the sign of G_{Lx}, $L_p > 0$, $L_w < 0$. Therefore, $L_x L_p \geq 0$ if and only if $G_{Lx} \geq 0$ and $L_x L_w \leq 0$ if and only if $G_{Lx} \geq 0$.

Adopting the obvious notation we can also show that $M_{\lambda p} = M_{\lambda w} = 0$ and that, if $G_{Lx} \geq 0$, then $M_{xp} = \pi_{xp}(p, w; x) = \partial y(x, p, w)/\partial x \geq 0$, $M_{xw} = \pi_{xw}(p, w; x) = -\partial L(x, p, w)/\partial x \leq 0$.
Therefore, by the Proposition, $x_p(t: p) \geq 0$ and $x_w(t; w) \leq 0$ for all t.

We can also show that

$$M'_{xp} = G_x(x, \hat{L}) > 0, \quad M'_{xw} = 0,$$

and

$$M'_{\lambda p} = M'_{\lambda w} = 0. \tag{26}$$

Again by the Proposition $x'_p(t; p) \geq 0$ and $x'_w(t; w) \leq 0$ for all t. (In fact, $x'_w(t; w) = 0$.) This completes the proof of (20).

Finally $y(t; p) = G(x(t; p), L(x, p, w))$ and $y'(t; p) = G(x'(t; p), \hat{L})$. Therefore, $y_p(t; p_0) = G_x x_p(t; p_0) + G_L(L_x x_p + L_p) \geq G_x x_p(t; p_0) \geq G_x x'_p(t; p_0) = y'_p(t; p_0)$ if $G_{Lx} \geq 0$, i.e., $y_p(t; p_0) \geq y'_p(t; p_0)$ for all t.

To prove that x_q is uniformly nonpositive proceed as follows: consider the profiles $x(t; q)$ and $x(t; q + \Delta)$, Δ a small positive number. If it is not the case that $x(t; q + \Delta) \leq x(t; q)$ for all $0 \leq t \leq T$, then there exist t_1 and t_2 such that $x(t_i; q) = x(t_i; q + \Delta)$, $i = 1, 2$ and $x(t; q + \Delta) > x(t; q)$ for $t_1 < t < t_2$. Thus it is sufficient to prove that with a fixed planning horizon and given initial and terminal capital stocks, an increase in q cannot uniformly raise the optimal capital stock profile. In fact an analysis of the phase plane of the appropriate variational system shows that $\dot{x}_q(0; q) \leq 0$. The arguments for the constrained capital stock profile and for the parameter "a" are similar.

REFERENCES

1. A. ARAUJO AND J. A. SCHEINKMAN, Smoothness, comparative dynamics and the turnpike property, *Econometrica* **45** (1977), 601–620.
2. G. A. CALVO, Efficient and optimal utilization of capital services, *Amer. Econ. Rev.* **65** (1975), 181–186.
3. W. E. DIEWERT, Applications of duality theory, *in* "Frontiers of Quantitative Economics" (M. D. Intrilligator and D. A. Kendrick, Eds.), Vol. II, North-Holland, Amsterdam, 1974.
4. S. KUSUMOTO, Extensions of the Le Chatelier Samuelson Principle and their application to analytical economics—constraints and economic analysis, *Econometrica* **44** (1976), 509–535.
5. M. J. P. MAGILL AND J. A. SCHEINKMAN, "Stability of Regular Equilibria and the Correspondence Principle for Symmetric Variational Problems," Report 7736, University of Chicago, June 1977.
6. Y. MUNDLAK, On the microeconomic theory of distributed lags, *Rev. Econ. Statist.* **48** (1966), 51–60.
7. K. NAGATANI, "On the Comparative Statics of Temporary Equilibrium," Discussion Paper 76-30, University of British Columbia, October 1976.
8. K. NAGATANI, "On Comparative Dynamics," paper presented to the Unapplied Economics Workshop, University of British Columbia, 1976.

9. H. Oniki, Comparative dynamics (sensitivity analysis) in optimal control theory, *J. Econ. Theory* **6** (1973), 265–283.
10. L. S. Pontryagin, "Ordinary Differential Equations," Addison–Wesley, Reading, Mass., 1962.
11. P. A. Samuelson, "Foundations of Economic Analysis," Harvard University Press, Cambridge, Mass., 1947.
12. A. B. Treadway, On rational entrepreneurial behaviour and the demand for investment, *Rev. Econ. Stud.* **36** (1969), 227–239.
13. A. B. Treadway, Adjustment costs and variable inputs in the theory of the competitive firm, *J. Econ. Theory* **2** (1970), 329–347.

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Correlation and Duopoly*

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INTRODUCTION

The basic difficulty raised by the oligopoly problem may be attributed to the fact that even though each oligopolistic firm selects an action (price and/or quantity) which maximizes its own individual profit, all firms are interdependent and they are aware of it. Thus, an oligopolist's best choice is necessarily dependent upon the assumptions it makes about rivals decisions and reactions. Of course, different alternative assumptions may plausibly be entertained.

The first attempt to deal with the oligopoly problem by Cournot [4] was based upon the postulate that in an homogeneous oligopoly (all products are perfect substitutes in the mind of the buyers) each firm chooses to market the quantity of its output which maximizes its own profit assuming the quantity marketed by others to be fixed. In this initial model the firms quantity of output are their strategic variables with the market establishing (through some process) a price sufficient to equalize total output and total demand. The heart of the model is preserved by assuming that prices are the firms' strategic variables, with the buyers deciding how much to purchase at these prices, at least under the assumption that there exists some degree of differentiation in the oligopoly so that different firms are able to sell simultaneously in the same market at different prices.

In any case, the major feature of Cournot equilibrium is that once the oligopolists have reached it, they have no incentive to move out. However there are in general moves which would allow an increase in all firm's profits

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simultaneously. Cournot [4] neglected these moves for the reason that the firms' strategic choices had to be thought as entirely independent with any form of collusion being excluded. Chamberlin [3] challenged that point of view by pointing out that

> ... the assumption of independence cannot be constructed as requiring the sellers to compete as though their fortunes were independent, for this is to belie the very problem of (oligopoly) itself (pp. 46-47)

And that, in an oligopoly,

> since the result of a cut (in price) by anyone is inevitably to decrease his own profits, no one will cut, and although the sellers are entirely independent, the equilibrim result is the same *as though* there were a monopolistic agreement between them (p. 48).

Chamberlin was actually suggesting that there are for oligopolies other solutions that the Cournot equilibrium which are based on "mutual-dependence-recognized" property—a sort of tacit-cooperation mechanism—and which provide to the firms better profits than at the equilibrium.[1] Similar point of view was developped by Fellner [5] under the vocable of "implicit agreement" or "quasi-agreement." It meets a game-theoretical investigation of cooperation in terms of self-enforced arrangements as initiated by Harsanyi [1] and Schelling [13].

Among all possible arrangements[2] which can be designed in an oligopoly, let us single out those which avoid explicit communication between the participants. One example of such arrangement is present in the "electrical equipment conspiracy of the 1950s" [14, p. 31] which involved 29 U.S. companies selling different sorts of products. For the power-switching equipement, sealed-bid competitions were sponsored by the U.S. government. Through some preliminary secret negotiation, each other was assigned a share of all sealed-bid business. The sellers then coordinated their bidding so that each one of them was low hidder in just enough transactions to gain its predetermined share of the market. This was achieved by dividing the territory into four quadrants assigned different sellers to each quadrant and letting the sellers in a quadrant rotate their bids. A "phases of the moon" system was used to decide of the low-bidding privileges. Thus, the result was an ostensibly random process conveying the impression of independent behavior.

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[1] Chamberlin [3, p. 47] was clearely thinking about the selection of an outcome collectively efficient for the firms.

[2] "Collusion to secure monopolistic ... prices is a venerable if not venerated institution" and "the variety of collusive ... arrangements in industry is limited only by the bounds of human ingenuity." Sherer [14, p. 30.]

The concept of correlated equilibrium as introduced by Aumann [1] provides a suitable formalization for collusive arrangements wjthout explicit communication among the participants.[3] The concept can be introduced by the following 2 × 2 game[4] which captures essential features of the oligopoly problem:

0 0	3 2
2 3	1 1

In this game the conflictual choice between (3, 2) and (2, 3) requires some players' coordination if they want to avoid outcomes (1, 1) and (0, 0). This can be achieved through a "random mediator" sending signals to both players which depend upon the selected outcome (3, 2) or (2, 3). On this basis each player may simply guess the other's expected strategic behavior. It gives an equilibrium in the sense that each one of the players is self-enforced to comply with the outcome suggested by the mediator; i.e., no one has an incentive to deviate.

In the present paper we consider duopoly games where the players strategies (quantities or prices) belong to real intervals X and Y and where the payoffs are real-valued profit functions U and V defined on $X \times Y$. Assuming the existence of a Cournot equilibrium, we formalize the idea of a tendency towards the maximization of collective profits through collusive tacit arrangements by considering that both players may be better off at some nearly correlated equilibrium. Take the game of Fig. 1.

We have a Cournot equilibrium $(\bar{x}, \bar{y})$ and CM_1M_2 is the area of all outcomes where both players are better off than at the equilibrium. Consider (x^1, y^1) and (x^2, y^2) two outcomes in this area which are in a neighborhood of $(\bar{x}, \bar{y})$. A lottery which selects (x^1, y^1) with probability $\frac{1}{2}$ and (x^2, y^2) with probability $\frac{1}{2}$ is a correlated equilibrium whenever

$$\forall x \in X, \qquad \tfrac{1}{2}U(x^1, y^1) + \tfrac{1}{2}U(x^2, y^2) \geq \tfrac{1}{2}U(x, y^1) + \tfrac{1}{2}U(x, y^2);$$
$$\forall y \in Y, \qquad \tfrac{1}{2}V(x^1, y^1) + \tfrac{1}{2}V(x^2, y^2) \geq \tfrac{1}{2}V(x^1, y) + \tfrac{1}{2}V(x^2, y).$$

Our general theorem gives a characterization of those Cournot equilibria which are improvable upon by a self-enforcing (tacit) collusive agreement of the above type. When applied to various duopoly models it gives measures of the strategic interdependences which are required for the firms to be able

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[3] See also Rosenthal [12], Moulin and Vial [11], Gerard-Varet [6], and Moulin [8, 9].

[4] Row player's payoff: top left; Column player's payoff: bottom right.

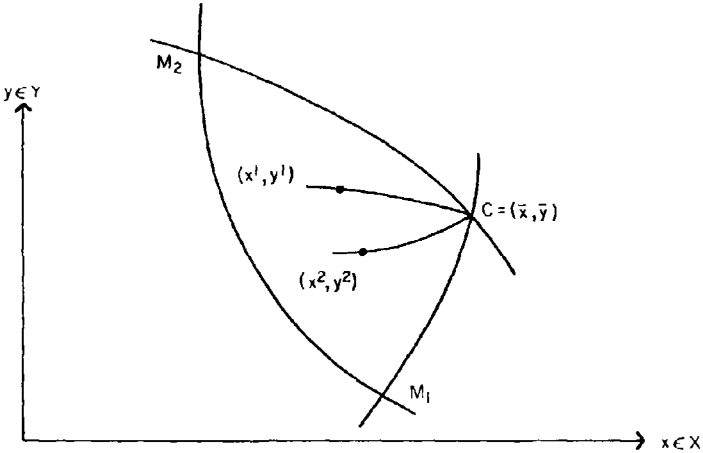

FIGURE 1

to make the tacit-cooperation move from a Cournot equilibrium. Different measures of these interdependencies have been given in the literature, mainly in terms of the cross elasticities of the demands with respect to prices or outputs.[5] As noted by Shubik [15], cross elasticities give information which is necessary but not efficient for a qualification of duopolistic interdependences. As a matter of fact we introduce measures which involve the computation—at the equilibrium—of (price or quantity) elasticities of the marginal receipt functions, derivatives of the market elasticities, or second-order market elasticities. We also see how marginal costs may facilitate or limit tacit cooperation.

1. IMPROVING UPON COURNOT EQUILIBRIUM BY CORRELATION

We start with a two-person game in normal form,

$$u, v : X \times Y \to \mathbb{R},$$

where X and Y are real compact intervals, and u, v are C^2-differentiable payoff functions on $X \times Y$.

A Cournot equilibrium is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\forall x \in X \qquad u(x, \bar{y}) \leqslant u(\bar{x}, \bar{y});$$
$$\forall y \in Y \qquad v(\bar{x}, y) \leqslant v(\bar{x}, \bar{y}).$$

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[5] See Triffin [16] and Bishop [2].

A *public lottery* is a discrete probability distribution on $X \times Y$: Thus a public lootery L is described by an integer n, n pairs $(x_1, y_1) \in X \times Y, \ldots, (x_n, y_n) \in X \times Y$ and n nonnegative numbers $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 + \cdots + \alpha_n = 1$. If we denote by $\delta_{x,y}$ the probability distribution with weight 1 on (x, y), then the lottery L can be written as

$$L = \sum_{i=1}^{n} \alpha_i \delta_{(x_i, y_i)}.$$

If w is a payoff function (defined on $X \times Y$) we denote by $[L, w]$ the expected value of w with respect to L,

$$[L, w] = \sum_{i=1}^{n} \alpha_i w(x_i, y_i).$$

We also denote by L_X (resp. L_Y) the "private lottery"—or mixed strategy—induced by L on X (resp. Y), that is, the marginal distribution of L on X (resp. Y)

$$L_X = \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \qquad L_Y = \sum_{i=1}^{n} \alpha_i \delta_{y_i}.$$

Finally we denote by $u(x, L_Y)$ (resp. $v(L_X, y)$) the expected value of u when player **X** uses his pure strategy x and player **Y** his mixed strategy L_Y (resp. player **X** uses his mixed strategy L_X and player **Y** his pure strategy y)

$$u(x, L_Y) = \sum_{i=1}^{n} \alpha_i u(x, y_i),$$

$$v(L_X, y) = \sum_{i=1}^{n} \alpha_i v(x_i, y).$$

DEFINITION 1. The public lottery L is said to be correlated equilibrium if we have

$$\begin{aligned} \forall x \in X \quad & [L, u] \geq u(x, L_Y), \\ \forall y \in Y \quad & [L, v] \geq v(L_X, y). \end{aligned} \tag{1}$$

We say that the public lottery L is a *strict* correlated equilibrium if the above inequalities are strict.

$$\begin{aligned} \forall x \in X \quad & [L, u] > u(x, L_Y), \\ \forall y \in Y \quad & [L, v] > v(L_X, y). \end{aligned} \tag{1$^{\text{bis}}$}$$

Let us interpret the above system. The players **X** and **Y** agree to (collectively) construct the public lottery L. A referee (possibly a machine) is called to select an outcome (x, y) according to the probability distribution L (i.e., (x_i, y_i) is selected with probability s_i), and to *selectively* inform the players of the result: Player **X** is informed of the particular strategy x selected by the referee, but ignores the corresponding strategy y; at the same time player **Y** is informed of the particular y selected by the referee, but ignores the corresponding x. Now, the tacit agreement of the players when constructing the lottery L is to follow the suggestions of the machine (to play "as the referee says"). System (1) precisely expresses that this agreement is self-enforced *as long as no communication arises between the players*. Namely if player **Y** faithfully plays as the referee suggests –and ignores player **X**'s behavior – whereas player **X** secretly betrays the agreement by using his pure strategy x (and is no longer informed of the outcome selected by the referee –see Remark 1 below) then the corresponding expected payoff is $u(x, L_Y)$. A symmetrical argument holds for unilateral betrayals of the agreement by player **Y**.

Particular examples of correlated equilibriums are any Cournot equilibrium $(\bar{x}, \bar{y})$ (as identified to the public lottery $\delta_{(\bar{x},\bar{y})}$) and any Nash equilibrium (μ, ν) in mixed strategies (as identified with the public lottery $\mu \otimes \nu$).

The main interest of correlated equilibria is that they may improve upon a Cournot equilibrium.

DEFINITION 2. A Cournot equilibrium is said to be *improvable by correlation* if there exists a correlated equilibrium L such that

$$[L, u] > u(\bar{x}, \bar{y}),$$

$$[L, v] > v(\bar{x}, \bar{y}).$$

Remark 1. As we define it, a "correlated equilibrium" is actually a "simple correlation scheme" of Moulin and Vial [11]. The differences with Aumann's concept of correlated equilibrium [1] are studied in this paper and also in Moulin [8, 9]. Roughly, in Aumann's correlated equilibrium a betraying player is still informed of the suggestion of the referee, which in turn gives him some information about the strategical choice of the other player, therefore making betrayal more attractive.

Remark 2. The restrictive assumption that the public lottery is discrete could be easily dropped out: Public lotteries are then Radon measures on $X \times Y$ and their carrier is not necessarily finite. Although every definition and every result of this paper still hold true in this more general framework, the interpretative power of the model gains so little that the corresponding increase in mathematical complexity is practically useless.

2. MAIN THEOREM

The aim of this section is to characterize whether or not a Cournot equilibrium is improvable by correlation. We give to this property a local meaning. We state the characterization for any 2-person game

$$u, v : X \times Y \to \mathbb{R},$$

where X, Y are real compact intervals and u, v are C^2-differentiable. Moreover we consider only those Cournot equilibria $(\bar{x}, \bar{y})$ verifying the following assumptions:

$$\begin{aligned} x \neq \bar{x} &\Rightarrow u(\bar{x}, \bar{y}) > u(x, \bar{y}), \\ y \neq \bar{y} &\Rightarrow v(\bar{x}, \bar{y}) > v(\bar{x}, y), \end{aligned} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2}(\bar{x}, \bar{y}) < 0, \qquad \frac{\partial^2 v}{\partial y^2}(\bar{x}, \bar{y}) < 0, \tag{3}$$

$$\bar{x} \text{ (resp. } \bar{y}\text{) is an interior element of } X \text{ (resp. } Y\text{)}, \tag{4}$$

$$\frac{\partial u}{\partial y}(\bar{x}, \bar{y}) \neq 0, \qquad \frac{\partial v}{\partial x}(\bar{x}, \bar{y}) \neq 0. \tag{5}$$

DEFINITION 3. A Cournot equilibrium $(\bar{x}, \bar{y})$ is said to be locally improvable by correlation if there exists an integer n, n continuous mappings from $[0, 1]$ into $X \times Y$, and n continuous mappings from $(0, 1)$ into $\mathbb{R}_+$

$$\begin{aligned} t &\to (x_i(t), y_i(t)), & i &= 1, \ldots, n, \\ t &\to \alpha_i(t), & i &= 1, \ldots, n, \end{aligned}$$

such that $L(0) = \delta_{(\bar{x}, \bar{y})}$, such that $L(t) = \sum_{i=1}^{n} \alpha_i(t)\, \delta_{(x_i(t), y_i(t))}$ is a strict correlated equilibrium for every $t \in]0, 1]$ and such that the payoffs $t \to [L(t), u]$ and $t \to [L(t), v]$ are strictly increasing on $[0, 1]$.

THEOREM. *Let* $(\bar{x}, \bar{y})$ *be a Cournot equilibrium verifying* (2), (3), (4), (5). *We set*

$$\rho = \left[\frac{\partial^2 u}{\partial x \partial y}(\bar{x}, \bar{y}) \cdot \frac{\partial^2 v}{\partial x \partial y}(\bar{x}, \bar{y})\right] \Big/ \left[\frac{\partial^2 u}{\partial x^2}(\bar{x}, \bar{y}) \cdot \frac{\partial^2 v}{\partial y^2}(\bar{x}, \bar{y})\right]$$

(i) *If* $\rho < \frac{1}{4}$ *then* $(\bar{x}, \bar{y})$ *is not locally improvable by correlation.*

(ii) *If* $\rho > \frac{1}{4}$ *then* $(\bar{x}, \bar{y})$ *is locally improvable by correlation.*

Moreover we may choose $L(t)$ *as the half-sum of two Dirac measures only,*

$$L(t) = \tfrac{1}{2}\delta_{(x_1(t), y_1(t))} + \tfrac{1}{2}\delta_{(x_2(t), y_2(t))},$$

where $t \to (x_i(t), y_i(t))$ *are continuous and* $(x_i(0), y_i(0)) = (\bar{x}, \bar{y})$ *for* $i = 1, 2$.

Notice that the number ρ remains unchanged when the function u is transformed to $\phi o u$, where ϕ is increasing and twice differentiables on $\mathbb{R}$. Accordingly ρ does depend only on the underlying preference preordering of u. The same argument applies for v.

In the next section we illustrate the inequalities $\rho \lesseqgtr \frac{1}{2}$ on various duopoly models.

3. APPLICATION TO DUOPOLY MODELS

In the present section we intend to show how the theorem may be applied to the analysis of several classes of duopoly models. We successively study homogeneous duopolies with the firms' outputs as strategic variables and differentiated duopolies with the prices as strategic variables. In both cases the payoffs are the duopolists' profit functions.

Assuming that there exists a Cournot equilibrium we derive conditions implying that such an equilibrium is (or is not) locally improvable upon. In the present context, the coefficients involve second-order derivatives of the individual profits. However, by studying different simplified cases, we exhibit conditions which actually involve derivatives of the elasticities of the firms' market demand and, whenever they exist, the marginal costs, all being taken at the equilibrium. These conditions are interpretable as giving second-order measures of the duopolists' "strategic interdependences." In general and loosely speaking, a Cournot equilibrium is locally improvable upon when there are sufficient interdependences in these terms between the two players.

1. *Homogeneous Output-Setting Duopolies*

Let two firms be selling the same commodity to a competitive market. They compete by independently selecting the quantity they produce and sell. Let $q_1 \in \mathbb{R}_+$ and $q_2 \in \mathbb{R}_+$ be the outputs produced and sold, respectively, by the first and the second firm. The price $p \in \mathbb{R}_+$ of the commodity is given as a function of the total supply on the market $q = q_1 + q_2$. This function (also called inverse demand function of the market) is denoted $p = g(q)$.

Let the firms revenue be functions from $\mathbb{R}_+^2$ to $\mathbb{R}$ defined by letting

$$r_1(q_1, q_2) = g(q) \cdot q_1 \qquad \text{and} \qquad r_2(q_1, q_2) = g(q) \cdot q_2 .$$

In general, each firm has a total cost which is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ denoted $c_1(q_1)$ for the first and $c_2(q_2)$ for the second. Thus, the duopolists' payoffs are the real-valued profit functions

$$\forall (q_1, q_2) \in \mathbb{R}_+^2, \qquad \begin{aligned} U(q_1, q_2) &= r_1(q_1, q_2) - c_1(q_1), \\ V(q_1, q_2) &= r_2(q_1, q_2) - c_2(q_2). \end{aligned}$$

We shall make the following assumptions:

(6) the inverse demand is a function g from $\mathbb{R}_+$ to $\mathbb{R}$ which is C^2-differentiable and such that the price is a decreasing function of the total output

$$\frac{dg}{dq} = \frac{\partial g}{\partial q_1} = \frac{\partial g}{\partial q_2} < 0.$$

Furthermore, there exists a strictly positive output $q^0 \in \mathbb{R}_+ - \{0\}$ such that $g(q) = 0$ for every $q \geqslant q^0$.

(7) The total costs c_1 and c_2 are functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ which are C^2-differentiable.

(8) The profit functions U and V are such that

$$\frac{\partial^2 U}{\partial q_1^2} = \frac{d^2 g}{dq^2} q_1 + 2\frac{dg}{dq} - \frac{d^2 c_1}{dq_1^2} < 0,$$

$$\frac{\partial^2 V}{\partial q_2^2} = \frac{d^2 g}{dq^2} q_2 + 2\frac{dg}{dq} - \frac{d^2 c_2}{dq_2^2} < 0.$$

Let, under assumption (6), $X = Y = [0, q^0]$. With assumptions (6), (7), and (8), the payoff functions U and V are C^2-differentiable on $X \times Y$ and strictly concave with respect to the corresponding strategic variables. Thus a Cournot equilibrium exists and is a pair $(\bar{q}_1, \bar{q}_2) \in X \times Y$ which necessarily satisfies

$$\begin{aligned} \frac{\partial U}{\partial q_1}(\bar{q}_1, \bar{q}_2) &= \frac{dg}{dq}(\bar{q}_1 + \bar{q}_2)\,\bar{q}_1 + g(\bar{q}_1 + \bar{q}_2) - \frac{dc_1}{dq_1}(\bar{q}_1) = 0, \\ \frac{\partial V}{\partial q_2}(\bar{q}_1, \bar{q}_2) &= \frac{dg}{dq}(\bar{q}_1 + \bar{q}_2)\,\bar{q}_2 + g(\bar{q}_1 + \bar{q}_2) - \frac{dc_2}{dq_2}(\bar{q}_2) = 0. \end{aligned} \tag{9}$$

Condition (9) gives

$$\bar{q}_i = \left[\frac{dx_i}{dq_i}(\bar{q}_i) - g(\bar{q}_1 + \bar{q}_2)\right] \Big/ \left[\frac{dg}{dq}(\bar{q}_1 + \bar{q}_2)\right] \qquad \text{for } i = 1 \text{ or } 2. \tag{10}$$

Condition (10) provides a Cournot equilibrium $(\bar{q}_1, \bar{q}_2)$ which is an interior element of $X \times Y$ and by assumption (8) we have

$$\frac{\partial^2 U}{\partial q_1^2}(\bar{q}_1, \bar{q}_2) < 0 \qquad \text{and} \qquad \frac{\partial^2 V}{\partial q_2^2}(\bar{q}_1, \bar{q}_2) < 0.$$

Since relations (2), (3), (4), and (5) are easily satisfied, the theorem gives a criterion for such an equilibrium to (or not) be locally improvable upon. We see by studying two different cases that this criterion has an interpretation in terms of basic characteristics of the market: namely, marginal costs and elasticity of the marginal receipt functions (or second-order elasticity of the market demand taken at the equilibrium.

a. *A Case with Linear Demand and Quadratic Costs*

Let us first study the situation when the two firms face the same linear demand and have quadratic total costs. More precisely, the inverse market demand is the linear function

$$p = g(q_1 + q_2) = \alpha - \beta(q_1 + q_2) \qquad \text{with } \alpha > 0 \text{ and } \beta > 0.$$

It gives the revenue functions

$$\forall (q_1, q_2) \in \mathbb{R}_+^2, \forall i = 1, 2, \qquad V_i(q_1, q_2) = [\alpha - \beta(q_1 + q_2)]\, q_i$$

implying

$$\frac{\partial v_1}{\partial q_1}(q_1, q_2) = -2\beta q_1 - \beta q_2 < 0,$$

$$\frac{\partial v_2}{\partial q_2}(q_1, q_2) = -2\beta q_2 - \beta q_1 < 0,$$

and

$$\frac{\partial^2 v_1}{\partial q_1^2}(q_1, q_2) = \frac{\partial^2 v_2}{\partial q_2^2}(q_1, q_2) = -2\beta < 0.$$

The revenues and the marginal revenues are strictly decreasing.

Let on another hand the total costs be

$$\forall (q_1, q_2) \in \mathbb{R}_+^2, \qquad c_1(q_1) = \delta_1 q_1 + \epsilon q_1^2 \qquad \text{and} \qquad c_2(q_2) = \delta_2 q_2 + \epsilon q_2^2$$

yielding

$$\frac{dc_1}{dq_1}(q_1) = \delta_1 + 2\epsilon q_1 \qquad \text{with } \delta_1 > 0,$$

$$\frac{dc_2}{dq_2}(q_2) = \delta_2 + 2\epsilon q_2 \qquad \text{with } \delta_2 > 0,$$

and

$$\frac{d^2 c_1}{dq_1^2}(q_1) = \frac{d^2 c_2}{dq_2^2}(q_2) = 2\epsilon.$$

Assumption (8) requires

$$-2\beta < 2\epsilon \tag{11}$$

meaning (if $\epsilon > 0$) that the marginal costs are strictly increasing or (if

$\epsilon < 0$) that the marginal costs are decreasing but fall more slowly than the marginal revenues.

We compute now the number ρ introduced in the statement of the theorem. We have

$$\frac{\partial^2 U}{\partial q_1^2}(\bar{q}_1, \bar{q}_2) = \frac{\partial^2 V}{\partial q_2^2}(\bar{q}_1, \bar{q}_2) = -2\beta - 2\epsilon < 0.$$

Since

$$\frac{\partial^2 U}{\partial q_1 \partial q_2}(\bar{q}_1, \bar{q}_2) = \frac{\partial^2 V}{\partial q_1 \partial q_2}(\bar{q}_1, \bar{q}_2) = -\beta,$$

we get

$$\rho = \left[\frac{\beta}{2(\beta + \epsilon)}\right]^2$$

and

$$\rho \lessgtr \frac{1}{4} \iff \left|\frac{\beta}{2(\beta + \epsilon)}\right| \lessgtr \frac{1}{2},$$

i.e.,

$$\rho \lessgtr \frac{1}{4} \iff |\beta| \lessgtr |\beta + \epsilon|.$$

Since $\beta > 0$ (by assumption) and $\beta + \epsilon > 0$ (by (6)),

$$\rho \lessgtr \frac{1}{4} \iff 0 \lessgtr \epsilon.$$

Thus, *the Cournot equilibrium is locally improvable upon when the marginal costs are strictly decreasing. The equilibrium is not locally improvable upon when the marginal costs are strictly increasing.*

Notice that when $\epsilon = 0$, i.e., *when marginal costs are constant* we cannot conclude anything about the Cournot equilibrium by using directly the criteria of the theorem. However in that case a direct argument (see Moulin [8, 9]) shows that the equilibrium is not improvable upon.

b. *A Case with Null Costs*

Let us consider now the situation where the duopolists face a market demand satisfying conditions (6), but with both production costs identically null. The individual payoff functions are now simply the revenue functions

$$\forall (q_1, q_2) \in \mathbb{R}_+^2, \qquad \begin{aligned} U(q_1, q_2) &= g(q_1 + q_2) \cdot q_1, \\ V(q_1, q_2) &= g(q_1 + q_2) \cdot q_2. \end{aligned}$$

Assumption (8) becomes

$$\frac{\partial^2 U}{\partial q_1^2} = \frac{d^2 g}{dq^2} q_1 + 2\frac{dg}{dq} < 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial q_2^2} = \frac{d^2 g}{dq^2} q_2 + 2\frac{dg}{dq} < 0.$$

According to condition (10) a Cournot equilibrium is a pair $(\bar{q}_1, \bar{q}_2) \in \mathbb{R}_+^2$ which is necessarily such that

$$\forall i = 1, 2, \qquad \bar{q}_i = -[g(\bar{q}_1 + \bar{q}_2)] \Big/ \left[\frac{dg}{dq}(\bar{q}_1 + \bar{q}_2)\right].$$

It implies that we must have $\bar{q}_1 = \bar{q}_2 = \bar{q}$. Thus, a Cournot equilibrium is simply denoted $(\bar{q}, \bar{q})$.

Let us compute coefficient ρ. Under (12) we have

$$\frac{\partial^2 U}{\partial q_1^2}(\bar{q}, \bar{q}) = \frac{\partial^2 V}{\partial q_2^2}(\bar{q}, \bar{q}) = \frac{d^2 g}{dq^2}(2\bar{q})\, \bar{q} + 2\frac{dg}{dq}(2\bar{q}) < 0.$$

Since

$$\frac{\partial^2 U}{\partial q_1\, \partial q_2}(\bar{q}, \bar{q}) = \frac{\partial^2 V}{\partial q_1\, \partial q_2}(\bar{q}, \bar{q}) = \frac{d^2 g}{dq^2}(2\bar{q})\, \bar{q} + \frac{dg}{dq}(2\bar{q}),$$

we get

$$\rho = \left[\frac{d^2 g}{dq^2}(2\bar{q})\, \bar{q} + \frac{dg}{dq}(2\bar{q})\right]^2 \Big/ \left[\frac{d^2 g}{dq^2}(2\bar{q})\, \bar{q} + 2\frac{dg}{dq}(2\bar{q})\right]^2.$$

According to the theorem Cournot equilibrium $(\bar{q}, \bar{q})$ is locally improvable upon when $\rho > \frac{1}{4}$, which equivalently gives here

$$\left|\frac{d^2 g}{dq^2}(2\bar{q})\, \bar{q} + 2\frac{dg}{dq}(2\bar{q})\right| < 2\left|\frac{d^2 g}{dq^2}(2\bar{q})\, \bar{q} + \frac{dg}{dq}(2\bar{q})\right|. \tag{13}$$

One possible case for (13) to be true is, with our assumptions,

$$\frac{d^2 g}{dq^2}(2\bar{q}) \cdot \bar{q} < 0. \tag{14}$$

A second possible case for (13) to be true is

$$-\frac{4}{3}\frac{dg}{dq}(2\bar{q}) < \frac{d^2 g}{dq^2}(2\bar{q}) \cdot \bar{q} < -2\frac{dg}{dq}(2\bar{q}). \tag{15}$$

Alternately, a Cournot equilibrium $(\bar{q}, \bar{q})$ is not locally improvable upon when $\rho < \frac{1}{4}$, which amounts to the following inequalities

$$0 < \frac{d^2 g}{dq^2}(2\bar{q}) \cdot \bar{q} < -\frac{4}{3}\frac{dg}{dq}(2\bar{q}). \tag{16}$$

To give an interpretation to conditions (14), (15), and (16), let us introduce the elasticity of the marginal receipt function be the function η from $\mathbb{R}_+$ to $\mathbb{R}$ defined by letting,

$$\forall q \in \mathbb{R}_+, \qquad \eta(q) = -\frac{d^2 g}{dq^2}(q)\left[q \Big/ \frac{dg}{dq}(q)\right].$$

Since both costs are null, assumption (8) is satisfied whenever

$$\forall q \in \mathbb{R}_+ \qquad \eta(q) < -1. \tag{18}$$

Furthermore, at the equilibrium $(\bar{q}, \bar{q})$, we have

$$\eta(2\bar{q}) = -\frac{de}{dq}(2\bar{q})\,\bar{q} - 3.$$

Thus, assuming (18), conditions (14), (15), and (16) give *that the Cournot equilibrium* $(\bar{q}, \bar{q})$ *is locally improvable upon when, at the equilibrium, the elasticity of the marginal receipt is such that*

$$-4 < \eta(2\bar{q}) < -\tfrac{8}{3};$$

and it cannot be the case when

$$-\tfrac{8}{3} < \eta(2\bar{q}) < -1.$$

One may qualitatively state our conclusions by saying that for output-setting homogeneous duopoly with null costs, the Cournot equilibrium is locally improvable upon if the elasticity of the marginal receipt function is sufficiently high when the elasticity of the marginal receipt is null. Marginal costs and the value at the equilibrium of the marginal receipt functions are parameters of the market structure which measure strategic interdependences of the firms.

2. *Differentiated Price-Setting Duopolies with Null Costs*

Let us consider now duopoly models where the outputs of the two firms are differentiated but substituable commodities. Let us assume furthermore that the two firms compete by independently selecting a price to charge. The consumers now select a quantity for both commodities that the firms may sell at that prices. Let $q_1 \in \mathbb{R}_+$ and $q_2 \in \mathbb{R}_+$ be the outputs produced by the two firms, $p_1 \in \mathbb{R}_+$ and $p_2 \in \mathbb{R}_+$ be their respective prices. Firms i $(i = 1, 2)$ are facing a well defined demand function h_i from $\mathbb{R}_+^2$ to $\mathbb{R}_+$ such that

$$q_i = h_i(p_1, p_2).$$

The firms' revenue functions coincide with their payoff functions and are

$$\forall (p_1, p_2) \in \mathbb{R}_+^2, \qquad U(p_1, p_2) = p_1 \cdot h_1(p_1, p_2),$$

and

$$V(p_1, p_2) = p_2 \cdot h_2(p_1, p_2).$$

We assume: Both demands h_1 and h_2 are C^2-differentiable for those pairs of prices for which they are well-defined (i.e. nonnegative[6] and the first derivatives are such that

$$\frac{\partial h_1}{\partial p_1} < 0, \qquad \frac{\partial h_1}{\partial p_2} > 0, \qquad \frac{\partial h_2}{\partial p_2} < 0, \qquad \frac{\partial h_2}{\partial p_1} \geq 0. \tag{19}$$

These imply that the firms do not sell homogeneous commodities. Furthermore, we assume the existence of a pair $(p_1^0, p_2^0) \in \mathbb{R}_+^2$ of prices for which both demands are well defined and such that for any other pair of prices $(p_1, p_2) \in R_2^+$, where demands remain well defined we have

$$p_1 \geq p_1^0, p_2 \geq p_2^0 \qquad \text{and} \qquad h_1(p_1^0, p_2^0) = h_2(p_1^0, p_2^0) = 0.$$

If any firm should increase its price beyond p_i^0, the effect on both firms would be null.[7]

Finally we assume: The payoff functions U and V are such that

$$\begin{aligned} \frac{\partial^2 U}{\partial p_1^2} &= p_1 \cdot \frac{\partial^2 h_1}{\partial p_1^2} + 2\frac{\partial h_1}{\partial p_1} < 0 \\ \frac{\partial^2 V}{\partial p_2^2} &= p_2 \cdot \frac{\partial^2 h_2}{\partial p_2^2} + 2\frac{\partial h_2}{\partial p_2} < 0 \end{aligned} \tag{20}$$

Let $X =^{\text{def}} \{p_1 \in \mathbb{R}_+ ; p_1 \leq p_1^0\}$ and $Y =^{\text{def}} \{p_2 \subset \mathbb{R}_+ ; p_2 \leq p_2^0\}$ the payoffs U and V are C^2-differentiable on $X \times Y$, which is compact, and with assumption (20), a Cournot equilibrium does exists. It is a pair $(\bar{p}_1, \bar{p}_2) \in X \times Y$ which necessarily satisfies

$$\frac{\partial U}{\partial p_1}(\bar{p}_1, \bar{p}_2) = \frac{\partial h_1}{\partial p_1}(\bar{p}_1, \bar{p}_2)\, \bar{p}_1 + h_1(\bar{p}_1, \bar{p}_2) = 0,$$

$$\frac{\partial V}{\partial p_2}(\bar{p}_1, \bar{p}_2) = \frac{\partial h_2}{\partial p_2}(\bar{p}_1, \bar{p}_2)\, \bar{p}_2 + h_2(\bar{p}_1, \bar{p}_2) = 0.$$

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[6] Let $A_i = \{(p_1, p_2) \in \mathbb{R}_+^2, h_i(p_1, p_2) \geq 0\}$ be set of pairs of prices for which the demand for firm i is nonnegative. Functions h_1 and h_2 have to be C^2-differentiable respectively on the interiors $\mathring{A}_1$ and $\mathring{A}_2$ of A_1 and A_2.

[7] Let $A = A_1 \cap A_2$ with A_i as defined in footnote 6. One may simply assume that A is bounded. As a result attention may be restricted to the compact set of pairs of prices in $\mathbb{R}_+^2$ not larger than (p_1^0, p_2^0).

This condition gives

$$\forall i = 1, 2 \quad \bar{p}_i = - \frac{h_i(\bar{p}_1, \bar{p}_2)}{(\partial h_i/\partial p_i)(\bar{p}_1, \bar{p}_2)} + \frac{dc_i}{dq_i}(h_i(\bar{p}_1, \bar{p}_2)). \tag{21}$$

Clearly this equilibrium verifies (2) to (5).

To compute coefficient ρ of the theorem let us define the *elasticities of the individual demand with respect to the individual strategic variables* i.e. the *price elasticities*. It is, for firm 1, a function from $\mathbb{R}_+^2$ to $\mathbb{R}$, denoted E_{11}, and defined as being:

$$E_{11} \overset{\text{def}}{=} \frac{\partial h_1}{\partial p_1} \cdot \frac{p_1}{h_1}.$$

Assuming (19), we necessarily have $E_{11} < 0$. The elasticity similarly defined for firm 2 is denoted E_{22}. All first derivatives for E_{11} and E_{22} are well-defined continuous functions.

We have

$$\frac{\partial U}{\partial p_1} = h_1(1 + E_{11}) \qquad \frac{\partial V}{\partial p_1} = h_2(1 + E_{22}).$$

Clearly

$$E_{11}(\bar{p}_1, \bar{p}_2) = E_{22}(\bar{p}_1, \bar{p}_2) = -1.$$

Furthermore we get

$$\frac{\partial^2 U}{\partial p_1^2}(\bar{p}_1, \bar{p}_2) = h_1(\bar{p}_1, \bar{p}_2) \frac{\partial E_{11}}{\partial p_1}(\bar{p}_1, \bar{p}_2),$$

$$\frac{\partial^2 V}{\partial p_2^2}(\bar{p}_1, \bar{p}_2) = h_2(\bar{p}_1, \bar{p}_2) \frac{\partial E_{22}}{\partial p_2}(\bar{p}_1, \bar{p}_2).$$

We also have

$$\frac{\partial^2 U}{\partial p_1 \partial p_2}(\bar{p}_1, \bar{p}_2) = h_1(\bar{p}_1, \bar{p}_2) \frac{\partial E_{11}}{\partial p_2}(\bar{p}_1, \bar{p}_2)$$

and

$$\frac{\partial^2 V}{\partial p_1 \partial p_2}(\bar{p}_1, \bar{p}_2) = h_2(\bar{p}_1, \bar{p}_2) \frac{\partial E_{22}}{\partial p_1}(\bar{p}_1, \bar{p}_2).$$

Consequently coefficient ρ is given by

$$\rho = \frac{h_1(\bar{p}_1, \bar{p}_2)\, h_2(\bar{p}_1, \bar{p}_2) \frac{\partial E_{11}}{\partial p_2}(\bar{p}_1, \bar{p}_2) \frac{\partial E_{22}}{\partial p_1}(\bar{p}_1, \bar{p}_2)}{h_1(\bar{p}_1, \bar{p}_2)\, h_2(\bar{p}_1, \bar{p}_2) \frac{\partial E_{11}}{\partial p_1}(\bar{p}_1, \bar{p}_2) \frac{\partial E_{22}}{\partial p_2}(\bar{p}_1, \bar{p}_2)}. \tag{22}$$

From the theorem and from (22) we may conclude that the *Cournot equilibrium* $(\bar{p}_1, \bar{p}_2)$ *given by condition* (21) *is locally improvable upon when*

$$4\frac{\partial E_{11}}{\partial p_2}(\bar{p}_1, \bar{p}_2)\cdot\frac{\partial E_{22}}{\partial p_1}(\bar{p}_1, \bar{p}_2) > \frac{\partial E_{11}}{\partial p_1}(\bar{p}_1, \bar{p}_2)\cdot\frac{\partial E_{22}}{\partial p_2}(\bar{p}_1, \bar{p}_2).$$

The Cournot equilibrium $(\bar{p}_1, \bar{p}_2)$ *is not locally improvable upon when*

$$4\frac{\partial E_{11}}{\partial p_2}(\bar{p}_1, \bar{p}_2)\frac{\partial E_{22}}{\partial p_1}(\bar{p}_1, \bar{p}_2) < \frac{\partial E_{11}}{\partial p_1}(\bar{p}_1, \bar{p}_2)\cdot\frac{\partial E_{22}}{\partial p_2}(\bar{p}_1, \bar{p}_2).$$

Take $\partial E_{ii}/\partial p_i$ as a measure of the "direct effect" of firm i strategic choice on its own demand and $\partial E_{ii}/\partial p_j$ as a measure of the "cross effect" of its rival's strategic choice on the same demand. The result says that, *at the equilibrium, the product of the two direct effects must be sufficiently smaller than the product of the two cross effects to allow the duopolists to locally improve upon the equilibrium.*

This conclusion has direct implications in some simple cases. Consider, for example, a costless duopoly where

$$\forall(p_1, p_2) \in \mathbb{R}_+^2, \qquad \begin{aligned} h_1(p_1, p_2) &= 1 - \alpha_1 p_1 + \beta_1 p_2, & \alpha_1 \geqslant 1, \beta_1 \geqslant 1,\\ h_2(p_1, p_2) &= 1 - \alpha_2 p_2 + \beta_2 p_1, & \alpha_2 \geqslant 1, \beta_2 \geqslant 1. \end{aligned}$$

Parameters β_1 and β_2 are the "cross-effects" parameters, and parameters α_1 and α_2 are the "substituability" parameters. There exists a Cournot equilibrium[8] $(\bar{p}_1, \bar{p}_2)$ given by (21). This equilibrium is locally improvable upon when

$$\beta_1\beta_2 > \alpha_1\alpha_2,$$

i.e., when the product of the two substituability parameters is smaller than the product of the two cross-effects parameters. The equilibrium is not locally improvable upon when

$$\beta_1\beta_2 < \alpha_1\alpha_2.$$

Notice that it is the case when the two goods are imperfect substitutes ($\alpha_1 \neq \alpha_2$, $\alpha_1 > 1$, $\alpha_2 > 1$) and if the cross effects are negligible ($\beta_1 = \beta_2 = 1$).

One could have used in stating the conclusions the direct and cross elasticities of the firms marginal receipt functions[9] instead of the derivatives of the

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[8] It is easily computed to be

$$\bar{p}_1 = \frac{2\alpha_2 + \beta_1}{4\alpha_1\alpha_2 - \beta_1\beta_2}, \qquad \bar{p}_2 = \frac{2\alpha_1 + \beta_2}{4\alpha_1\alpha_2 - \beta_1\beta_2}.$$

[9] Define

elasticities of the individual demand functions. By that means one would have captured essentially the same phenomenon namely that the interdependences between the firms in terms of cross effects has to be sufficiently high for a Cournot equilibrium to be locally improvable upon.

Price Setting Duopoly with Null Costs—A Particular Case

Let us conclude this section by the discussion of a differentiated duopoly arising from the fact that the market for a simple commodity has been shared between two firms according to a given rule. Let the market demand be a C^2-differentiable function denoted $q = h(p)$ and such that $dh/dp > 0$. The sharing rule is such that when they compete by independently setting their prices, the firms face well-defined individual demands which are function of the first degree differential of the prices. In other words we have a differentiated duopoly with the two demand functions being.[10]

$$q_1 = h_1(q_1, q_2) = h(p_2 - p_1) \quad \text{and} \quad q_2 = h_2(p_1, p_2) = h(p_1 - p_2).$$

We have

$$\begin{aligned} \frac{\partial h_1}{\partial p_1} &= -\frac{dh}{dp} < 0, & \frac{\partial h_1}{\partial p_2} &= +\frac{dh}{dp} > 0, \\ \frac{\partial h_2}{\partial p_2} &= -\frac{dh}{dp} < 0, & \frac{\partial h_2}{\partial p_1} &= +\frac{dh}{dp} > 0. \end{aligned} \tag{23}$$

The two firms clearly have identical shares of the market at identical prices. Both production costs being identically null, the payoffs are

$$U(p_1, p_2) = h(p_2 - p_1) \cdot p_1 \quad \text{and} \quad V(p_1, p_2) = h(p_1 - p_2) \cdot p_2 .$$

A Cournot equilibrium is a pair $(\bar{p}_1, \bar{p}_2)$ which is necessarily such that

$$\frac{\partial U}{\partial p_1}(p_1, p_2) = h(p_2 - p_1) - p_1 \frac{dh}{dp}(p_2 - p_1) = 0,$$

$$H_{ii} = \frac{\partial^2 h_i}{\partial p_i^2} \frac{p_i}{\dfrac{\partial h_i}{\partial p_i}} \quad \text{and} \quad H_{ij} = \frac{\partial^2 h_i}{\partial p_i \partial p_j} \frac{p_i}{\dfrac{\partial h_i}{\partial p_j}}.$$

With these, formula (22) becomes

$$\rho = \frac{(\partial h_1/\partial p_2(\bar{p}_1, \bar{p}_2)[1 + H_{12}(\bar{p}_1, \bar{p}_2)])(\partial h_2/\partial p_1(\bar{p}_1, \bar{p}_2)[1 + H_{21}(\bar{p}_1, \bar{p}_2)])}{(\partial h_1/\partial p_1(\bar{p}_1, \bar{p}_2)[1 + H_{11}(\bar{p}_1, \bar{p}_2)])(\partial h_2/\partial p_2(\bar{p}_1, \bar{p}_2)[1 + H_{22}(\bar{p}_1, \bar{p}_2)])}.$$

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[10] Hotelling (1929) has studied a particular version of this model where the individual demands are given by $q_1 = \alpha_1 + \delta_1$, $q_2 = \alpha_2 + \delta_2$ with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\delta_1 = \frac{1}{2}[\lambda - \alpha_1 - \alpha_2 + (p_2 - p_1)/\gamma]$, $\delta_2 = \frac{1}{2}[\lambda - \alpha_1 - \alpha_2 + (p_1 - p_2)/\gamma]$, $\gamma > 0$, $\lambda > 0$.

and

$$\frac{\partial V}{\partial p_2}(p_1, p_2) = h(p_1 + p_2) + p_2 \frac{dh}{dp}(p_1, p_2) = 0.$$

Let us focus on symmetrical equilibria: $p_1 = p_2 = p$. The above equations provide a unique of these equilibria $(\bar{p}, \bar{p})$ which is such that

$$\bar{p} = -h(o) \Big/ \frac{dh}{dp}(o)$$

We compute

$$\frac{\partial^2 U}{\partial p_1{}^2}(\bar{p}, p) = \frac{\partial^2 V}{\partial p_2{}^2}(\bar{p}, \bar{p}) = \bar{p}\frac{d^2h}{dp^2}(o) - 2\frac{dh}{dp}(o) < 0$$

and

$$\frac{\partial^2 U}{\partial p_1 \partial p_2}(\bar{p}, \bar{p}) = \frac{\partial^2 V}{\partial p_1 \partial p_2}(\bar{p}, \bar{p}) = \bar{p}\frac{d^2h}{dp^2}(o) + \frac{dh}{dp}(o).$$

We get

$$\rho = \frac{\left(\bar{p}\dfrac{d^2h}{dp^2}(o) - \dfrac{dh}{dp}(o)\right)^2}{\left(\bar{p}\dfrac{d^2h}{dp^2}(o) - 2\dfrac{dh}{dp}(o)\right)^2}.$$

According to the theorem, the Cournot equilibrium $(\bar{p}, \bar{p})$ is locally improvable upon when $\rho > \frac{1}{4}$ which implies here

$$\left|\bar{p}\frac{d^2h}{dp^2}(o) - 2\frac{dh}{dp}(o)\right| < 2\left|\bar{p}\frac{d^2h}{dp^2}(o) - \frac{dh}{dp}(o)\right|.$$

A discussion similar to the one given above would lead to the conclusion that $\rho > \frac{1}{4}$ if and only if

$$\bar{p}\frac{d^2h}{dp^2}(o) < 0 \qquad \text{or} \qquad \frac{4}{3}\frac{dh}{dp}(o) < \bar{p}\frac{d^2h}{dp^2}(o) < 2\frac{dh}{dp}(o). \tag{24}$$

On the other hand, the equilibrium $(\bar{p}, \bar{p})$ is not locally improvable upon when

$$0 < \bar{p}\frac{d^2h}{dp^2}(o) < \frac{4}{3}\frac{dh}{dp}(o). \tag{25}$$

Let now the *price-elasticity of the marginal receipt function* be the function h from $\mathbb{R}_+$ to $\mathbb{R}$ defined as

$$H \overset{\text{def}}{=} \frac{d^2h}{dp^2} \cdot \left(p \Big/ \frac{dh}{dp}\right)$$

According to condition (24), *the equilibrium* $(\bar{p}, \bar{p})$ *is locally improvable upon when the market demand is such that*

$$H(\bar{p}) < 0 \qquad \text{or} \qquad \tfrac{4}{3} < H(\bar{p}) < 2.$$

Condition (25) implies that the equilibrium *is not locally improvable if we have*

$$0 < H(\bar{p}) < \tfrac{4}{3}H(\bar{p}).$$

One can see these conclusions are qualitatively similar to those obtained in the previous paragraphs: the possibility (or impossibility) of improving upon the Cournot equilibrium depends upon a measure of strategic interdependences in terms of the value—at the equilibrium—of the elasticity of the marginal receipt function.[11]

4. PROOF OF THE THEOREM

Step 1. We prove first assertion (i). Define

$$\phi(x, y) = u(x, y) - u(\bar{x}, y); \qquad \psi(x, y) = v(x, y) - v(x, \bar{y}).$$

In view of (4) we have

$$\frac{\partial\phi}{\partial x}(\bar{x}, \bar{y}) = \frac{\partial\phi}{\partial y}(\bar{x}, \bar{y}) = \frac{\partial\psi}{\partial x}(\bar{x}, \bar{y}) = \frac{\partial\psi}{\partial y}(\bar{x}, \bar{y}) = 0.$$

Moreover $\phi(\bar{x}, \bar{y}) = \psi(\bar{x}, \bar{y}) = 0$.

Then for every positive numbers λ, μ, the function $\lambda\phi + \mu\psi$ has zero value and zero gradient at $(\bar{x}, \bar{y})$. Let us compute the Hessian at $(\bar{x}, \bar{y})$ (second-order derivative) of $\lambda\phi + \mu\psi$:

$$D^2(\lambda\phi + \mu\psi) = \lambda D^2\phi + \mu D^2\psi = \begin{bmatrix} \lambda \dfrac{\partial^2 u}{\partial x^2}, & \lambda \dfrac{\partial^2 u}{\partial x\,\partial y} + \mu \dfrac{\partial^2 v}{\partial x\,\partial y} \\ \lambda \dfrac{\partial^2 u}{\partial x\,\partial y} + \mu \dfrac{\partial^2 v}{\partial x\,\partial y}, & \mu \dfrac{\partial^2 v}{\partial y^2} \end{bmatrix}.$$

At point $(\bar{x}, \bar{y})$ we have $\lambda\, \partial^2 u/\partial x^2\, (\bar{x}, \bar{y}) < 0$, $\mu\, \partial^2 v/\partial y^2\, (\bar{x}, \bar{y}) < 0$ (remember that $\lambda > 0$ and $\mu > 0$). Therefore the Hessian $D^2(\lambda\phi + \mu\psi)$ will be negative at $(\bar{x}, \bar{y})$ if and only if at point $(\bar{x}, \bar{y})$

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[11] One can see that in the Hotelling case (footnote above) the Cournot equilibrium has to be given by $\bar{p}_1 = \gamma[\lambda + (\alpha_1 - \alpha_2)/3]$ and $\bar{p}_2 = \gamma[\lambda - (\alpha_1 - \alpha_2)/3]$. We have $d^2h/dp^2 = 0$, the market demand is (and all individual demands are) inelastic with respect to price, and we cannot conclude since $H(\bar{p}) = 0$. However the argument of Moulin [8, 9] leads to the conclusion that the equilibrium is not improvable upon (globally or locally).

$$\lambda\mu \cdot \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 v}{\partial y^2} > \left(\lambda \frac{\partial^2 u}{\partial x\, \partial y} + \mu \frac{\partial^2 v}{\partial x\, \partial y}\right)^2 \tag{26}$$

(all these derivative being taken at $(\bar{x}, \bar{y})$).

Let us set.

$$\alpha = \frac{\partial^2 u}{\partial x^2}(\bar{x}, \bar{y}) \cdot \frac{\partial^2 v}{\partial y^2}(\bar{x}, \bar{y}) > 0,$$

$$a = \frac{\partial^2 u}{\partial x\, \partial y}(\bar{x}, \bar{y}) \qquad b = \frac{\partial^2 v}{\partial x\, \partial y}(\bar{x}, \bar{y}).$$

As an elementary lemma we claim

$$(\exists \lambda, \mu > 0, (\lambda a + \mu b)^2 - \alpha\lambda\mu < 0) \Leftrightarrow (ab)/\alpha < \tfrac{1}{4}$$

(we leave the proof to the reader: remember that $\alpha > 0$).

Remark that $\rho = (ab)/\alpha$. Then, assumption $\rho < \frac{1}{4}$ implies that for some positive λ, μ the Hessian of $\lambda\phi + \mu\psi$ is definite negative at $(\bar{x}, \bar{y})$. Then $(\bar{x}, \bar{y})$ is a (strict) local maximum of $\lambda\phi + \mu\psi$ thus proving the existence of an open neighborhood U of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$\forall (x, y) \in U : \lambda\phi(x, y) + \mu\psi(x, y) \leq 0. \tag{27}$$

Suppose now that $(\bar{x}, \bar{y})$ is locally improvable and let:

$$t \to L(t) = \sum_{i=1}^{n} \alpha_i(t)\, \delta_{(x_i(t), y_i(t))}$$

be the associated path of strict correlated equilibriums (see Definition 3). By the continuity of $t \to x_i(t)$ and $t \to y_i(t)$, for t small enough, the carrier of $L(t)$: $\{(x_i(t), y_i(t))\}_{i=1,\ldots,n}$ is contained in U.

The two preceding relations yield, for t small enough

$$[L(t), \lambda\phi + \mu\psi] = \lambda[L(t), \phi] + \mu[L(t), \psi] \leq 0. \tag{28}$$

On the other hand because $L(t)$ is a strict correlated equilibrium, for every nonzero t, we have

$$\begin{aligned} &[L(t), u] > u(\bar{x}, L_Y), \\ &[L(t), v] > v(L_X, \bar{y}). \end{aligned} \tag{29}$$

Clearly we have

$$[L(t), \phi] = [L(t), u - u(\bar{x}, \cdot)] = [L(t), u] - u(\bar{x}, L_Y).$$

Similary

$$[L(t), \psi] = [L(t), v] = v(L_X, \bar{y}).$$

Thus inequalities (28) and (29) yield a contradiction, and $(\bar{x}, \bar{y})$ cannot be locally improvable upon.

Step 2. Let a, b, a', b', be four real numbers. Because $(\bar{x}, \bar{y})$ is an interior element of $X \times Y$, then for t small enough the following $L(t)$:

$$L(t) = \tfrac{1}{2}\delta_{(x+ta, y+tb)} + \tfrac{1}{2}\delta_{(x+ta', y+tb')} \tag{30}$$

is a lottery on $X \times Y$. Moreover its carrier varies continuously with respect to t, and $L(0) = \delta_{(\bar{x}, \bar{y})}$. We now determine a, b, a', b' so that $t \to L(t)$ verifies the remaining properties of Definition 3 for t small enough. This clearly implies property (ii) of the theorem.

We compute

$$\begin{aligned} 2\frac{d}{dt}([L(t), u])_{t=0} &= \frac{d}{dt}(u(\bar{x} + ta, \bar{y} + tb) + u(\bar{x} + ta', \bar{y} + tb'))_{t=0} \\ &= \frac{\partial u}{\partial y}(\bar{x}, \bar{y}) \cdot (b + b'). \end{aligned} \tag{31}$$

Similary.

$$2\frac{d}{dt}([L(t), v])_{t=0} = \frac{\partial v}{\partial x}(\bar{x}, \bar{y}) \cdot (a + a').$$

Therefore $t \to [L(t), u]$ and $t \to [L(t), v]$ are strictly increasing for t small enough if we have

$$\begin{aligned} &\frac{\partial u}{\partial y}(\bar{x}, \bar{y}) \cdot (b + b') > 0, \\ &\frac{\partial v}{\partial x}(\bar{x}, \bar{y}) \cdot (a + a') > 0. \end{aligned} \tag{32}$$

We set now

$$\begin{aligned} \alpha(t, x) = [L(t), u] - u(x, L_Y(t)) &= \tfrac{1}{2}[u(\bar{x} + ta, \bar{y} + tb) - u(x, \bar{y} + tb)] \\ &\quad + \tfrac{1}{2}[u(\bar{x} + ta', \bar{y} + tb') - u(x, \bar{y} + tb')]. \end{aligned}$$

Thus $\alpha(t, x)$ is defined for $x \in X$ and every t such that $L(t)$ is a lottery on $X \times Y$. Clearly $\alpha(0, \bar{x}) = 0$. Now an easy computation gives

$$\frac{\partial \alpha}{\partial t}(0, \bar{x}) = \frac{\partial \alpha}{\partial x}(0, \bar{x}) = 0,$$

$$\frac{\partial^2 \alpha}{\partial x^2}(0, \bar{x}) = -\frac{\partial^2 u}{\partial x^2}(\bar{x}, \bar{y}),$$

$$\frac{\partial^2 \alpha}{\partial t\,\partial x}(0, \bar{x}) = \frac{1}{2}\frac{\partial^2 u}{\partial x\,\partial y}(\bar{x}, \bar{y})\cdot(b + b'),$$

$$\frac{\partial^2 \alpha}{\partial t^2}(0, \bar{x}) = \frac{1}{2}\left[(a^2 + a'^2)\frac{\partial^2 u}{\partial x^2}(\bar{x}, \bar{y}) + 2(ab + a'b')\cdot\frac{\partial^2 u}{\partial x\,\partial y}(\bar{x}, \bar{y})\right].$$

Inview of assumption (3): $\partial^2\alpha/\partial x^2(0, \bar{x}) > 0$. Thus, the pair $(0, \bar{x})$ is a local strict minimum of α if we have

$$\frac{\partial^2 \alpha}{\partial x^2}\cdot\frac{\partial^2 \alpha}{\partial t^2} > \left(\frac{\partial^2 \alpha}{\partial t\,\partial x}\right)^2 \tag{33}$$

(all these derivatives being taken at $(0, \bar{x})$). (Remark that inequality (33) implies $\partial^2\alpha/\partial t^2 > 0$).

In view of the above computation we obtain that condition (33) is equivalent to

$$(a^2 + a'^2)\frac{\partial^2 u}{\partial x^2} + 2(ab + a'b')\frac{\partial^2 u}{\partial x\,\partial y} + \frac{1}{2}(b + b')^2\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2\Big/\frac{\partial^2 u}{\partial x^2} > 0$$

with all derivatives taken at $(\bar{x}, \bar{y})$.

On the other hand, if $(0, \bar{x})$ is a local strict minimum of α we may find an open neighborhood V of $\bar{x}$ in X and a number $t_0 > 0$ such that

$$\forall t : 0 < t < t_0, \qquad \inf_{x\in V}\alpha(t, x) > 0. \tag{35}$$

By assumption (2), we have

$$\forall x \in X, \qquad x \notin V : u(x, \bar{y}) < u(\bar{x}, \bar{y}).$$

Since the complement of V in X is compact and since u is continuous there exists a positive $\eta(\eta > 0)$ such that:

$$\forall x \in X, \qquad x \notin V : u(x, \bar{y}) \leq u(\bar{x}, \bar{y}) - \eta. \tag{36}$$

Because of the continuity of the carrier of $L(t)$ with respect to t, we deduce:

$$\lim_{t\to 0}\,[L(t), u] = u(\bar{x}, \bar{y}),$$

$$\forall x \in X, \qquad \lim_{t\to 0} u(x, L_Y(t)) = u(x, \bar{y}).$$

In view of (12) we get the existence of a number $t_1 > 0$ such that

$$\forall t : 0 < t < t_1, \qquad \sup_{\substack{x\in X\\ x\notin V}} u(x, L_Y(t)) < [L(t), u]. \tag{37}$$

On the other hand, the definition of $\alpha(x, t)$ implies that (35) is equivalent to

$$\forall t : 0 < t < t_0, \qquad \sup_{x \in X} u(x, L_Y(t)) < [L(t), u]. \tag{38}$$

Comparing properties (37) and (38), we obtain that for some $t_2 > 0$,

$$\forall t : 0 < t < t_2, \qquad \sup_{x \in X} u(x, L_Y(t)) < [L(t), u]. \tag{39}$$

Thus we have just proved that inequality (34) implies the existence of t_2 such that (39) holds true. A symetrical proof gives that: if the following inequality holds true

$$\frac{1}{2}(a + a')^2 \left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2 \Big/ \frac{\partial^2 v}{\partial y^2} + 2(ab + a'b')\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial y^2}(b^2 + b'^2) > 0 \tag{40}$$

(with all derivatives being taken at $(\bar{x}, \bar{y})$) then there exists $t_3 > 0$ such that

$$\forall t : 0 < t < t_3, \sup_{y \in Y} v(L_X(t), y) < [L(t), v]. \tag{41}$$

Properties (39) and (41) together mean that $L(t)$ is a strict correlated equilibrium for t small enough.

This is exactly property (iii) of the Definition (up to a suitable normalization of t). The conclusion of step 2 is the following : if we can find a, b, a', b', such that (32), (39), and (41) hold true then the lottery $L(t)$ given by (30) achieves a local improvement of $(\bar{x}, \bar{y})$ of the required type.

Step 3. It remains to be shown that $\rho > \frac{1}{4}$ implies the existence of a solution a, b, a', b' of system (32), (39), and (41).

Let

$$\epsilon = \operatorname{sign}\left(\frac{\partial u}{\partial y}(\bar{x}, \bar{y})\right),$$

$$\epsilon' = \operatorname{sign}\left(\frac{\partial v}{\partial x}(\bar{x}, \bar{y})\right),$$

$$\sigma = -\frac{\partial^2 u}{\partial x\,\partial y}\Big/\frac{\partial^2 u}{\partial x^2}(\bar{x}, \bar{y}),$$

$$\epsilon'' = \operatorname{sign}(\sigma),$$

$$\sigma' = \frac{\partial^2 v}{\partial x\,\partial y}\Big/-\frac{\partial^2 v}{\partial y^2}(\bar{x}, \bar{y}).$$

These quantities are well defined by assumptions (3) and (5). System (32), (39), and (41) can be reformulated as:

$$\begin{gathered}\epsilon \cdot (b + b') > 0, \qquad \epsilon' \cdot (a + a') > 0, \\ (a^2 + a'^2) + 2(ab + a'b')\,\sigma + \tfrac{1}{2}(b + b')\,\sigma^2 < 0, \\ \tfrac{1}{2}(a + a')^2\,\sigma'^2 + 2(ab + a'b')\,\sigma' + (b^2 + b'^2) < 0.\end{gathered}$$

We must prove the existence of a solution as soon as $\rho = \sigma \cdot \sigma' > \frac{1}{4}$.

We set

$$\tilde{a} = \frac{a}{\sigma}, \qquad \tilde{a}' = \frac{a'}{\sigma}.$$

The above system is then equivalent to

$$\begin{gathered}\epsilon \cdot (b + b') > 0, \qquad \epsilon' \cdot \sigma \cdot (\tilde{a} + \tilde{a}') > 0, \\ (\tilde{a}^2 + \tilde{a}'^2) + 2(\tilde{a}b + \tilde{a}'b') + \tfrac{1}{2}(b + b')^2 < 0, \\ \tfrac{1}{2}(\tilde{a} + \tilde{a}')^2\,\rho^2 + 2(\tilde{a}b + \tilde{a}'b')\,\rho + (b^2 + b'^2) < 0.\end{gathered} \tag{42}$$

We seek for solutions of the above system of the type

$$\begin{aligned}\tilde{a} &= \frac{\epsilon' \cdot \epsilon''}{2} + \lambda, & b &= \frac{\epsilon}{2} + \mu, \\ \tilde{a}' &= \frac{\epsilon' \cdot \epsilon''}{2} - \lambda, & b' &= \frac{\epsilon}{2} - \mu.\end{aligned} \tag{43}$$

In order that the above numbers verify system (42) it is clearly sufficient that

$$\begin{gathered}\frac{1}{2} + 2\lambda^2 + 2\left(\frac{\epsilon\epsilon'\epsilon''}{2} + 2\lambda\mu\right) + \frac{1}{2} < 0, \\ \frac{\rho^2}{2} + 2\left(\frac{\epsilon\epsilon'\epsilon''}{2} + 2\lambda\mu\right) \cdot \rho + \frac{1}{2} + 2\mu^2 < 0.\end{gathered} \tag{44}$$

We distinguish two cases

Case 1. $\epsilon\epsilon'\epsilon'' = -1$. System (44) becomes

$$\lambda^2 + 2\lambda\mu < 0 \tag{45}$$

and

$$\frac{(\rho - 1)^2}{4} + 2\rho\lambda\mu + \mu^2 < 0. \tag{46}$$

We choose an arbitrary $\mu > 0$ and $\lambda = \lambda(\mu)$

$$\lambda = -\frac{1}{2\rho\mu}\left(\mu^2 + \frac{(\rho - 1)^2}{4} + 1\right).$$

Clearly inequality (46) holds true for every $\mu > 0$. Let us show that for μ large enough, (45) is also true. Namely when μ goes to infinity, we have

$$\lambda(\mu) \sim -\frac{\mu}{2\rho}$$

so that $\lambda(\mu)$ is negative for μ large enough. Moreover we have

$$\lambda(\mu) + 2\mu \sim -\frac{\mu}{2\rho} + 2\mu = 2\left(1 - \frac{1}{4\rho}\right)\mu.$$

By assumption: $1 - 1/4\rho > 0$. Hence, for μ large enough $\lambda^2 + 2\lambda\mu$ is positive. Thus $(\lambda(\mu), \mu)$ verifies (45) and (46) for μ large enough. This provides the required solution of system (42) by means of formulas (43).

Case 2. $\epsilon\epsilon'\epsilon'' = +1$. System (44) becomes

$$\lambda^2 + 2\lambda\mu + 1 < 0,$$

$$\frac{(\rho + 1)^2}{4} + 2\rho\lambda\mu + \mu^2 < 0.$$

We choose $\mu > 0$ and

$$\lambda(\mu) = -\frac{1}{2\rho\mu}\left(\mu^2 + \frac{(\rho + 1)^2}{4} + 1\right).$$

An argument similar to case 1 shows that $(\lambda(\mu), \mu)$ is a solution of the above system for μ large enough.

This concludes the proof of the theorem.

CONCLUSION

This paper provides a characterization of those two-person games on real intervals in which a pure noncooperative equilibrium is locally improvable upon by correlation. In duopoly games the correlated equilibrium stands for a tacit collusive agreement allowing both firms to be better off than at the equilibrium.

We have restricted ourselves to those duopoly models for which there exist a pure Cournot equilibrium. The analysis could be extended to mixed equilibria. It is well known [15] that only mixed equilibria do exist for several price setting homogeneous duopolies where different firms cannot sell in the same market at different prices. Such equilibria are usually interpreted as representing some price instability on the market. It is easy to see that in the present context a mixed equilibrium is only a degenerated form of a correlated equilibrium. This gives a new outlook on mixed equilibria. Indeed, we may consider any equilibrium as a pair of prices such that if each firm maximizes its (expected) profit given the information it has about its rivals price policy, than it will not have any incentive to follow another price policy. In case of a correlated equilibrium the two firms share some information instead of at a mixed equilibrium where they only have their own information.

Tacit cooperation as we have defined it in terms of correlation coincides with only one of all forms of oligopolistic cooperation which can be observed. Assuming as we did a tendency in an oligopoly toward the maximization of collective profits, it remains that different processes of coordination and communication may operate to resolve conflicts. Sherer [14] points out that price (or output) leaderships, or such rules of behavior as those supporting the "kinked demand" approach, are different forms of oligopolistic coordination. One could consider the lotteries on two pure outcomes that we have introduced here as defining an "alternate leadership." However, a more detailed study of the coordination processes such as leadership is needed to take account how exchanges of information are organized between the firms. This, in turn, involves such phenomena as "threats" and "commitments." As a matter of fact, tacit cooperation in terms of correlation and collusion through threats and commitments may actually be considered as dual approaches[12] for the general oligopolistic coordination problem.

REFERENCES

1. R. J. AUMANN, Subjectivity and correlation in randomized strategies, *J. Math. Econ.* **1** (1974), 67–96.
2. R. L. BISHOP, Elasticities, cross-elasticities, and market relationships, *Amer. Econ. Rev.* **13** (1952), 779–803.
3. F. H. CHAMBERLIN, "The Theory of Monopolistic Competition," Harvard Univ. Press, Cambridge, Mass., 1933.
4. A. A. COURNOT, "Recherches sur les principes mathématiques de la théorie des richesses," Hachette, Paris, 1838; English translation: "Researches into the Mathematical Principles of the Theory of Wealth," Macmillan Co. New York, 1897.
5. W. FELLNER, "Competition Among the Few," A. Kelley, New York, 1949.
6. L. A. GERARD-VARET, "Non-cooperative Equilibrium Concepts for Oligopoly Theory," Technical Report, No. 192, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1975.
7. J. C. HARSANYI, "A General Solution for Finite Non cooperative Games Based on Risk-Dominance," Annals of Mathematical Studies No 52, Princeton Univ. Press, Princeton, N.J., 1963.

[12] See Moulin [10], where it is shown that in an homogeneous quantity-setting duopoly with linear demand and quadratic costs, there exists a collective efficient outcome which is stable in terms of mutual threats when the marginal costs are increasing, i.e., when the equilibrium is not locally improvable through correlation.

8. H. Moulin, Cooperation in mixed equilibrium, *Math. Operation Res.* 1 (1976).
9. H. Moulin, "Correlated and Prudent Equilibria of a 2 Person Game," Cahiers de Mathématiques de la Décision, No. 7605, 1976.
10. H. Moulin, "Deterrence and Cooperation. A Classification of Two-Person Games," E.N.S.A.E., Jan. 1977.
11. H. Moulin and J. P. Vial, Strategically sero-sum games, *Int. J. Game Theory*, in press.
12. R. W. Rosenthal, Correlated equilibria in some classes of two-person games. *Int. J. of Game Theory* 3 (1974), 119-128.
13. T. C. Schelling, "The Strategy of Conflict," 4th ed., Havard Univ. Press, Cambridge, Mass., 1970.
14. F. M. Sherer, "Industrial Pricing. Theory and Evidence," Rand MacNally, Chicago, 1970.
15. M. Shubik, "Strategy and Market Structure," Wiley, New York, 1959.
16. R. Triffin, "Monopolistic Competition and General Equilibrium Theory," Harvard Univ. Press, Cambridge, Mass., 1940.

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Measuring the Technical Efficiency of Production*

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I. INTRODUCTION

Production technology is commonly modelled by means of a production function, which in the scalar output case specifies the maximum output obtainable from an input vector. The degree to which the actual output of a production unit approaches its maximum is called the technical efficiency of production. A technically efficient unit must operate on its production function, although this condition is not sufficient; a technically inefficient unit may operate beneath its production function, although this condition is not necessary. If the notion of technical efficiency is to have empirical content, it must be based on a proper measure, or index, of the technical efficiency of a production unit. It is the purpose of this paper to specify a set of properties such an efficiency measure should satisfy, to show that the widely used measure proposed by Farrell [4] does not satisfy these properties, and to introduce a new measure that does satisfy these properties.

The pioneering work on technical efficiency is that of Farrell. Inspired by the work of Debreu [2] and Koopmans [7], Farrell suggested a measure of technical efficiency that can be interpreted, somewhat loosely for the moment, in either of two ways: as the ratio of technically minimal to actual inputs, given output and the input mix, or as the ratio of actual to technically

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maximal output, given inputs. Virtually all subsequent work on technical efficiency is based to some extent on Farrell's seminal work.[1]

There are, however, some difficulties with Farrall's measure. First, as is already noted, it is open to two interpretations, which except under the most favorable of circumstances give rise to different inferences concerning technical efficiency. Second, it measures technical efficiency relative to an isoquant rather than to an efficient subset, a practice that can lead to the identification of a unit as being technically efficient when it is not. Third, it is a radial measure, in the sense that it assumes a given input mix, although there is no reason to measure technical efficiency radially, even for homothetic technologies. In fact, such a practice is likely to produce erroneous inferences concerning technical efficiency. Finally Farrell's restrictive assumptions on the production function severely limit the types of technology to which his measure can be applied.

The plan of the paper is as follows. In Section 2 we describe our technological assumptions, which are considerably less restrictive than those of Farrell, and we introduce some useful terminology. In Section 3 we investigate the two interpretations of the Farrell measure. In Section 4 we propose a set of properties that an efficiency measure should satisfy, and we introduce a new measure that satisfies these properties under our weak technological assumptions. Concluding remarks follow in Section 5.

2. THE PRODUCTION TECHNOLOGY

A production technology transforming inputs $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ into net output $u \in R_+$ is modelled by a production function $\phi\colon R_+^n \to R_+$ or inversely by an input correspondence $u \to L(u) \subset 2^{R_+^n}$, where $\phi(x)$ is the maximal output obtainable from an input vector x and $L(u)$ denotes all input vectors yielding at least output rate u. The production function ϕ and the input correspondence L are inversely related by

$$L(u) := \{x \mid \phi(x) \geqq u\} \qquad \text{and} \qquad \phi(x) := \max\{u \mid x \in L(u)\}.$$

The production function ϕ is assumed to have the following properties:[2]

$\phi.1$ $\phi(0) = 0$ and $\phi(x) > 0$ for some $x > 0$;
$\phi.2$ $\phi(x)$ is finite for x bounded;
$\phi.3$ $\phi(\lambda \cdot x) \geqq \phi(x)$ for $\lambda \geqq 1$;
$\phi.4$ If $\phi(\lambda \cdot x) > 0$, then $\phi(\lambda \cdot x) \to +\infty$ as $\lambda \to +\infty$;
$\phi.5$ ϕ is upper semi-continuous.

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[1] Recent empirical studies that make use of Farrell's approach to the measurement of technical efficiency include Aigner, Lovell and Schmidt [1], Førsund and Jansen [6], Meeusen and van den Broeck [9], and Schmidt and Lovell [10]. An extensive bibliography is available from the authors on request.

[2] For a discussion of the significance of these properties see, e.g., Shephard and Färe [13].

In addition to the production function ϕ and the input correspondence L the following two subsets of $L(u)$ are important:

The isoquant: $\text{Isoq}\, L(u) := \{x \mid x \in L(u),\ \lambda \cdot x \notin L(u),\ \lambda \in [0, 1)\}$ for $u > 0$, and $\text{Isoq}\, L(0) := 0$.

The efficient subset: $\text{Eff}\, L(u) := \{x \mid x \in L(u),\ y \leq x \Rightarrow y \notin L(u)\}$ for $u > 0$, and $\text{Eff}\, L(0) := 0$.[3]

Clearly the efficient subset of the input set $L(u)$ is a subset of the isoquant for each output rate u, but the converse is in general not true. Although the popular constant elasticity of substitution (CES) family of production functions with strictly positive elasticity of substitution satisfies this property, the Leontief function and subfamilies of variable elasticity of substitution (VES) and weak input disposability (WDI) functions are examples of production functions whose isoquants are not contained in their efficient subsets (see [3], [8]).

The distance function defined by

$$\psi(u, x) := [\min\{\lambda \mid \lambda \cdot x \in L(u)\}]^{-1}$$

is also of importance. This function, introduced by Shephard [11], [12], fully characterizes the production technology, and in particular

$$\begin{aligned} L(u) &= \{x \mid \psi(u, x) \geqq 1\}, \\ \text{Isoq}\, L(u) &= \{x \mid \psi(u, x) = 1\}, \\ \text{Eff}\, L(u) &= \{x \mid \psi(u, x) = 1,\ \psi(u, y) < 1,\ \forall y \leq x\}. \end{aligned}$$

3. THE FARRELL MEASURES OF TECHNICAL EFFICIENCY

In his pathbreaking study of production efficiency, Farrell did not state a formal definition of what we have called "the Farrell measure" of the technical efficiency of a production unit. Indeed his analysis leads quite easily to two different measures of technical efficiency. Although under the strong technological assumptions maintained by Farrell the two measures are equivalent, within the more general framework of $\phi.1$–$\phi.5$ for a production technology they typically assign different efficiency values to the same observation.[4] For this reason two definitions are offered.

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[3] $y \leq x$ means $y \leqq x$ but $y \neq x$.

[4] The two interpretations follow from a reading of Farrell's analysis at [4; 254, 258]. Farrell's technological assumptions include strong free disposability of inputs ($x' \geqq x \Rightarrow \phi(x') \geqq \phi(x)$) and constant returns to scale. It should be noted that Farrell was fully aware of the divergence between his two measures in the absence of constant returns to scale. Førsund and Hjalmarsson [5] have also noticed this divergence.

DEFINITION 1. The function $\mathscr{F}: R_{++} \times R_+^n \to R_+$ defined by $\mathscr{F}(u, x) := \min\{\lambda \mid \lambda \cdot x \in L(u)\}$, $x \in L(u)$ and $\mathscr{F}(u, x) = +\infty$, $x \notin L(u)$, for $u > 0$, is called the Farrell input measure of technical efficiency.

$\mathscr{F}(u, x)$ is a radial measure of technical efficiency; it gives the maximum amount by which an input vector x yielding the output rate u can be proportionally decreased while still producing u.[5] It is clear that the Farrell input measure is the inverse of the distance function for $x \in L(u)$, and that $\mathscr{F}(u, x) \in (0, 1]$ for $u > 0$ and $x \in R_+^n$. It is also true that $\{x \mid \mathscr{F}(u, x) = 1\} = \text{Isoq } L(u)$, so that an observation (u, x) is judged efficient by the Farrell input measure if and only if $x \in \text{Isoq } L(u)$.

An important implication of Definition 1 is that while $x \in \text{Eff } L(u)$ implies $\mathscr{F}(u, x) = 1$, the converse is not true. This is clear from the following example. Let

$$\phi(x_1, x_2) = \min\{x_1, x_2\} \tag{1}$$

be the production function. Then $(x_1, x_2) = (1, 2) \in L(1)$ and $\mathscr{F}(1, 1, 2) = 1$, but $(1, 2) \notin \text{Eff } L(1)$ because $(1, 2) \geq (1, 1) \in L(1)$. Thus the observation $(1, 1, 2)$ is labeled technically efficient by the Farrell input measure even though it does not belong to the efficient subset. Consequently it is of interest to determine the restrictions on the production structure under which the set $\{x \mid \mathscr{F}(u, x) = 1\}$ coincides with the efficient subset $\text{Eff } L(u)$, and hence when the Farrell input measure relates the efficiency of $x \in L(u)$ to efficient elements of the input set. The conditions for this to hold are obvious from the above definitions:

PROPOSITION 1. $\{x \mid \mathscr{F}(u, x) = 1\} = \text{Eff } L(u)$ *if and only if* $\text{Isoq } L(u) = \text{Eff } L(u)$, $u > 0$.

The gist of this proposition is that unless $\text{Isoq } L(u) = \text{Eff } L(u)$, the Farrell input measure may term an input vector efficient even when it is not. As pointed out above, the popular CES family of production functions satisfies the requirement of Proposition 1, provided the elasticity of substitution is strictly positive. But there are important classes of production functions that do not. For this reason a generalization of Farrell's input measure is introduced in the next section. But before pursuing that issue consider Farrell's second efficiency measure:

DEFINITION 2. The function $\mathscr{G}: R_{++} \times R_+^n \to R_+$ defined as $\mathscr{G}(u, x) := \min\{\theta \mid x \in L(u/\theta)\}$, $x \in L(u)$, and $\mathscr{G}(u, x) = +\infty$, $x \notin L(u)$, for $u > 0$, is called the Farrell output measure of technical efficiency.

$\mathscr{G}(u, x)$ is also a radial measure of efficiency; it gives the maximum amount by which output rate u can be increased and still be producible by input vector x.[6] For $x \in L(u)$ it is clear that $\mathscr{G}(u, x) \in (0, 1]$. It is also of interest to note that $\mathscr{G}(u, x)$ is the inverse of the output distance function, for $x \in L(u)$, defined by (see [12])

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[5] An equivalent way of expressing the Farrell input measure is $\mathscr{F}(u, x) := \|\lambda^0 \cdot x\| / \|x\|$, where $\lambda^0 = \min\{\lambda \mid \lambda \cdot x \in L(u)\}$, $x \in L(u)$, $u > 0$. This clearly shows $\mathscr{F}(u, x)$ to be the ratio of two distances computed along the same ray.

$$\Omega(u, x) := [\max\{\theta \mid \theta \cdot u \in P(x)\}]^{-1},$$

where $P(x) := \{u \mid x \in L(u)\}$, i.e., $x \to P(x)$ is the output correspondence inverse to the input correspondence $u \to L(u)$. Here, where output is a scalar, $P(x) = [0, \phi(x)]$. To show that $\mathscr{G}(u, x)$ is the inverse of $\Omega(u, x)$, note that for $x \in L(u)$,

$$\begin{aligned} \mathscr{G}(u, x) &= \min\{\theta \mid x \in L(u/\theta)\} \\ &= \max\{\mu \mid x \in L(u \cdot \mu)\} \\ &= \max\{\mu \mid \mu \cdot u \in P(x) = [0, \phi(x)]\}, \\ &= [\Omega(u, x)]^{-1}. \end{aligned}$$

Clearly the importance of this observation is related to the multiproduct case.

To continue the investigation of Farrell's output measure, it is first noted from the earlier example (1), that $\mathscr{G}(u, x) = 1$ does not imply that $x \in \operatorname{Eff} L(u)$. In addition the next example shows that $x \in \operatorname{Eff} L(u)$ does not imply that $\mathscr{G}(u, x) = 1$. Let the input correspondence be of the form

$$L(u) = \begin{Bmatrix} (x_1, x_2) \mid x_1 + x_2 \geqslant u, u \in [0, 1) \\ (x_1, x_2) \mid u \cdot x_1 + x_2 \geqslant u, u \in [1, 2] \\ (x_1, x_2) \mid x_1 + x_2 \geqslant u, u \in (2, +\infty) \end{Bmatrix}. \tag{2}$$

From this example it is clear that the input vector $(1, 0) \in \operatorname{Eff} L(1)$ but

$$\mathscr{G}(1, 1, 0) = \min\{\theta \mid (1, 0) \in L(1/\theta)\} = 1/2,$$

showing that $x \in \operatorname{Eff} L(u)$ does not imply that $\mathscr{G}(u, x) = 1$. It should be noted that for the input correspondence (2), $\psi(u, 1, 0) = \psi(1, 1, 0)$ for all $u \in [1, 2]$. Thus $x \in \operatorname{Eff} L(u)$ is neither necessary nor sufficient for $\mathscr{G}(u, x) = 1$. The following proposition establishes conditions under which efficiency on the basis of the Farrell output measure and membership in the efficient subset coincide.

Proposition 2. $x \in \operatorname{Eff} L(u) \Leftrightarrow \mathscr{G}(u, x) = 1$ *if and only if for all* $x \in R_+^n$, $\psi(\theta \cdot u, x) < \psi(u, x)$, $\theta > 1$ *and* $\operatorname{Eff} L(u) = \operatorname{Isoq} L(u)$, $u > 0$.

Proof. (Necessity) If $x \in \operatorname{Eff} L(u)$, $u > 0$, then by assumption $x \in \operatorname{Isoq} L(u)$, and consequently $\psi(u, x) = 1$. Now by definition

$$\mathscr{G}(u, x) = \min\{\theta \mid \psi(u/\theta, x) = 1\},$$

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[6] An equivalent way of expressing the Farrell output measure as the ratio of two distances computed along the same ray is $\mathscr{G}(u, x) := \| u \| / \| u/\theta^0 \|$, where $\theta^0 = \min\{\theta \mid x \in L(u/\theta)\}$, $x \in L(u)$, $u > 0$.

and $\theta \leqslant 1$ since $x \in L(u)$. Assuming $\theta < 1$, by the first assumption $\psi(u/\theta, x) < \psi(u, x) = 1$, i.e., $x \notin L(u/\theta)$, implying that $\theta = 1$. Thus the conditions for all $x \in R_+^n$, $\psi(\theta \cdot u, x) < \psi(u, x)$, $\theta > 1$ and $\operatorname{Eff} L(u) = \operatorname{Isoq} L(u)$, $u > 0$, are sufficient for $x \in \operatorname{Eff} L(u) \Leftrightarrow \mathscr{G}(u, x) = 1$, $u > 0$.

(Sufficiency) Assume for all $x \in R_+^n$, $\psi(\theta \cdot u, x) < \psi(u, x)$, $\theta > 1$ holds but $\operatorname{Eff} L(u) \neq \operatorname{Isoq} L(u)$. Then there exists an $x \in \operatorname{Isoq} L(u)$, for some $u > 0$, such that $x \notin \operatorname{Eff} L(u)$. Note that $\operatorname{Eff} L(0) = \operatorname{Isoq} L(0)$ by definition and that $\operatorname{Eff} L(u) \subset \operatorname{Isoq} L(u)$ always. Since $x \in \operatorname{Isoq} L(u)$, $\psi(u, x) = 1$ and

$$\mathscr{G}(u, x) = \min\{\theta \mid \psi(u/\theta, x) = 1\} = 1$$

as a consequence of ψ being strictly increasing in u. Thus $\operatorname{Eff} L(u) = \operatorname{Isoq} L(u)$ is necessary.

Next assume that $\operatorname{Eff} L(u) = \operatorname{Isoq} L(u)$, $u > 0$, holds but that for all $x \in R_+^n$, $\psi(\theta \cdot u, x) < \psi(u, x)$, $\theta > 1$ does not. Then for some $(\theta \cdot u) > 0$, $\psi(u, x) = \psi(\theta \cdot u, x)$ with $x \in L(u)$. Now $(\lambda \cdot x) \in \operatorname{Eff} L(u) = \operatorname{Isoq} L(u)$ for some scalar $\lambda > 0$, and hence $\psi(u, \lambda \cdot x) = \psi(\theta \cdot u, \lambda \cdot x) = 1$ since the distance function is homogeneous of degree $+1$ in x. Consequently,

$$\mathscr{G}(u, x) = \min\{\mu \mid \psi(u/\mu, \lambda \cdot x) = 1\} < 1$$

and thus $x \in \operatorname{Eff} L(u)$, $u > 0$, does not imply $\mathscr{G}(u, x) = 1$, showing the necessity of the property $\operatorname{Eff} L(u) = \operatorname{Isoq} L(u)$ for $x \in \operatorname{Eff} L(u) \Leftrightarrow \mathscr{G}(u, x) = 1$. Q.E.D.

We note that the requirements of Proposition 2 on $\mathscr{G}(u, x)$ are more stringent than the requirements of Proposition 1 on $\mathscr{F}(u, x)$, although the CES family of production functions with strictly positive elasticity of substitution does satisfy both sets of requirements.

We now turn to a comparison of the two Farrell efficiency measures. We asserted above that in general $\mathscr{F}(u, x)$ and $\mathscr{G}(u, x)$ do not assign the same efficiency value to a particular observation (u, x). However if ϕ is homogeneous of degree $+1$,

$$\begin{aligned} \mathscr{F}(u, x) &= \min\{\lambda \mid \lambda \cdot x \in L(u)\} \\ &= \min\{\lambda \mid x \in L(u/\lambda)\} \\ &= \mathscr{G}(u, x). \end{aligned}$$

Thus for constant returns to scale production technologies the two Farrell measures coincide.

Conversely assume that $\mathscr{F}(u, x) = \mathscr{G}(u, x)$ for $u > 0$ and $x \in L(u)$. Then from the relationships between Farrell's two measures and the input and output distance functions it follows that $\psi(u, x) = \Omega(x, u)$. Moreover, since both distance functions are homogeneous of degree +1 in their second subvector, it follows that ψ is homogeneous of degree +1 in u. This gives

$$\begin{aligned}\phi(\lambda\cdot x) &= \max\{u \mid \lambda\cdot x \in L(u)\}\\ &= \max\{u \mid \psi(u, \lambda\cdot x) \geqslant 1\}\\ &= \max\{u \mid \psi(u/\lambda, x) \geqslant 1\}\\ &= \lambda\cdot\max\{v \mid \psi(v, x) \geqslant 1\} \quad (v = u/\lambda)\\ &= \lambda\cdot\phi(x),\end{aligned}$$

proving

PROPOSITION 3. *The Farrell input and output measures assign the same efficiency value for* (u, x), $u > 0$, $x \in L(u)$, *if and only if the production function* ϕ *is homogeneous of degree* +1.

To continue the comparison of the two measures of efficiency introduce

DEFINITION 3. The input correspondence $u \to L(u)$ is sub-homogeneous (for increases in output) if $L(\theta\cdot u) \subset \theta\cdot L(u)$, $\theta \geqslant 1$.

DEFINITION 4. The input correspondence $u \to L(u)$ is super-homogeneous (for increases in output) if $L(\theta\cdot u) \supset \theta\cdot L(u)$, $\theta \geqslant 1$.
The first of these concepts is commonly used to describe non-increasing returns to scale, while the second is used to describe non-decreasing returns to scale. Under these restrictions on the production structure it can be proved:

PROPOSITION 4. *If* $u \to L(u)$ *is sub- (super-) homogeneous (for increases in output), then and only then is* $\mathscr{F}(u, x) \leqslant (\geqslant)\ \mathscr{G}(u, x)$.

Proof. It suffices to prove the case of sub-homogeneity, since the proof for super-homogeneity is similar. Let $x \in L(u)$, $u > 0$, to obtain

$$\begin{aligned}\mathscr{G}(u, x) &= \min\{\theta \mid x \in L(u/\theta)\}\\ &\geqslant \min\{\theta \mid x \in \theta^{-1}L(u)\}\\ &= \min\{\theta \mid \theta\cdot x \in L(u)\}\\ &= \mathscr{F}(u, x).\end{aligned}$$

Conversely, assume $\mathscr{G}(u, x) \geqslant \mathscr{F}(u, x)$. Then for $x \in L(u)$, $u > 0$,

$$\begin{aligned}\theta^* := \min\{\theta \mid x \in L(u/\theta)\} &= \mathscr{G}(u, x)\\ \geqslant \mathscr{F}(u, x) = \min\{\lambda \mid \lambda\cdot x \in L(u)\} &= \lambda^*.\end{aligned}$$

Also, $L(u/\theta^*) = L(u)/\lambda^* \subset L(u)/\theta^*$ since $\theta^* \geqslant \lambda^*$. Therefore since $\theta^* \in (0, 1]$, $L(\theta\cdot u) \subset \theta\cdot L(u)$ where $\theta = 1/\theta^* \geqslant 1$. Q.E.D.

We summarize the results of this section with three remarks.

(1) $x \in \text{Eff } L(u)$ is sufficient but not necessary for $\mathscr{F}(u, x) = 1$, so that the Farrell input measure may term an input vector efficient when it is not.

(2) $x \in \text{Eff } L(u)$ is neither necessary nor sufficient for $\mathscr{G}(u, x) = 1$, so that the Farrell output measure may term an inefficient input vector efficient, and may also term an efficient input vector inefficient.

(3) $\mathscr{F}(u, x) = \mathscr{G}(u, x)$ if and only if production technology is linearly homogeneous.

4. A GENERALIZATION OF THE FARRELL INPUT EFFICIENCY MEASURE

It was shown above that even the Farrell input efficiency measure, which is to be preferred to the Farrell output efficiency measure, cannot in general distinguish between efficient and inefficient production units. This is a rather serious drawback for an efficiency measure. Our next task therefore is to provide a new measure that is capable of distinguishing between efficient and inefficient production units. The measure we shall propose is a nonradial measure that contains Farrell's input measure as a special case. Before doing so, however, we list the requirements an efficiency measure should meet.

Let $E: R_+ \times R_+^n \rightarrow R_+$ be an efficiency measure. Then $E(u, x)$, $x \in L(u)$, $u \geq 0$, must satisfy the following properties:

E.1 If $x \in L(u)$, $u \geq 0$, then $E(u, x) = 1 \Leftrightarrow x \in \text{Eff } L(u)$.

E.2 If $x \in L(u)$, $u \geq 0$, and $x \notin \text{Eff } L(u)$, then $E(u, x)$ should compare x to some $x^* \in \text{Eff } L(u)$.

E.3 If $x \in L(u)$, and $\lambda \cdot x \in L(u)$, $u \geq 0$, then $E(u, \lambda \cdot x) = \lambda^{-1} E(u, x)$ for all $\lambda \in [\lambda^0, +\infty)$, where $\lambda^0 \cdot x$ satisfies $\lambda^0 \cdot x \in \text{Isoq } L(u)$.

E.4 If $x \in L(u)$, $u \geq 0$, and if $x' \geq x$, then $E(u, x) \geq E(u, x')$.

Property *E*.1 has already been discussed, and requires a measure to call (u, x) efficient if and only if x belongs to the efficient subset of $L(u)$. Property *E*.2 addresses the issue of specifying an input vector x^* with which the observed input vector x should be compared in computing the efficiency of (u, x). It requires x^* to belong to the efficient subset of $L(u)$, but it does not specify the location of $x^* \in \text{Eff } L(u)$. Property *E*.3 is a homogeneity property, and requires a feasible scaling of an input vector to lead to an inverse scaling of the efficiency of that input vector. Property *E*.4 is a monotonicity property requiring efficiency to vary inversely with input usage. The conjunction of these properties guarantees that for $x \in L(u)$, $u \geq 0$, the efficiency measure $E(u, x)$ attains its maximum value of unity for all $x \in \text{Eff } L(u)$, declines monotonically for any increase in input usage, and declines equiproportionately for any proportionate increase in input usage.

It is clear from the preceding section that the preferred Farrell input measure satisfies only the homogeneity property $E.3$. We now provide a generalization of $\mathscr{F}(u, x)$ that satisfies $E.1$–$E.4$. For this purpose let $(x_{\gamma_1}, \ldots, x_{\gamma_k}; \ldots)$ with $k \in \{1, 2, \ldots, n\}$ denote a vector $x \in R_+^n$ with $x_{\gamma_i} > 0$ $(i = 1, 2, \ldots, k)$ and $x_{\gamma_{k+1}} = x_{\gamma_{k+2}} = \cdots = x_{\gamma_n} = 0$. Now introduce

DEFINITION 5. The function $\mathscr{R}: R_{++} \times R_+^n \to R_+ \cup \{+\infty\}$ defined as $\mathscr{R}(u, x) := \min\{(\sum_{i=1}^{k} \lambda_{\gamma_i})/k \mid (\lambda_{\gamma_1} \cdot x_{\gamma_1}, \lambda_{\gamma_2} \cdot x_{\gamma_2}, \ldots, \lambda_{\gamma_k} \cdot x_{\gamma_k}; \ldots) \in L(u), \lambda_{\gamma_i} \in (0, 1]\ (i = 1, 2, \ldots, k), k \in \{1, 2, \ldots, n\}\}$, $u > 0$, for $\{y \mid 0 \leqslant y \leqslant x\} \cap L(u)$ not empty and $\mathscr{R}(u, x) = +\infty$ for $\{y \mid 0 \leqslant y \leqslant x\} \cap L(u)$ empty is called the Russell measure of input efficiency.

The unweighted arithmetic mean of the scalars $\lambda_{\gamma_i} \in (0, 1]$, $i = 1, 2, \ldots, k$, is minimized in the Russell measure, although other functions (such as an unweighted geometric mean) may serve the same purpose. Different functions lead to the calculation of the efficiency of (u, x) relative to different elements of Eff $L(u)$. Since each element of $x \in L(u)$ can be weighted by a different scalar, the Russell measure can be a nonradial measure.[7] The Russell measure clearly generalizes the Farrell input measure, with the latter being the special case of $\lambda_{\gamma_i} = \lambda$, $i = 1, 2, \ldots, k$. For the Russell measure we have

PROPOSITION 5. *The Russell measure $\mathscr{R}(u, x)$ satisfies $E.1$–$E.4$.*

Proof of $E.1$. Let $x \in \text{Eff}\, L(u)$ for $u > 0$; thus by property $\phi.1$ of the production function, $x \geqslant 0$. Assume first that $x > 0$; then clearly $\mathscr{R}(u, x) = 1$, since there is no $y \in L(u)$, $y \leqslant x$. Next assume $x_{\gamma_i} > 0$ $(i = 1, 2, \ldots, k)$, $k \in \{1, 2, \ldots, n\}$. Now since $x \in \text{Eff}\, L(u)$, there is no $y \in L(u)$ with $y_{\gamma_i} < x_{\gamma_i}$ for any $i = 1, 2, \ldots, k$. Thus $\mathscr{R}(u, x) = 1$.

Conversely, assume $x \in L(u)$ but $x \notin \text{Eff}\, L(u)$, $u > 0$. Then there is a $y \in L(u)$, $0 \leqslant y \leqslant x$, i.e., $0 < y_i < x_i$ for some $i \in \{1, 2, \ldots, n\}$ implying that $\mathscr{R}(u, x) < 1$.

Finally, assume $x \notin L(u)$. Then if $\{y \mid 0 \leqslant y \leqslant x\} \cap L(u)$ is not empty, $\mathscr{R}(u, x) < 1$, and if $\{y \mid 0 \leqslant y \leqslant x\} \cap L(u)$ is empty, $\mathscr{R}(u, x) = +\infty$ by definition.

Proof of $E.2$. Assume that $(\lambda^*_{\gamma_1}, \lambda^*_{\gamma_2}, \ldots, \lambda^*_{\gamma_k})$ is a vector of scalars that minimizes $\{(\sum_{i=1}^{k} \lambda_{\gamma_i})/k \mid (\lambda_{\gamma_1} \cdot x_{\gamma_1}, \lambda_{\gamma_2} \cdot x_{\gamma_2}, \ldots, \lambda_{\gamma_k} \cdot x_{\gamma_k}; \ldots) \in L(u), \lambda_{\gamma_i} \in (0, 1]$

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[7] An equivalent way of expressing the Russell measure is $\mathscr{R}(u, x) := \|\lambda^0 \cdot y\| / \|x\|$, where $\lambda^0 = \min\{\sum_{i=1}^{k} \lambda_{\gamma_i})/k \mid (\lambda_{\gamma_1} \cdot x_{\gamma_1}, \lambda_{\gamma_2} \cdot x_{\gamma_2}, \ldots, \lambda_{\gamma_k} \cdot x_{\gamma_k}; \ldots) \in L(u), \lambda_{\gamma_i} \in (0, 1]$ $(i = 1, 2, \ldots, k)$, $k \in \{1, 2, \ldots, n\}\}$, $u > 0$, for $\{x' \mid 0 \leqslant x' \leqslant x\} \cap L(u)$ not empty, and where $y = [(\lambda_{\gamma_1}/\lambda^0) \cdot x_{\gamma_1}, (\lambda_{\gamma_2}/\lambda^0) \cdot x_{\gamma_2}, \ldots, (\lambda_{\gamma_k}/\lambda^0) \cdot x_{\gamma_k}; \ldots]$. This clearly shows $\mathscr{R}(u, x)$ to be the ratio of two distances computed along rays that diverge unless $\lambda_{\gamma_i} = \lambda$, $i = 1, 2, \ldots, k$, in which case $\mathscr{R}(u, x) = \mathscr{F}(u, x)$.

$(i = 1, 2,\ldots, k)$, $k \in \{1, 2,\ldots, n\}\}$ for $x \in L(u)$ and $u > 0$. It has to be shown that $(\lambda^*_{\gamma_1} \cdot x_{\gamma_1}, \lambda^*_{\gamma_2} \cdot x_{\gamma_2},\ldots, \lambda^*_{\gamma_k} \cdot x_{\gamma_k};\ldots) \subset \text{Eff}\, L(u)$ so that $x \in L(u)$ is compared with an efficient input vector. Assume that $(\lambda^*_{\gamma_1} \cdot x_{\gamma_1}, \lambda^*_{\gamma_2} \cdot x_{\gamma_2},\ldots, \lambda^*_{\gamma_k} \cdot x_{\gamma_k};\ldots) \notin \text{Eff}\, L(u)$. Then there must exist a $y \in L(u)$, $0 \leq y \leq (\lambda^*_{\gamma_1} \cdot x_{\gamma_1}, \lambda^*_{\gamma_2} \cdot x_{\gamma_2},\ldots, \lambda^*_{\gamma_k} \cdot x_{\gamma_k};\ldots)$ implying that $(\lambda^*_{\gamma_1}, \lambda^*_{\gamma_2},\ldots, \lambda^*_{\gamma_k})$ does not minimize the above expression. Thus $(\lambda^*_{\gamma_1} \cdot x_{\gamma_1}, \lambda^*_{\gamma_2} \cdot x_{\gamma_2},\ldots, \lambda^*_{\gamma_k} \cdot x_{\gamma_k};\ldots)$ is efficient.

Proof of E.3. Let $(\lambda^*_{\gamma_1}, \lambda^*_{\gamma_2},\ldots, \lambda^*_{\gamma_k})$ minimize $\{(\sum_{i=1}^k \lambda_{\gamma_i})/k \mid (\lambda_{\gamma_1} \cdot x_{\gamma_1}, \lambda_{\gamma_2} \cdot x_{\gamma_2},\ldots, \lambda_{\gamma_k} \cdot x_{\gamma_k};\ldots) \in L(u), \lambda_{\gamma_i} \in (0, 1]\ (i = 1, 2,\ldots, k), k \subset \{1, 2,\ldots, n\}\}$ for $x \in L(u)$ and $u > 0$. Then $(\lambda^*_{\gamma_1} \cdot x_{\gamma_1}, \lambda^*_{\gamma_2} \cdot x_{\gamma_2},\ldots, \lambda^*_{\gamma_k} \cdot x_{\gamma_k};\ldots) \in \text{Eff}\, L(u)$ and $\mathscr{R}(u, x) = \sum_{i=1}^k \lambda^*_{\gamma_i}/k$. If the vector $(x_{\gamma_1}, x_{\gamma_2},\ldots, x_{\gamma_k};\ldots)$ is replaced with the vector $(\lambda \cdot x_{\gamma_1}, \lambda \cdot x_{\gamma_2},\ldots, \lambda \cdot x_{\gamma_k};\ldots)$, where $\lambda \in [\lambda^0, +\infty)$ and $\lambda^0 \cdot x \in \text{Isoq}\, L(u)$, then clearly the vector $[(\lambda^*_{\gamma_1}/\lambda) \cdot (\lambda \cdot x_{\gamma_1}), (\lambda^*_{\gamma_2}/\lambda) \cdot (\lambda \cdot x_{\gamma_2}),\ldots, (\lambda^*_{\gamma_k}/\lambda) \cdot (\lambda \cdot x_{\gamma_k});\ldots] \in \text{Eff}\, L(u)$ and $\mathscr{R}(u, \lambda \cdot x) = \sum_{i=1}^k (\lambda^*_{\gamma_i}/\lambda)/k = \lambda^{-1} \cdot \sum_{i=1}^k \lambda^*_{\gamma_i}/k = \lambda^{-1}\mathscr{R}(u, x)$.

Proof of E.4. Let $x \in L(u)$, $u > 0$, with $\mathscr{R}(u, x) = \sum_{i=1}^k \lambda^*_{\gamma_i}/k$. Since $x \leq x'$, i.e., $0 \leq x_i \leq x_i'$ for some $i \in \{1, 2,\ldots, n\}$ it follows directly that $\mathscr{R}(u, x') \leq \mathscr{R}(u, x)$. Q.E.D.

Thus far we have assumed that ϕ satisfies a weak input disposability property ϕ.3. However if ϕ.3 is strengthened to a free input disposability property, the next proposition shows that the Russell measure completely characterizes the production technology.

PROPOSITION 6. *If $\phi(x') \supseteq \phi(x)$ for $x' \geq x$, then for $u > 0$, $L(u) = \{x \mid 0 < \mathscr{R}(u, x) \leq 1\}$.*

Proof. Since $L(u) \subset \{x \mid 0 < \mathscr{R}(u, x) \leq 1\}$ in general assume $x \notin L(u)$, $u > 0$. Then since inputs are freely disposable, there is no $(\lambda_{\gamma_1}, \lambda_{\gamma_2},\ldots, \lambda_{\gamma_k}) \leq 1$, $k \in \{1, 2,\ldots, n\}$ such that $(\lambda_{\gamma_1} \cdot x_{\gamma_1}, \lambda_{\gamma_2} \cdot x_{\gamma_2},\ldots, \lambda_{\gamma_k} \cdot x_{\gamma_k};\ldots) \in L(u)$; thus $\mathscr{R}(u, x) = +\infty$. Q.E.D.

Note that by defining $\mathscr{R}(0, x) = 1$, Proposition 6 holds for all $u \in R_+$.

In general $\mathscr{F}(u, x)$ and $\mathscr{R}(u, x)$ do not assign the same efficiency value to a given input vector $x \in L(u)$, $u > 0$, although for $x \in \text{Eff}\, L(u)$ it is true that $\mathscr{F}(u, x) = \mathscr{R}(u, x) = 1$. However $\mathscr{F}(u, x)$ and $\mathscr{R}(u, x)$ are difficult to compare when $x \in L(u)$ but $x \notin \text{Eff}\, L(u)$. This is because $\mathscr{F}(u, x)$ relates $x \in L(u)$ radially to an element of Isoq $L(u)$, whereas $\mathscr{R}(u, x)$ relates $x \in L(u)$ nonradially to an element of Eff $L(u)$. The radial-nonradial distinction causes no problems; the Isoq $L(u)$ — Eff $L(u)$ distinction does, because the efficiency of $x \in L(u)$ is being computed relative to different subsets of $L(u)$ by the two measures. There are two sets of circumstances in which a meaningful comparison can be made between the two measures: (1) When Eff $L(u)$ = Isoq $L(u)$, the two measures can be compared for all points (u, x), $x \in L(u)$.

$u \geq 0$; (2) When Eff $L(u) \neq$ Isoq $L(u)$, the two measures can be compared for some, but not all, points (u, x), $x \in L(u)$, $u \geq 0$.

PROPOSITION 7. *If* Eff $L(u) =$ Isoq $L(u)$, *then* $\mathscr{F}(u, x) \geq \mathscr{R}(u, x)$ *for all* (u, x) *such that* $x \in L(u)$, $u \geq 0$.

Proof. Since Eff $L(u) =$ Isoq $L(u)$, both $\mathscr{F}(u, x)$ and $\mathscr{R}(u, x)$ compute the efficiency of (u, x) relative to the same subset of $L(u)$. $\mathscr{R}(u, x)$ is of course free to select $\lambda_{\nu_i} = \lambda^0$, $i = 1, 2, \dots, k$, where $\lambda^0 = \min\{\lambda \mid \lambda \cdot x \in L(u)\}$, in which case $\mathscr{R}(u, x) = \mathscr{F}(u, x)$. But $\mathscr{R}(u, x)$ would never select $\lambda^{+}_{\nu_i}$, $i = 1, 2, \dots, k$, such that $(\sum_{i=1}^{k} \lambda_{\nu_i}/k) > \lambda^0$, since such a selection of weights would not minimize $(\sum_{i=1}^{k} \lambda_{\nu_i}/k)$ when λ^0 was available. Hence $\mathscr{R}(u, x)$ must select $\lambda^0_{\nu_i}$, $i = 1, 2, \dots, k$, such that $(\sum_{i=1}^{k} \lambda^0_{\nu_i}/k) \leq \lambda^0$, and so $\mathscr{R}(u, x) \leq \mathscr{F}(u, x)$. Q.E.D.

Thus when Eff $L(u) =$ Isoq $L(u)$, the two measures are comparable for all $x \in L(u)$, $u \geq 0$, and the Russell measure never exceeds the Farrell input measure. To illustrate, consider the production function

$$\phi(x_1, x_2) = x_1^{1/4} \cdot x_2^{3/4} \tag{3}$$

for which Eff $L(u) =$ Isoq $L(u)$, $u \geq 0$. It is clear from Definition 1 that $\mathscr{F}(16, 20, 20) = 0.8$ with $\lambda = 0.8$. It also can be verified by means of Definition 5 that $\mathscr{R}(16, 20, 20) = 0.7048$ with $\lambda_1 = 0.4096$ and $\lambda_2 = 1$. This example is illustrated in Figure 1. The firm is operating inefficiently at point A,

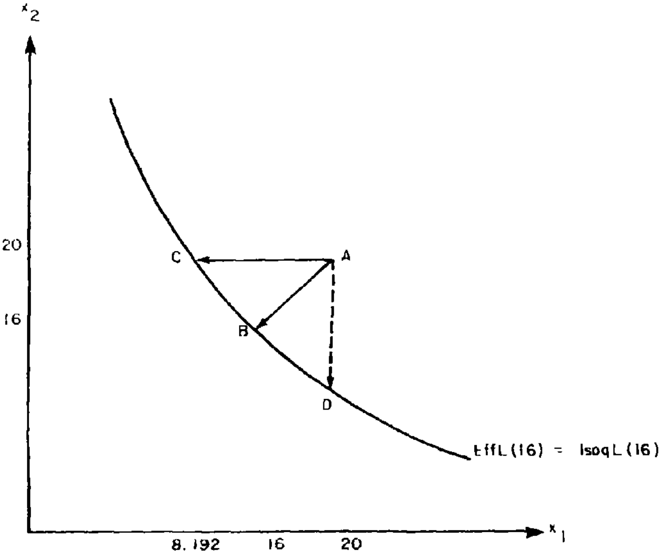

FIGURE 1

where $(u, x) = (16, 20, 20)$. $\mathscr{F}(u, x)$ relates point A to point B, where $(16, 16) \in \text{Isoq } L(16)$. $\mathscr{R}(u, x)$ could conceivably relate point A to any point of $\text{Eff } L(16)$ between points C and D inclusive, but the minimization process imbedded in $\mathscr{R}(u, x)$ selects point C, where $(20, 8.192) \in \text{Eff } L(16)$.

PROPOSITION 8. *If* $\text{Eff } L(u) \neq \text{Isoq } L(u)$, *then for all* (u, x) *such that* $x \in L(u)$, $u \geq 0$, *and such that* $(\lambda \cdot x) \in \text{Eff } L(u)$ *for some* $\lambda \in (0, 1]$, $\mathscr{F}(u, x) = \mathscr{R}(u, x)$.

After taking into consideration the restricted domain for x, the proof is identical to the proof of Proposition 7, and is omitted.

The gist of Proposition 8 is that if $\text{Eff } L(u) \neq \text{Isoq } L(u)$ the Russell and Farrell input measures are not directly comparable for all $x \in L(u)$, $u \geq 0$. The two measures are comparable only for those $x \in L(u)$, $u \geq 0$, such that $(\lambda \cdot x) \in \text{Eff } L(u)$ for some $\lambda \in (0, 1]$, in which case $\mathscr{F}(u, x) = \mathscr{R}(u, x)$. To illustrate, consider the production function (1). Off the ray $x_1 = x_2$ the two measures are not comparable; along the ray $x_1 = x_2$ we have $\mathscr{F}(u, x) = \mathscr{R}(u, x)$.

5. CONCLUSIONS

If the structure of production technology is restricted sufficiently, then Farrell's notions of technical efficiency have merit. Thus, for example, if the production function ϕ is continuous on R_+^n, and if ϕ satisfies a strong free disposability property $(x' \geq x \Rightarrow \phi(x') \geq \phi(x))$, then $\text{Eff } L(u) = \text{Isoq } L(u)$ and $\mathscr{F}(u, x)$ satisfies $E.1$–$E.4$, although $\mathscr{G}(u, x)$ still fails $E.3$. If, in addition, ϕ is homogeneous of degree $+1$, then $\mathscr{G}(u, x)$ satisfies $E.1$–$E.4$, and $\mathscr{F}(u, x) = \mathscr{G}(u, x)$ for $x \in L(u)$, $u \geq 0$. Thus under strong free disposability and constant returns to scale Farrell's two measures become one, and that measure is well-behaved on our criteria $E.1$–$E.4$.

The purpose of introducing $\mathscr{R}(u, x)$ is to provide an efficiency measure that is well-behaved under the less restrictive technological assumptions $\phi.1$–$\phi.5$. This undertaking is justified by the dependence on Farrell's ideas of a growing number of empirical studies dealing with the computation or estimation of production frontiers, not all of which satisfy the strong technological assumptions under which Farrell's ideas are valid. In practice the implementation of $\mathscr{R}(u, x)$ is no more difficult than implementation of either $\mathscr{F}(u, x)$ or $\mathscr{G}(u, x)$. Once the frontier is computed, the efficiency of each observation (u, x), $x \in L(u)$, $u \geq 0$, is calculated by solving the minimization problem specified in Definition 5 rather than the minimization problems specified in Definition 1 or Definition 2.

REFERENCES

1. D. J. AIGNER, C. A. K. LOVELL, AND P. SCHMIDT, Formulation and estimation of stochastic frontier production function models, *J. Econometrics* **6** (1977), 21–37.
2. G. DEBREU, The coefficient of resource utilization, *Econometrica* **19** (1951), 272–292.
3. R. FÄRE AND L. JANSSON, On VES and WDI Production Functions, *Int. Econ. Rev.* **16** (1975), 245–250.
4. M. J. FARRELL, The measurement of productive efficiency, *J. Royal Stat. Soc. Ser. A* **120** (1957), 253–290.
5. F. R. FØRSUND AND L. HJALMARSSON, On the measurement of productive efficiency, *Swedish J. Econ.* **76** (1974), 141–154.
6. F. R. FØRSUND AND E. S. JANSEN, On estimating average and best practice homothetic production functions via cost functions, *Int. Econ. Rev.* **18** (1977), 463–476.
7. T. C. KOOPMANS (Ed.), "Activity Analysis of Production and Allocation," Wiley, New York, 1951.
8. C. A. K. LOVELL, Estimation and prediction with CES and VES production functions, *Int. Econ. Rev.* **14** (1973), 676–692.
9. W. MEEUSEN AND J. VAN DEN BROECK, Efficiency estimation from Cobb–Douglas production functions with composed error, *Int. Econ. Rev.* **18** (1977), 435–444.
10. P. SCHMIDT AND C. A. K. LOVELL, Estimating technical and allocative inefficiency relative to stochastic production and cost frontiers, *J. Econometrics*, in press.
11. R. W. SHEPHARD, "Cost and Production Functions," Princeton Univ. Press, Princeton, N.J., 1953.
12. R. W. SHEPHARD, "Theory of Cost and Production Functions," Princeton Univ. Press, Princeton, N.J., 1970.
13. R. W. SHEPHARD AND R. FÄRE, The law of diminishing returns, *Z. Nationalokonomie* **34** (1974), 69–90.

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Optimal Savings and Exhaustible Resource Extraction in an Open Economy

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1. INTRODUCTION

Economic analysis of natural resources has traditions going back to Malthus and Ricardo. The basis for the modern treatment of the best use of an exhaustible natural resource is the classical article by Hotelling [3]. Lately the problem has also been analyzed from a macroeconomic point of view by, e.g., Koopmans [5] and Vousden [8]. In these models either all consumption in the economy is provided from the resource, or an additional source of consumption, outside the model, is postulated. The assumption of no alternative sources of consumption is extreme and unrealistic. The assumption of an alternative, exogenous source of consumption has been introduced by Vousden in [8] as "a convenient simplification of the relevance of the rest of the economy to the resource-use decision." But except for foreign aid, and the most primitive subsistence agriculture, the time path of the alternative consumption will depend on the stock of physical capital in the economy and the savings ratio together with the growth in labor supply and technological progress. It seems reasonable to think that, e.g., the optimal savings ratio will depend on the availability of natural resources in the economy. On the other hand it does not seem reasonable to assume that the optimal path of resource depletion will be completely independent of, e.g., the stock of physical capital in the economy and the resulting potentiality for consumption from sources other than the current resource extraction. This shows the need for an integrated model of the economy where optimal savings and resource extraction can be determined *simultaneously*. The purpose of this paper is to present and analyze such a model for a small, open economy. The interrelationship between the optimal rate of investment and the optimal depletion of natural resources is explored by Heal and Dasgupta [2]. However, their model is rather different from the following model, which has the recent petroleum discoveries in the North Sea as its background. We consider an open economy where the resource good is exchanged for other goods in the

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world market. The problem of optimal extraction of a nonrenewable resource in an open economy has earlier been analyzed by Vousden [9], Kemp and Suzuki [4], Van Long [7], and Strøm [6]. Common to all these contributions is the fact that there is no physical capital accumulation in their models, so that the problem of determining the optimal accumulation of capital, together with resource extraction, disappears. In this paper the central issues therefore are:

(1) What is the optimal intertemporal pattern of physical capital accumulation in an open economy with a considerable stock of an exhaustible resource?

(2) What is the optimal intertemporal pattern of extraction of this resource?

(3) How are the decisions implicit in (1) affected by conditions in the resource sector?

(4) How are the decisions implicit in (2) affected by conditions in the rest of the economy?

2. THE MODEL

The following variables are used:

$c(t)$	Total consumption per capita
$\bar{c}$	An exogenous source of consumption
$r(t)$	Resource extraction per capita
$s(t)$	The (average) savings ratio
$k(t)$	Physical capital per capita
$f(k)$	Production per capita, exclusive of resource extraction
$p(t)$	The price of the resource relative to the "price" of other goods in the world market
$b(t)$	Total extraction costs per capita
U	Social welfare
ρ	The social rate of discount (constant)
n	The rate of growth in total population
μ	The rate of depreciation of physical capital
$x(t)$	The stock of the resource per capita
$\Pi(t)$	Net proceeds per capita from resource extraction

The problem is then

$$\max \int_0^\infty U(c(t))\, e^{-\rho t}\, dt$$

s.t.

(i) $c(t) = (1 - s(t))[f(k(t)) + H(t)] + \bar{c}$,

(ii) $\dot{k}(t) = s(t)[f(k(t)) + H(t)] - \lambda k(t)$,

(iii) $H(t) = p(t)v(t) - b(v(t))$,

(iv) $\lambda = n + \mu$,

(v) $-\dot{x}(t) = v(t) + nx(t)$,

(vi) $0 \leq s(t) \leq 1$,

(vii) $0 \leq v(t) \leq \bar{v}$,

(viii) k_0, x_0 given,

(ix) $\lim_{t \to \infty} x(t) \geq 0$, $\lim_{t \to \infty} k(t)$ free,

(x) ρ, n, μ, $\bar{c}$, and $p(t)$ exogenously given.

Stated in words, the problem posed is to find such paths over time for resource extraction and total savings that the present value of total social welfare is maximized. The planning horizon is infinity. Instantaneous welfare depends on consumption per capita, and we assume that $U' > 0$ and $U'' < 0$. Total population is assumed to grow at the same rate as the labor force. Consumption per capita is given by (i), where H is defined in (iii). All net earnings from resource extraction are used for import, so that the current account is always balanced. $p(t)$ is assumed to be independent of the amount exported ("small country" argument). Relation (ii), the expression for the increase in capital intensity, is familiar from ordinary growth theory. (iii) expresses net earnings from resource extraction, where $b' > 0$ and $b'' > 0$. (v) says that the stock of the resource per capita is reduced by the extraction per capita v, and is also diluted by nx because of the growth in population. By (vi) s must be nonnegative and it cannot exceed one. (vii) says that the resource extraction is irreversible and that there is some upper bound $\bar{v}$ on extraction per unit of time, due to, e.g., limited pipeline capacity or limited loading capacity for tankers at the production platforms, caused by climatic and/or geographical conditions. In addition to the assumption of a balanced current account, the structure of the model above also assumes:

(a) No search activity for new resources.

(b) No uncertainties. In particular the future relative price of the resource is assumed known.

(c) The stock of the resource does not affect social welfare or the extraction conditions, except that it restricts total extraction.

(d) External effects are disregarded. Examples might be pollution due to oil spill, blowouts, reduced fishing possibilities, or the fact that two (or more) countries are extracting petroleum from the same reservoir.

(e) The producing country does not use the resource as an input.

Assumptions (a) to (e) are not trivial. Still, this model contains aspects from "real life" not found in any of the contributions quoted above.

3. OPTIMALITY CONDITIONS AND POLICY REGIMES

To analyze the problem, form the (present value) Hamiltonian function

$$H = e^{-\rho t}\{U[(1 - s(t))(f(k(t)) + \Pi(t)) + \bar{c}] + q_1(t)(s(t)[f(k(t)) + \Pi(t)] - \lambda k(t)) - q_2(t)(v(t) + nx(t))\}, \tag{1}$$

where we have substituted for c in U from (i). $q_1(t)$ and $q_2(t)$ are the so-called co-state variables, associated with $k(t)$ and $x(t)$, respectively. According to Pontryagin's maximum principle, a solution to the problem posed must satisfy the following necessary conditions[1] :

(a) There exist continuous functions of time, q_1 and q_2 , such that

$$\dot{q}_1 = -U'(1 - s)f' - [sf' - (\rho + \lambda)]\, q_1 , \tag{2}$$

$$\dot{q}_2 = (\rho + n)\, q_2 . \tag{3}$$

(b) For all t, H is maximized in s and v, so that

$$\begin{aligned} s(t) &= 1 && \text{if } q_1(t) > U', & v(t) &= \bar{v} && \text{if } q_2(t) < q_1\Pi' \\ &\in [0, 1] && \text{if } q_1(t) = U', & &\in [0, \bar{v}] && \text{if } q_2(t) = q_1\Pi' \\ &= 0 && \text{if } q_1(t) < U'; & &0 && \text{if } q_2(t) > q_1\Pi' \end{aligned} \tag{4}$$

A number of policies are therefore available to the economy; see Table I.

Regimes G, I, and J are of limited economic relevance and will not be referred to any more.

Equation (3) yields

$$q_2(t) = q_{2,0}e^{(\rho + n)t}; \qquad q_{2,0} > 0.^2 \tag{3'}$$

q_2 is the shadow price of the resource, so that $q_2(t)$ denotes the addition to the optimal value of the criterion function of leaving the marginal unit of the resource at t unexploited. q_1 is the shadow price of physical capital per worker. By (4), if the solution for v is interior, the shadow price of the resource, q_2 , is equal to the marginal proceeds from extraction times the shadow

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[1] Such a solution will be really optimal, since (a) the Hamiltonian is concave in k, x, s, and v for given $q_1(t)$, $q_2(t)$, and t, and (b) it is shown later that $x(t)$ will be exhausted in finite time and that $q_1(t)$ and $k(t)$ approach finite limits as t goes to infinity.

[2] $q_{2,0}$ is of course not an exogenously given constant—it is determined in the optimization process. $q_{2,0} < 0$ is disregarded as economically uninteresting.

TABLE I

Regimes	s	v	c	q_1	q_2
A	1	$\bar{v}$	$\bar{c}$	$\geq U'$	$\leq \Pi'(\bar{v})q_1$
B	1	$\in [0, \bar{v}]$	$\bar{c}$	$\geq U'$	$= \Pi'(v)q_1$
C	1	0	$\bar{c}$	$\geq U'$	$\geq \Pi'(0)q_1$
D	$\in [0, 1]$	$\bar{v}$	$(1 - s)(f(k) + \Pi(\bar{v})) + \bar{c}$	$= U'$	$\leq \Pi'(\bar{v})q_1$
E	$\in [0, 1]$	$\in [0, \bar{v}]$	$(1 - s)(f(k) + \Pi(v)) + \bar{c}$	$= U'$	$= \Pi'(v)q_1$
F	$\in [0, 1]$	0	$(1 - s)(f(k)) + \bar{c}$	$= U'$	$\geq \Pi'(0)q_1$
G	0	$\bar{v}$	$f(k) + \Pi(\bar{v}) + \bar{c}$	$\leq U'$	$\leq \Pi'(\bar{v})q_1$
I	0	$\in [0, \bar{v}]$	$f(k) + \Pi(v) + \bar{c}$	$\leq U'$	$= \Pi'(v)q_1$
J	0	0	$f(k) + \bar{c}$	$\leq U'$	$\geq \Pi'(0)q_1$

price of capital. For an interior solution for s, this shadow price is equal to the marginal utility of consumption per worker. In that case, the shadow price of the resource is equal to the marginal utility of resource extraction in terms of consumption per worker $\Pi' U'$. Given k, q_1, and q_2, (4) determines the optimal $s(t)$ and $v(t)$ so that total consumption and capital-accumulation are determined. Let us study each policy in turn, assuming for the moment that $p(t)$ is a constant p.

3.1. *The Interior Solution* (*Regime E*)

Since $q_1 = U'$ in E, it follows from (2) that

$$\dot{q}_1 = (-f' + \rho + \lambda)\, q_1 \tag{5}$$

so that

$$\begin{aligned} \dot{q}_1 &> 0 \quad \text{if } k > k^*, \\ &= 0 \quad \text{if } k = k^*, \\ &< 0 \quad \text{if } k < k^*, \end{aligned} \tag{6}$$

where k^*, the modified golden rule capital intensity, is uniquely defined by $f'(k^*) = \rho + \lambda$, *irrespective of conditions in the resource-extraction sector.* Differentiating $q_1 = U'$ with respect to t, and using (5),

$$\dot{c} = (U'/U'')[-f' + \rho + \lambda]. \tag{7}$$

(7) is familiar from optimal growth theory and says that along the optimal path consumption is increasing (decreasing) as long as the capital intensity of the economy is below (above) the optimal steady state capital intensity k^*,

defined above. Regarding resource extraction, differentiating $\Pi'(v)q_1 = q_2$ with respect to time and solving for $\dot v$, we obtain

$$\dot v = (1/q_1\Pi'')(\dot q_2 - \Pi'\dot q_1).$$

Inserting for $\dot q_2$ and $\dot q_1$ from (3) and (5), respectively, and using (4), this simplifies to

$$\dot v = (\Pi'/\Pi'')(f' - \mu). \tag{8}$$

Due to the assumptions on the cost function, (8) says that, along the optimal path, resource extraction should decrease (increase) as long as k is less (greater) than $\hat k$, where $\hat k = k^*$ is defined by $f'(\hat k) = \mu$. In addition to the cost structure in the extraction sector *the optimal extraction path also depends on the capital intensity of the economy*. If the economy is growing $(k < k^*)$, the shadow price of capital (equal to the marginal utility of consumption) is falling and the user cost of the resource is increasing, both contributing to a falling optimal path of resource extraction. In a contracting economy $(k > k^*)$, the shadow price of capital (equal to the marginal utility of consumption) is increasing, counteracting the growing user cost of the resource. For $k = \hat k$ the relative rates of growth in q_1 and q_2 are equal, thus the rise in user cost is exactly offset by the rise in the shadow price of capital. For $k > \hat k$, $\dot q_1/q_1 > \dot q_2/q_2$. In a contracting economy resource extraction is therefore increasing for $k > \hat k$, it is constant for $k = \hat k$, and it is decreasing for $k < \hat k$. Furthermore, when $k \neq \hat k$, (8) shows that when $b'' \to 0$, $\dot v(t) \to -\infty$ which means that *when marginal extraction costs are constant for the permissible values of v, the interior solution cannot last for more than "an instant of time"*[3]; i.e., this regime cannot be part of an optimal policy sequence when $b'' = 0$. From (5) and (6) it is intuitively plausible that there is a unique equilibrium situation (q^*, k^*) in this regime. This equilibrium is a saddlepoint; i.e., to every intial k_0 there is a unique optimal path leading to (q^*, k^*). (The proof is given in presentations of the one-sector optimal growth model; see, e.g., Burmeister and Dobell [1, Chap. 11]). Along this optimal path (7) and (8) hold, so, for $k_0 < k^*$, c is increasing while v is decreasing. While this equilibrium is reached only asymptotically, regime E will last only for a finite time until extraction stops and the resource is exhausted. From (4), optimal resource extraction in E is governed by

$$(p - b'(v(t)))\, U'[(1 - s(t))(f(k(t)) + \Pi(v(t))) + \bar c] = q_{2,0}e^{(\rho+n)t}. \tag{9}$$

In (9), $\lim_{t\to\infty}$ RHS $= \infty$, while LHS$(t) \leq$ LHS$(0) < \infty$. Thus (9) cannot hold for $t \to \infty$. There is some finite t, T, where v goes to zero and the extraction period is over. With a finite T, the transversality condition for $x(T)$ is

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[3] This can be seen more clearly by assuming $\Pi'(v)q_1 = q_2$, with $b'' = 0$, to last for some interval of time. This is only possible for $f' = \mu$. But k has no stationary in $\hat k$, hence E cannot last for more than "an instant of time."

$$e^{-\rho T} q_2(T)\, x(T) = 0; \qquad e^{-\rho T} q_2(T) \geq 0. \tag{10}$$

Since $q_2(t)$ in E is always positive from (4), (10) is only satisfied for $x(T) = 0$. The same conclusions hold for regimes B and I. *At T the resource will therefore be exhausted.*

3.2. *Other Regimes*

B: In this regime the only source of consumption is the exogenous component, which provides the society with a subsistence level of consumption. At the same time, capital scarcity is extreme; this is why all production is saved. Equations (5), (6), and (8) hold also in this regime, and again, if extraction costs are constant, this regime cannot be part of an optimal policy.

A, C: These policies are boundary policies in the sense that the values of the control variables are at their boundary points. For these policies the exact behavior of the system may be inferred from these boundary values together with the initial conditions on k and x. Since $s = 1$, (5) and (6) hold also in these regimes.

D, F: In D resource extraction is maximal. In F there is no extraction; (5), (6), and (7) hold in both cases. As in regime E it can be shown that there is a unique equilibrium, which is a saddlepoint, for each of the regimes D and F.

Having studied the behavior of the economy within each possible policy regime, we now proceed to an analysis of possible switches between these regimes to find optimal policy sequences over time.

4. OPTIMAL POLICY SEQUENCES

The necessary conditions for the various switches to take place between the different policies are summarized in Table II. Piecing the different policy regimes together is particularly simple in this model, even if we have two co-state variables, since q_2 is growing exponentially. As an illustration, follow the possible optimal sequence from an initial situation in regime A. Such an economy is extremely poor in physical capital. $k_0 < k^*$ and q_1 is falling. The shadow price of physical capital capital is higher than the marginal utility of consumption. All production is accumulated in order to expand the capital stock of the economy. On the other hand the economy is rich in the resource, which is extracted at a maximal rate. Since $c = \bar{c}$ is a constant in A, $U'(c)$ is a constant. Over time q_1 is falling and q_2 is increasing so that U' and $q_2/\Pi'(\bar{r})$ are approached "from above" by q_1. Policy A therefore cannot be sustained.

TABLE II

Necessary Conditions for Policy Switches

Switches from \ Switches to	B	C	D	E	F
A	$q_1 > U'(\bar{c})$ $\frac{q_2}{\Pi'(\bar{v})} = q_1$	0	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(\bar{v})} < q_1$	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(\bar{v})} = q_1$	0
B	·	$q_1 < U'(\bar{c})$ $\frac{q_2}{\Pi'(0)} = q_1$	0	$q_1 = U'(\bar{c})$ $\frac{q_2}{\Pi'(v)} = q_1$	$q_1 < U'(\bar{c})$ $\frac{q_2}{\Pi'(0)} = q_1$
C	0	·	0	0	$q_1 < U'(\bar{c})$ $\frac{q_2}{\Pi'(0)} > q_1$
D	0	0	·	$q_1 = U'[(1-s)(f(k)+\Pi(\bar{v}))+\bar{c}]$ $\frac{q_2}{\Pi'(\bar{v})} = q_1$	0
E	0	0	0	·	$q_1 = U'[(1-s)(f(k)+\Pi(0))+\bar{c}]$ $\frac{q_2}{\Pi'(0)} = q_1$

If, at some t, $U'(\bar{c}) = q_1$, while still $q_2 < H'(\bar{v})\, q_1$, the economy switches to *D*, where now consumption is above subsistence level, while the resource is extracted at a maximal rate. Total consumption is now increasing over time and $q_1 = U'$ is falling. The economy is accumulating capital also in this regime. Capital scarcity is less extreme than in regime *A*, and the resource is abundant. Of course, the initial situation of a real economy is usually in *D* or *E*. A plausible example of an initial situation in *D* is the United Kingdom after the recent petroleum discoveries in the North Sea. Policy *D* is also a transient policy since $q_2/H'(\bar{v})$ is increasing while $U' = q_1$ is falling. Sooner or later the economy will therefore switch to *E*, with resource extraction at less than a maximal rate. Consumption is still increasing even if resource extraction is now decreasing towards zero over time. An example of an economy where the initial situation seems to be in this regime is Norway, where initial oil production is well below the possible maximum. When extraction ends, the resource is exhausted and the economy switches to *F* which is the final policy, which, once established, will last forever. In *F* the optimal development is the same as in the one-sector optimal growth model, with consumption and the capital intensity increasing towards their optimal steady-state levels.

Alternatively, if, at some t, $q_2 = H'(\bar{v})\, q_1$, with $q_1 > U'(\bar{c})$, the economy switches from regime *A* to *B*, where now resource extraction is less than its maximum and falling, while still all production is used for accumulation purposes so that consumption per capita equals $\bar{c}$. *B* may of course also be the initial situation. *B* is also a transient regime. q_2 will be increasing over time while q_1 is falling. From *B* the economy must switch to either *E* or *C*, or, by accident, directly to *F*. If q_1 "reaches" $U'(\bar{c})$ while $q_2 = H'(v)\, q_1$, the economy switches to *E*, with the optimal path as explained above. The other possibility is that $q_1 > U'(\bar{c})$ while $q_2 = H'(0)\, q_1$. In that case the economy switches from *B* to *C* where resource extraction has ended while capital accumulation is still maximal. Since q_1 is falling towards $U'(\bar{c})$, a switch from *C* to *F* must eventually take place. Finally, A may also by accident switch directly to *E*, if q_1 happens to be equal to $U'(\bar{c})$ exactly at the point in time where maximal resource extraction stops. From an initial situation in *A*, the optimal policy sequence will therefore be one of the following:

(a) *ADEF*, (b) *ABEF*, (c) *ABCF*, (d) *AEF*, (e) *ABF*.

In all sequences extraction is gradually reduced. Maximal extraction until extraction stops is inoptimal, since such a policy would require a jump in one of the co-state variables, which is impossible by the maximum principle. In Fig. 1 the sequence *DEF* is illustrated graphically by q_1^0 and q_2^0. The economy is in regime *D* until t' with $q_1 = q_2/H'(\bar{v})$. From t' to t'' it it is in *E*, with $q_1 = q_2/H'(v)$, after which it is in *F* with $q_2/H'(0) > q_1$. Resource extraction

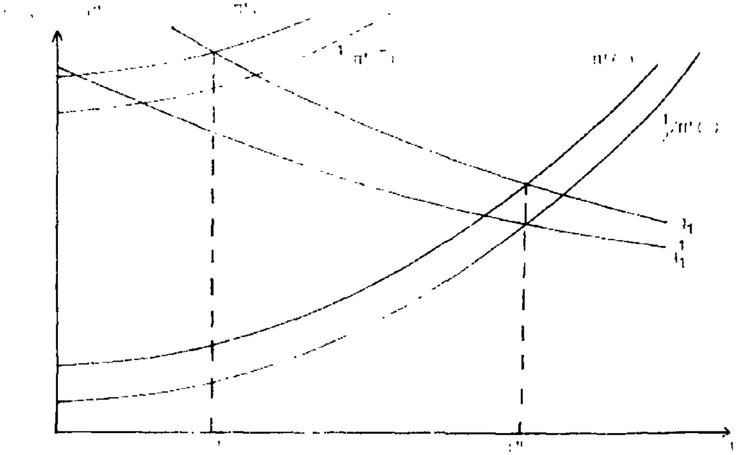

FIGURE 1

ends at t_1, when the resource is exhausted. An interesting question is now: How does the optimal extraction policy depend on the initial situation?

Consider first a positive shift in k_0, with everything else equal. From the one-sector optimal growth model we know that a higher k_0 would imply a lower $q_1(t)$ everywhere. A higher k_0 would cet. par. also mean a lower $q_{2,0}$. If not, the intersection of $q_1(t)$ and $q_2(t)/\Pi'(0)$ would take place for $t < t''$ so that the resource extraction period would be shortened. From the figure it is seen that the time period when extraction is maximal (regime D) is reduced when cer. par. q_1 shifts down. Also, for $q_1 = q_2/\Pi'(v)$ to hold for every t with a lower $q_1(t)$, $\Pi'(v)$ must increase; i.e., $v(t)$ must be reduced for every t. But a shorter extraction period with reduced extraction everywhere cannot exhaust the resource, hence it is inoptimal by (10). $q_{2,0}$ must therefore fall when k_0 gets a positive shift. Suppose then that $q_{2,0}$ falls so that the extraction period is unchanged. The new situation is illustrated by q_1^1 and q_2^1 in the figure. $q_1^1(t)$ is less steep than $q_1^0(t)$; this follows from (5). $\dot{q}_2/q_2$ is unchanged. Because of this, going backwards from t'', the difference between the two curves is less in the new situation than in the old; i.e., $q_1^0 - q_2^0/\Pi'(0) > q_1^1 - q_2^1/\Pi'(0)$ for all $t < t_1$, and the inequality increases as t goes towards zero. Thus, for $q_1 = q_2/\Pi'(v)$ to hold, this means that $1/\Pi'(v)$ is less in the new situation, or that the extraction level is less now for all $t < t''$, except in the period when D is operative. But the time period when this regime is operative must fall—this follows from the same argument, and is clear from the figure. Again this would mean that the resource is not exhausted at t_1, *hence the extraction period must be lengthened when the initial capital intensity gets a positive shift.* Regarding a positive shift in x_0, it can be shown along the same lines that cet. par., $q_1(t)$ and $q_2(t)$ must both fall when x_0 increases. Resource extraction and consumption increase and the extraction period is increased.

A more general question is: Under what initial conditions in the economy are each of the sequences (a)–(c), or parts of them, optimal? Suppose initially that *EF* is optimal and that k_0 now gets a negative shift. We then know that $v(t)$ shifts up and the period when *E* is operative is reduced. Over time $v(t)$ is falling, so for a sufficiently great negative shift in k_0, $v(t)$ would be equal to $\bar{v}$ for some timeperiod, regime *D* would be operative, and the optimal sequence would be *DEF*. For still lower k_0, regime *A* might be optimal initially so that the whole sequence (a) would be relevant. For even lower k_0, q_1 might not have reached $U'(\bar{c})$ when extraction is reduced from its maximal level and (b) would be the optimal sequence. Finally, (disregarding (d) and (e)), sequence (c) would be optimal for the lowest k_0.

Consider next the effects of changes in ρ, the social rate of time preference. In the capital sector, an increase in ρ leads to a fall in k^*. From the theory of optimal growth in the one-sector model we know that, for a given k, this leads to a lower q_1 and to a lower s for each k. Also increased social preference for consumption "today" relative to "tomorrow" would tend to concentrate resource extraction more towards the beginning of the planning period and to reduce the extraction period. This is accomplished by keeping the economy in *E* for a shorter period when ρ increases. As ρ increases still more and t_1 decreases, the optimal policy sequence may change from *EF* to *DEF*, and so on. Changes in the exogenously given rate of growth in population, n, have the same effects as changes in ρ.

Finally, consider the effects of an exponential trend in $p(t)$ so that $p(t) = p_0 e^{\beta t}$. Instead of (8), we now obtain

$$\dot{v} = (\Pi'/\Pi'')(f' - \mu) - \dot{p}/\Pi'' \tag{11}$$

from which it follows that, cet. par. *an exponential rise (fall) in the relative price of the resource would tend to reduce (increase) the rate of fall in extraction along the optimal path.* For sufficiently high rates of increase in the relative price of the good the optimal extraction path may be rising. Inserting for $\dot{p}$ and Π' and rearranging, (11) shows that

$$\dot{v}(t) \lessgtr 0 \qquad \text{as } f'(k) \gtrless \beta/(1 - b'/p) + \mu. \tag{12}$$

From (4), $0 < b'/p < 1$, so that the RHS of (12) is always greater than $\beta + \mu$. As long as the marginal productivity of capital in the economy is greater than the RHS of (12), the optimal devlopment of the economy will in principle be the same as when $\beta = 0$. If the economy is more capital intensive, however, resource extraction will rise over time along the optimal path when there is a positive exponential trend in the relative price of the resource. In general (12) shows that the higher this trend is, the lower must the capital intensity be for falling resource-use to be optimal, or conversely, *the higher the capital intensity of the economy, the lower the price rise needed to make a rising extraction-path optimal.* The economic explanation is that when the relative price of the resource is rising, the increasing user cost of the resource and falling shadow price of capital are counteracted by the price rise. If capital is "scarce," however, its shadow price may fall so fast that it offsets the rate of increase in the resource price over and above the user cost. In that case optimal extraction would still be falling over time. Conditions for this case are given in (12).

5. COMPARING THE RESULTS WITH EARLIER MODELS

(a) *The Resource Model*

The results obtained on the optimal path of resource-use in this combined model are somewhat more general than those found in earlier contributions. The optimal extraction path now depends on conditions in the rest of the economy. When the relative price of the resource is given and marginal extraction costs are constant, we have shown that an interior solution to v and s simultaneously cannot be optimal. If the solution for the savings rate is interior, resource extraction should either be zero or at its maximum; and vice versa. This is so even if the social welfare function in the model is concave. A comparison with the results in the resource model by Vousden [8] is therefore not completely straightforward. In his model marginal extraction costs are constant. In a growth-model context, the alternative constant source of consumption postulated in his model must be interpreted as a steady-state consumption level, associated with some interior constant savings rate s^*. At the same time v is also interior and falling in his model, which is incompatible with the necessary conditions for optimality in the combined model, where v is either zero or $\bar{v}$ in a steady state. When marginal extraction costs are rising, however, an interior solution for s and v simultaneously is relevant in the combined model. With constant prices and the capital intensity less than or equal to the modified golden rule capital intensity, extraction is either constant for some initial period and then falling, or always falling. The extraction period is finite and the resource is always exhausted when extraction ends. All these results are similar to those found by Vousden in [8]. However, the result on the effect on the optimal extraction path from changes in the initial capital or resource stock bears little resemblance to the result in [8] that "the optimal time of exhaustion will increase as the alternative source of consumption, $\bar{C}$, falls."

Since the welfare function is the same in both models, the contrast in the solutions must be due to the difference in supply conditions. In the resource model the resource may be used for consumption purposes only. Extracting the resource is the only source of consumption in addition to the exogenous component. Since consumption from the resource is subject to a finite upper bound on cumulative extraction, society must—in order to survive—stretch out the use of the resource when the alternative source of consumption falls. In the combined model, the resource extracted may also be used for capital-formation purposes, and survival can be secured on basis of the physical capital stock alone. The future benefits to be derived from an extra unit of the resource extracted for investment purposes is higher the smaller the capital stock already attained. The optimal resource-use in the combined model is therefore slowed down when the initial capital intensity gets a positive shift.

Also, with an initial capital-intensity higher than the modified golden rule, a rising extraction is possible for some time. In such a contracting economy, an increasing shadow price of capital (equal to the marginal utility of consumption), which makes investment and consumption more valuable in the future, tends, cet. par., to make it optimal to postpone extraction; this effect must be balanced against a rising user cost of the resource. The combined effect may well be to keep back production for some time, in contrast to the ordinary case where a falling shadow price of capital and an increasing user cost of the resource both lead to a higher extraction now than in the future. While a contracting economy may be of limited practical interest, the relevance of the capital intensity on optimal resource-use is obvious in the case of a rising price of the resource. In that case the increasing user cost is counteracted by the rising resource price. In a model without capital, extraction rises if the rate of growth in price is higher than in the user cost. In a model with capital, this pattern is accentuated if capital is above its optimal steady-state level, since then the social value of capital (or consumption) over time increases as well. If capital is scarce, however, its shadow price is falling, and it may fall so fast that it offsets the rate of increase in the resource price over and above the user cost. In that case optimal extraction would still be falling over time. Such a country is too poor to afford to wait for the higher prices, at least for some initial period until the capital stock of the economy is built up.

(b) *The Capital Model*

The problem now is how the optimal path of consumption and capital accumulation in an economy is affected by conditions in the resource sector. To follow Vousden [8], assume that the economy initially, before resource extraction is started up at t_0, is in a steady state; i.e., $k = k^*$ and $s = s^* = \lambda k^*/f(k^*)$. Assume further that after extraction has started up, the economy is in regime E, so that extraction starts at less than its maximum. At the outset we may safety assume that consumption is a normal good for all t so that there is *some* increase in consumption when extraction starts. From (4) this means a negative shift in q_1 at t_0. Also, as in the one-sector optimal growth model it can be shown from (4) that $\partial s/\partial q_1 > 0$, so that *the savings rate falls when extraction begins*. But since the value of total production in the economy increases when extraction begins, *total* savings may a priori be increased or reduced (or stay constant). To show that total savings must rise, suppose the opposite. If they fall for $t = t_0 + \epsilon$ ($\epsilon > 0$ and sufficiently small), k falls below its steady-state level k^*. But from the one-sector optimal growth model a reduced k must imply a higher shadow-price q_1, so this is a contradiction. By the same argument, k cannot stay constant when q_1 gets a negative shift. *Hence total savings rise when extraction begins.* k increases above its steady-state level, so by (6), q_1 increases over time, which means that c is falling over time along the optimal path. Over time, k and c approach their optimal steady-state values asymptotically from above, while q_1 increases asymptotically towards its optimal stationary value. The optimal development of c, q_1, and k is illustrated in Fig. 2. At t_1, when extraction ends, c and k are both above their optimal stationary values.

Since the optimal steady state is unlikely to materialize in any actual economy (it is reached only asymptotically), the above thought-experiment is somewhat illegitimate. Consider therefore the case when $k < k^*$ when a new resource, like a petroleum reservoir, is discovered and exploited. We may

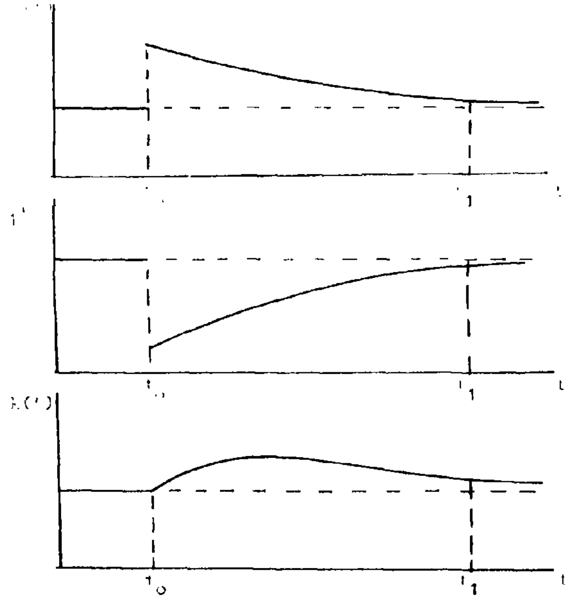

FIGURE 2

again safely assume that consumption gets a positive shift when extraction begins so that $q_1(t)$ shifts down. By (6), $q_1(t)$ must be falling, since $k < k^*$ also after extraction begins, which means that $c(t)$ is increasing. Two cases may be distinguished. The case in which $c(t) < c^*$ for all t is illustrated in Fig. 3, where the optimal paths of c, q_1, and k are shown. If, however, k_0 is sufficiently close to k^* and/or the new resource is sufficiently rich, $c(t)$ and $k(t)$ may increase above their optimal steady-state levels; see Fig. 4.

In this case, as in the case discussed in connection with Fig. 2, consumption shifts up above its optimal steady-state level when extraction begins, and, correspondingly, q_1 shifts below q_1^*. k is increasing. Since now $k(t_0) < k^*$, however, q_1 falls and $c(t)$ increases for some period after t_0, until, at $\hat{t}$, $k = k^*$. In $\hat{t}$, $\dot{q}_1 = 0$, and immediately after, when $k > k^*$, q_1 is increasing. This means that $c(t)$ has a maximum in $\hat{t}$. For $t > \hat{t}$ the optimal development of the economy is identical to the case when $k_0 = k^*$, discussed in connection with Fig. 2. (7) may be rewritten as

$$\dot{c}/c = (-f' + \rho + \lambda)/\breve{\omega}, \tag{13}$$

where $\breve{\omega} = -cU''/U'$ is the elasticity of marginal utility. When extraction starts, total savings increase so that k increases faster than before the extraction period. With $\breve{\omega}$ (approximately) constant in the relevant range, it follows from (13) that the relative growth in consumption is reduced when

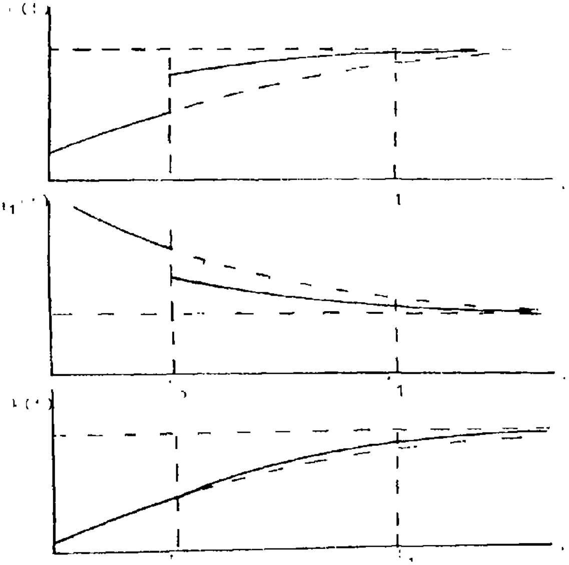

FIGURE 3

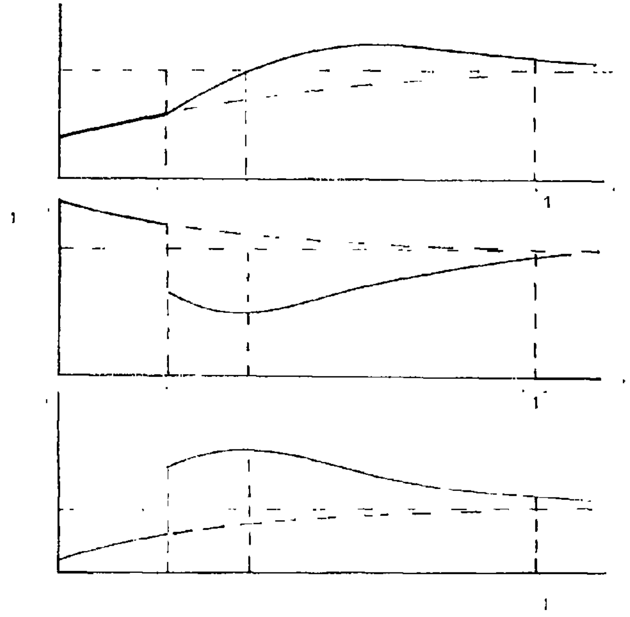

FIGURE 4

extraction starts. *The optimal reaction to a newly discovered resource is therefore a positive shift in initial consumption combined with a reduced relative growth in consumption.* Figures 2–4 show that *with* resource extraction $q_1(t)$ is less than $q_1(t)$ *without* extraction for all t, including $t > t_1$. This means that *also in the post-extraction period society will enjoy higher levels of capital and consumption per capita than it would have done without a resource extraction period.*

6. CONCLUSIONS

(i) A dynamic model for an open economy where savings and resource use can be optimized simultaneously has been analyzed. The results are somewhat more general than those found in earlier contributions. With constant prices and the capital intensity of the economy less than or equal to the modified golden rule, however, extraction is either constant for some initial period and then falling, or always falling, along any of the possible optimal sequences for the economy.

(ii) When the price of the resource depends exponentially on time, it is optimal if, and only if, the rate of increase in the price of the resource is greater than some critical value, determined partly by the capital intensity of the economy, to depart from the optimal sequences mentioned above. In that case resource extraction is increasing over time, and it may be optimal to leave the resource in the ground for some initial period. When the capital intensity of the economy increases, the price rise needed to make such a policy optimal is reduced.

(iii) It has been shown that as the initial capital stock of the economy increases, the extraction period is lengthened and the extraction level is reduced for every t. The resource is exhausted when extraction ends and the extraction period is always finite. Extraction should be reduced gradually towards zero, where extraction ends.

(iv) If a resource is discovered and exploited, the optimal savings rate in the economy falls, while total savings increase. Compared to a situation without resource extraction, consumption gets an initial positive shift, while its relative rate of growth along the optimal path is reduced. Consumption and the capital stock will be higher also in the postextraction period.

(v) Finally, it should be noted that with constant marginal extraction costs—an assumption often made in the literature—an interior solution for savings and resource extraction at the same time cannot be optimal.

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REFERENCES

1. E. BURMEISTER AND A. R. DOBELL, "Mathematical Theories of Economic Growth," Macmillan, New York/London, 1970.
2. G. M. HEAL AND P. DASGUPTA, The optimal depletion of exhaustible resources, *in* "Rev. Econ. Stud. Symposium, 1974," pp. 3–28.
3. H. HOTELLING, The economics of exhaustive resources, *J. Pol. Econ.* **39** (1931), 137–175.
4. M. C. KEMP AND H. SUZUKI, International trade with a wasting but possibly replenishable resource, *Int. Econ. Rev.* **16** (1975), 712–732.
5. T. C. KOOPMANS, "Some Observations on Optimal Economic Growth and Exhaustible Resources," Discussion Paper No. 356, Cowles Foundation, 1973.
6. S. STRØM, Utnytting av naturressurser, *Sosialøkonomen* **8** (1974), 5–12.
7. NGO VAN LONG, International borrowing for resource extraction, *Int. Econ. Rev.* **15** (1974), 168–183.
8. N. VOUSDEN, Basic theoretical issues of resources depletion, *J. Econ. Theory* **6** (1973), 126–143.
9. N. VOUSDEN, International trade and exhaustible resources: A theoretical model, *Int. Econ. Rev.* **15** (1974), 149–167.

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Axioms for Approval Voting: Direct Proof

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1. INTRODUCTION

Approval voting is a procedure for multicandidate elections in which each voter votes for as many candidates as he wishes without ranking his choices. The candidate with the largest vote total wins the election. Ties occur if two or more candidates have the same largest vote total. Although it is assumed that the ultimate winner is one of the tied candidates in such cases, specific tie-breaking procedures will not be a primary concern in this paper.

Because approval voting restricts voters' responses rather severely as compared to weak order responses allowed by Arrowian methods [1, 3, 7], it is liable to criticisms based on its restricted ballot set. On the other hand it is an easy procedure for voters to understand and respond to, and it can be readily implemented on existing voting machines. In addition, Kellett and Mott [6] and Brams and Fishburn [2] argue that approval voting is superior in many ways to other simple procedures—such as the one vote per voter system—that are often used in multicandidate elections. Hence whatever its deficiencies are from a traditional social choice theory viewpoint, approval voting deserves serious consideration as a practical alternative to existing procedures.

Two variable-electorate axiomatizations of approval voting are presented in [4] as part of a more inclusive study on simple voting procedures. The one of these that I shall focus on here is closely related to the work of Smith [8] and Young [9, 10, 11] on point-count scoring procedures. The proof in [4] of the representation theorem for this axiomatization is based on a linear separation theorem and, as such, is indirect and nonintuitive. The purpose of the present note is to offer a mathematically simpler and more direct proof of this theorem. The new proof relates to the earlier proof in much the same way that Hansson and Sahlquist's proof [5] of Young's representation theorem for the Borda point-count procedure relates to Young's original proof [10].

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2. Axioms and Direct Proof

It will be assumed throughout that X is a fixed finite set of $m \geq 3$ candidates. The *ballot set* $\mathscr{A}$ will be taken to be the set of all subsets of X. Unlike [4], $\emptyset$ (abstain) and X (vote for all candidates) are included in $\mathscr{A}$ for completeness.

A *ballot response profile* is a function π from $\mathscr{A}$ into the nonnegative integers that has $\Sigma\pi(A) > 0$ so that there is at least one voter. The set of all such profiles is denoted as Π. A *social choice function* is here defined as a function f from Π into the nonempty subsets of X, with $f(\pi)$ interpreted as the "best" candidates in X from the standpoint of f operating on response profile π. Hence f applies to a fixed set of candidates and every positive finite number of voters or ballots. Since π does not retain voters' identities, f by definition is *anonymous* in the sense that it is invariant to permutations of ballots among the voters. This type of anonymity along with the neutrality condition defined shortly identifies f as a *symmetric* social choice function.

Let $\pi[x] = \Sigma\{\pi(A)\colon x \in A\}$ be the number of voters for π whose ballots contain $x \in X$. Then f is the *approval voting function* if and only if $f(\pi) = \{x \in X\colon \pi[x] \geq \pi[y]$ for all $y \in X\}$ for all $\pi \in \Pi$. We shall characterize the approval voting function by the axioms of neutrality, consistency, and disjoint equality. For any permutation λ on X and $\pi \in \Pi$ let $\lambda(\pi) \in \Pi$ be such that $\lambda(\pi)(A) = \pi(\lambda(A))$ where $\lambda(A) = \{\lambda(x)\colon x \in A\}$, for each $A \in \mathscr{A}$. When $A \neq B$, we shall say also that π consists of ballots $A, B \in \mathscr{A}$ if $\pi(A) = \pi(B) = 1$ and $\pi(C) = 0$ for all $C \in \mathscr{A}\backslash\{A, B\}$. Our three axioms, which apply to all $\pi, \pi' \in \Pi$, all permutations λ on X, and all $A, B \in \mathscr{A}$, can then be stated as follows:

$A1$ (neutrality): $f(\lambda(\pi)) = \lambda(f(\pi))$.

$A2$ (consistency): $f(\pi) \cap f(\pi') \neq \emptyset \Rightarrow f(\pi + \pi') = f(\pi) \cap f(\pi')$.

$A3$ (disjoint equality): *If π consists of nonempty ballots A and B with $A \cap B = \emptyset$, then $f(\pi) = A \cup B$.*

The first of these is the usual neutrality condition. The second is Young's consistency axiom [11] for the present context. It says that if the winning candidates for two disjoint groups of voters have a candidate in common, then the winning set for the two groups taken together is the intersection of the winning sets of the two groups taken separately. The third axiom, which applies only to two-voter response profiles in which the ballots of the two voters have no candidate in common, singles out the approval voting function from among all symmetric and consistent f on Π. Under the hypothesis of $A3$, the social choice set is to be the set of all candidates on the two ballots. In particular, note that $A3$ does not require $|A| = |B|$. Hence if one person votes for x and another votes for y and z, then $f(\pi) = \{x, y, z\}$ when there are no other voters. It is clear that $A3$ holds for the approval voting function since that function assigns a full vote or "point" to every candidate in a ballot regardless of how many candidates are in the ballot. It is also easily seen that the approval voting function satisfies $A1$ and $A2$. The remainder of this note will therefore be devoted to a "direct" sufficiency proof of the following theorem.

THEOREM 1. *f on Π is the approval voting function if and only if it satisfies $A1$, $A2$ and $A3$.*

In the ensuing proof, arguments of f will often contain elements in $\mathscr{A}$, as in $f(A)$, $f(\pi + A + B)$ and $f(\pi + A \cup B)$. Each such element in $\mathscr{A}$ denotes a one-ballot, one-voter response profile. For example, A in $f(A)$ stands for π with $\pi(A) = 1$ and $\pi(B) = 0$ for all $B \neq A$. If π consists of the ballots of n voters, then $\pi + A + B$ consists of the ballots of $n + 2$ voters, and $\pi + A \cup B$ consists of the ballots of $n + 1$ voters. For the latter, $(\pi + A \cup B)(A \cup B) = \pi(A \cup B) + 1$ and $(\pi + A \cup B)(C) = \pi(C)$ for all $C \neq A \cup B$. This notational convention will be used only in arguments of f.

Axioms $A1$, $A2$ and $A3$ are assumed to hold for f henceforth in this section. The proof that f is the approval voting function will be carried out in three main steps. The first step proves that f is faithful, dual and strongly Paretian. The second step shows that f satisfies a three-voter analogue of $A3$ and then notes that $f(\pi + A + B) = f(\pi + A \cup B)$ whenever $A \cap B = \varnothing$. The third step completes the proof.

Step 1

The *dual* $d(\pi)$ of response profile π is the response profile defined by $d(\pi)(A) = \pi(A^c)$ for all $A \in \mathscr{A}$, where $A^c = X \setminus A$, the complement of A. The following conditions, which apply to all $A \in \mathscr{A}$, all $\pi \in \Pi$, and all $x, y \in X$, will be shown to hold for f:

$A4$ (faithfulness): $f(A) = A$ unless $A = \varnothing$, and $f(\varnothing) = X$.

$A5$ (duality): $f(\pi) \cap f(d(\pi)) = \varnothing$ *or* $f(\pi) = f(d(\pi)) = X$.

$A6$ (strong Pareto): *If* $y \in A \Rightarrow x \in A$ *for all* A *for which* $\pi(A) > 0$, *and if* $x \in A$ and $y \notin A$ *for some* A *for which* $\pi(A) > 0$, *then* $y \notin f(\pi)$.

Proof of $A4$. It follows from those λ for which $\lambda(A) = A$ that $A1$ implies that $f(A) \in \{A, A^c, X\}$; and, by definition, $f(A) \neq \varnothing$. If $A \in \{\varnothing, X\}$ then $f(A) = X$. Assume henceforth that A is a nonempty proper subset of X. Suppose first that $|A| \leq m/2$, recalling that $m \geq 3$. Take $|B| = |A|$ with $B \subseteq A^c$. Then $A3$ implies $f(A + B) = A \cup B$; if $A^c \subseteq f(A)$, i.e. $f(A) = A^c$ or $f(A) = X$, then $B^c \subsetneq f(B)$ by $A1$ and then $A2$ implies that $f(A + B) = f(A) \cap f(B)$ since $A^c \cap B^c \neq \varnothing$. But in the latter case $f(A + B)$ then contains something not in $A \cup B$, contradicting $f(A + B) = A \cup B$. Therefore $A^c \not\subseteq f(A)$, and hence $f(A) = A$ when $0 < |A| \leq m/2$.

Now for any nonempty B that is disjoint from nonempty A for which $|A| < m/2$, $A3$ implies that $f(A + B) = A \cup B$. Then $A2$ implies that $f(B) \cap A = \varnothing$ for otherwise with $f(A) = A$ as just proved we get $f(A + B) = f(A) \cap f(B) \subsetneq A$, a contradiction. Therefore, since $f(B) \in \{B, B^c, X\}$ by $A1$, and since $f(B) \cap A = \varnothing$, we get $f(B) = B$. Since this is true for $B \subset X$ for which $|B| \geqslant m/2$, the proof of $A4$ is complete.

Proof of A5. By pairing up each ballot A in π with its complement A^c in $d(\pi)$, $A3$ and repeated applications of $A2$ imply that $f(\pi + d(\pi)) = X$. Hence if $f(\pi) \cap f(d(\pi)) \neq \varnothing$, $A2$ requires $f(\pi) = f(d(\pi)) = X$.

Proof of A6. Assume that x and y are as specified in the hypotheses of $A6$. If $x \in A$ and $y \notin A$ for all A for which $\pi(A) > 0$, then $A4$ and $A2$ imply that $x \in f(\pi)$ and $y \notin f(\pi)$. Thus assume henceforth that there is an A for which $\pi(A) > 0$ and either $x, y \in A$ or $x, y \in A^c$. Let π^* be the sum of all response profiles $\lambda(\pi)$ over the permutations λ on X that have $\lambda(x) = x$ and $\lambda(y) = y$, and let $Z = X \setminus \{x, y\}$. Then π^* is symmetric in Z and it follows from $A1$ that either $Z \subsetneq f(\pi^*)$ or $Z \cap f(\pi^*) = \varnothing$. Decompose π^* into π_1^* and π_2^* so that $\pi_1^* + \pi_2^* = \pi^*$ with $x \in A$ and $y \notin A$ for all A for which $\pi_1^*(A) > 0$, and either $x, y \in A$ or $x, y \in A^c$ for all A for which $\pi_2^*(A) > 0$. Both π_1^* and π_2^* are symmetric in Z so that either $Z \subsetneq f(\pi_i^*)$ or $Z \cap f(\pi_i^*) = \varnothing$ for $i \in \{1, 2\}$. In addition, $x \in f(\pi_1^*)$ and $y \notin f(\pi_1^*)$ by the first part of this proof, and either $\{x, y\} \subseteq f(\pi_2^*)$ of $\{x, y\} \cap f(\pi_2^*) = \varnothing$ by $A1$ since π_2^* is symmetric in $\{x, y\}$. That is, λ for which $\lambda(x) = y$ and $\lambda(y) = x$ and $\lambda(z) = z$ for all $z \in Z$ gives $\lambda(\pi_2^*) = \pi_2^*$.

Now if $x \in f(\pi_2^*)$ then $A2$ implies that $x \in f(\pi^*)$ and $y \notin f(\pi^*)$, hence that $y \notin f(\pi)$ since otherwise $A2$ would give $y \in f(\pi^*)$. On the other hand, if $x \notin f(\pi_2^*)$ then $f(\pi_2^*) = Z$. In this case $f(d(\pi_2^*)) = \{x, y\}$ by $A5$ and $A1$ (since $d(\pi_2^*)$ is symmetric in $\{x, y\}$), and $y \in f(d(\pi_1^*))$ and $x \notin f(d(\pi_1^*))$ by $A4$ and $A2$ (or by the first part of this proof). Therefore, since $d(\pi^*) = d(\pi_1^*) + d(\pi_2^*)$, $A2$ implies that $f(d(\pi^*)) = \{y\}$, and then $A5$ implies that $y \notin f(\pi^*)$. Hence it follows again from $A2$ that $y \notin f(\pi)$.

Step 2

The main task in this step is to prove the following three-voter analogue of $A3$:

A3.1: If $A, B, C \in \mathscr{A}$ and $A \cap B = A \cap C = B \cap C = \varnothing$ then $f(A + B + C) = A \cup B \cup C$.

Since this follows easily from $A3$ and from $A2$ and $f(\varnothing) = X$ if one or more of A, B and C is $\varnothing$, we shall assume that none of A, B and C is empty. Then, if $|A| = |B| = |C|$, the conclusion of $A3.1$ follows immediately from $A1$

and $A6$, the latter of which is needed only if $A \cup B \cup C$ is a proper subset of X. Henceforth in the proof of $A3.1$ we shall suppose for definiteness that $|A| \geq |B| \geq |C|$ and that $|A| > |C|$, and will let $A' \subset A$ be such that $|A'| = |C|$.

Since $A3$ implies that $f(A' + B \cup C) = A' \cup B \cup C$ and $f(A + B \cup C) = A \cup B \cup C$, and since $A6$ and $A1$ imply that $f(A + B + C)$ equals the union of one or more of A, B and C —and similarly for $f(A' + B + C)$—application of $A2$ implies that

$$\begin{aligned} f(A + B + C) \cap (A' \cup B \cup C) &= f((A + B + C) + (A' + B \cup C)) \\ &= f((A' + B + C) + (A + B \cup C)) \\ &= f(A' + B + C) \cap (A \cup B \cup C) \\ &= f(A' + B + C). \end{aligned}$$

Now if $|B| = |C|$ then $f(A' + B + C) = A' \cup B \cup C$ by the first paragraph of this proof (or by $A1$ and $A6$), and therefore $f(A + B + C) \cap (A' \cup B \cup C) = f(A' + B + C)$ implies that $A' \cup B \cup C \subseteq f(A + B + C)$ and hence that $f(A + B + C) = A \cup B \cup C$. On the other hand, if $|B| > |C|$ then $|B| > |A'| = |C|$, and the result just proved (for $|A| \geq |B| = |C|$) implies that $f(A' + B + C) = A' \cup B \cup C$; then, as in the latter part of the preceding sentence, we conclude again that $f(A + B + C) = A \cup B \cup C$.

Thus the proof of $A3.1$ is complete. To complete the present step in the proof of Theorem 1 we shall verify

A7: If $A, B \in \mathscr{A}$, $A \cap B = \varnothing$ and $\pi \in \Pi$, then $f(\pi + A + B) = f(\pi + A \cup B)$.

This is true since $f(\pi + A + B) = f((\pi + A + B) + (A \cup B + (A \cup B)^c)) = f((\pi + A \cup B) + (A + B + (A \cup B)^c)) = f(\pi + A \cup B)$, where the first equality follows from $A2$ and from either $A3$ or $f(\varnothing + X) = X$, and the third equality is implied by $A2$ and $A3.1$.

Step 3

As in the definition of the approval voting function let $\pi[x]$ be the number of voters for π whose ballots contain x. If $\pi[x] = 0$ for all $x \in X$, i.e. if $\pi(A) > 0$ only for $A = \varnothing$, then $f(\pi) = X$ by $A4$ and $A2$, which gives the approval voting result for this total abstention case. Assume henceforth that $\pi[x] > 0$ for some $x \in X$ and, using $A2$, assume without loss of generality that $\pi(\varnothing) = 0$. By applying $A7$ repeatedly to π it follows that $f(\pi) = f(\pi^0)$ where $\pi^0[x] = \pi[x]$ for all $x \in X$ and $\pi^0(A) > 0$ only if $|A| = 1$. In other words, all ballots in π^0 contain exactly one candidate.

Let $X = \{x_1, x_2, \ldots, x_m\}$ and for definiteness take $\pi^0[x_k] = n_k$ with $n_1 \geq n_2 \geq \cdots \geq n_m$ and $n_1 > 0$. If all n_k are equal then $f(\pi^0) = X$ by $A1$. If all n_k aren't equal, let j be such that $n_1 = \cdots = n_j > n_{j+1}$. Then, using $A7$ repeatedly, $f(\pi^0) = f(\pi^*)$ where

$$\begin{aligned} &\pi^*(\{x_1, \ldots, x_j\}) = n_j - n_{j+1} \\ &\pi^*(\{x_1, \ldots, x_k\}) = n_k - n_{k+1} \text{ for } k = j+1, \ldots, m-1 \\ &\pi^*(X) = n_m \end{aligned}$$

and $\pi^*(A) = 0$ for all other A. Sequential application of $A4$ and $A2$ then implies that $f(\pi^*) = \{x_1, \ldots, x_j\}$. Hence $f(\pi) = \{x_1, \ldots, x_j\}$ since $f(\pi) = f(\pi^0) = f(\pi^*)$. Therefore $f(\pi)$ is the subset of candidates who have the largest vote total in π.

REFERENCES

1. K. J. ARROW, "Social Choice and Individual Values," 2nd ed., Wiley, New York, 1963.
2. S. J. BRAMS AND P. C. FISHBURN, Approval voting, *Amer. Polit. Sci. Rev.* **72** (1978), 831–847.
3. P. C. FISHBURN, "The Theory of Social Choice," Princeton Univ. Press, Princeton, N.J., 1973.
4. P. C. FISHBURN, Symmetric and consistent aggregation with dichotomous voting, *in* "Aggregation and Revelation of Preferences" (J.-J. Laffont, ed.), North-Holland, Amsterdam, 1978.
5. B. HANSSON AND H. SAHLQUIST, A proof technique for social choice with variable electorate, *J. Econ. Theory* **13** (1976), 193–200.
6. J. KELLETT AND K. MOTT, Presidential primaries: measuring popular choice, *Polity* **9** (1977), 528–537.
7. A. K. SEN, "Collective Choice and Social Welfare," Holden-Day, San Francisco, 1970.
8. J. H. SMITH, Aggregation of preferences with variable electorate, *Econometrica* **41** (1973), 1027–1041.
9. H. P. YOUNG, A note on preference aggregation, *Econometrica* **42** (1974), 1129–1131.
10. H. P. YOUNG, An axiomatization of Borda's rule, *J. Econ. Theory* **9** (1974), 43–52.
11. H. P. YOUNG, Social choice scoring functions, *SIAM J. Appl. Math.* **28** (1975), 824–838.

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Notes, Comments, and Letters to the Editor

On the Speed of Adjustment in the Classical Tatonnement Process: A Limit Result

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1. INTRODUCTION

As a natural consequence of the development of a long series of theorems concerning the existence of equilibrium, in the general equilibrium framework, the question arose of how to formulate a dynamic price adjustment process, with the property that the prices determined by this process eventually converge to the equilibrium price. In other words, the problem posed was to describe a reasonable adjustment process with respect to which the price equilibrium is stable. The problem was first approached by Walras, who formulated the so-called tatonnement process. Since this process was first proposed a considerable literature discussing the problems raised by it has developed, see, e.g., Negishi's survey article [2].

The basic idea of the tatonnement process is that, given a price system, each agent informs an auctioneer of his demand and supply of each commodity. Having computed the total excess demand for each commodity the auctioneer then announces a new price vector which is obtained from the previous one by increasing (decreasing) the price of commodities with positive (negative) excess demand. This process is repeated until equilibrium is reached. A fundamental feature is that no agent is allowed to trade any commodity until equilibrium is achieved.

The question now arises: Given this assumption of "no trade out of equilibrium," if the tatonnement process does converge to an equilibrium, can we then say anything about the speed of adjustment? For the no-trade assumption to make any sense the convergence should be "fast"; that is, the organization of the market prior to trade should not take long. This question has been posed several times in the literature (see Negishi [2]), but to the best of our knowledge no answer has been offered.

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The purpose of this paper is to provide a partial answer to this question by showing how a well-known result in the theory of ordinary differential equations gives, directly, an estimate of the speed of adjustment of the tatonnement process in the limit, (i.e., as the price approaches equilibrium).

2. THE BASIC MODEL

In this section we present the basic model. As our result is closely related to the "neo-classical" stability theorems, we adopt the model here which has been used in the vast majority of papers concerning price adjustment processes by a tatonnement procedure.

We consider a pure exchange economy with m consumers ($i = 1, \ldots, m$) and l commodities ($h = 1, \ldots, l$). The commodity space is described by $\mathbb{R}^l$. P is defined by $P := \{x \in \mathbb{R}^l \mid x \gg 0\}$, and $\bar{P}$ is the closure of P. Each agent is assumed to have some initial resources $\omega^i = (\omega_1{}^i, \ldots, \omega_l{}^i)$ belonging to $\bar{P}$ and ω, defined as the total initial resources $\omega := \Sigma\omega^i$, is assumed to lie in P.

A state $(x^1, \ldots, x^m) \subset \bar{P}^m$ is a specification of a commodity bundle for each agent, and a state is said to be attainable if $\sum_i x^i = \omega$.

A price system is a linear map $p: \mathbb{R}^l \to \mathbb{R}$ which to each commodity bundle $y = (y_1, \ldots, y_l)$ assigns its value, $p \cdot y$. We only consider price vectors $p \gg 0$, where we have used the canonical isomorphism between $\mathbb{R}^l$ and $(\mathbb{R}^l)^*$. Each agent has a preference–indifference relation $\succsim_i$ (reflexive, transitive, complete), defined on $\bar{P}$, and it is assumed monotone.

The ith agent's demand set at the price system p is defined as the maximal elements w.r.t. $\succsim_i$ in the budget set $\gamma^i := \{x \in \bar{P} \mid p \cdot x \leq p \cdot \omega^i\}$. If we impose the strong convexity assumption on $\succsim_i$ ($x \sim_i y$, $x \neq y \Rightarrow x <_i tx + (1-t)y$, $0 < t < 1$) the demand set will consist of exactly one point, so the ith agent's demand is a function ξ^i. $\bar{p}$ is called an equilibrium price vector, if supply equals demand at the price system $\bar{p}$, i.e. $\Sigma\xi^i(\bar{p}) = \omega$. $\xi := \sum_i \xi^i$ is the total demand function and $\zeta := \xi - \omega$ is called the total excess demand function.

This model has the following two features:

(1) $\forall i : \xi^i$ is a positive homogeneous function of degree 0 in p, i.e., $\xi^i(\alpha p) = \xi^i(p)$, $\forall \alpha > 0$;

(2) Walras' law, i.e., $p \cdot \zeta(p) = 0$, $\forall p$.

Next we want to describe the dynamical feature of the model. This is given by the following differential equation:

$$dp/dt = f(p), \tag{1}$$

where $f = (f_1, \ldots, f_l)$ is a map defined on P with values in $\mathbb{R}^l$, such that sign $f_h =$ sign ζ_h, $h = 1, 2, \ldots, l$.

This process describes the adjustment in the prices through time with the interpretation of a tatonnement process as given above in the introduction.

DEFINITION. $\Psi(t; p^0)$ is said to be a solution to (1) with initial value p^0, if for $h = 1,..., l$

$$\frac{\partial \Psi_h(t; p^0)}{\partial t} = f_h(\Psi(t; p^0)), \qquad \forall t \geq 0,$$

and

$$\Psi_h(0; p^0) = p_h^0.$$

$\bar{p}$ is an equilibrium for (1) if and only if $\bar{p}$ is an equilibrium price vector, i.e., $\zeta_h(\bar{p}) = 0$, $h = 1,..., l$.

DEFINITION. An equilibrium $\bar{p}$ is called globally stable, if for every p^0 and every solution $\Psi(t; p^0)$ to (1)

$$\lim_{t \to \infty} \Psi(t; p^0) = \bar{p}.$$

The adjustment process (1) is called globally stable if for every p^0 there exists an equilibrium $\bar{p}$ such that

$$\lim_{t \to \infty} \Psi(t; p^0) = \bar{p}.$$

An equilibrium $\bar{p}$ is called locally stable if there exists a neighborhood U of $\bar{p}$ such that for every $p^0 \in U$ we have

$$\lim_{t \to \infty} \Psi(t; p^0) = \bar{p}$$

for every solution to (1).

The literature concerning tatonnement processes presents a variety of theorems giving more or less similar conditions for which (global or local) stability is present. Here we state only one such theorem. This theorem is representative of the existing stability theory, and so it is reasonable to accept as the basic result for the following section.

To formulate the theorem we have to define the notion of commodities, which are strongly gross substitutes.

DEFINITION. The commodities $h = 1,..., l$ are said to be strongly gross substitutes if the total excess demand function $\zeta(p) = (\zeta_1(p),..., \zeta_l(p))$ is differentiable at all price vectors $p \gg 0$ and

$$\partial \zeta_h / \partial p_k > 0 \qquad \text{for } h \neq k, p \gg 0.$$

STABILITY THEOREM (Uzawa [3]). *Suppose*

(1) For any initial price vector $p^0 \gg 0$ process (1) has a solution $\Psi(t; p^0)$ *which is uniquely determined by* p^0 *and continuous* w.r.t. p^0.

(2) *There exists a strictly positive equilibrium price vector.*

(3) *All commodities are gross substitutes.*

Then the equilibrium price vector is uniquely (up to scalar multiplication) determined and the adjustment process (1) *is globally stable.*

3. THE SPEED OF ADJUSTMENT IN THE LIMIT

If the assumptions of the above theorem are satisfied, then it follows that given $\epsilon > 0$ there exists a t_0 such that for $t > t_0$ $|\Psi(t: p^0) - \bar{p}| < \epsilon$.

This formulation immediately leads to one possible way of describing the speed of adjustment, namely, by expressing t_0 as a function of ϵ and the functions f_h. How this should be carried out precisely does not seem quite clear. Another approach to the problem would be to compare the convergence of the solution with the convergence of a well-known function, and this is exactly the task of this paper.

Let us start by considering the problem locally in a neighborhood of an equilibrium. Let $\bar{p}$ be a locally stable equilibrium price vector associated with the tatonnement process

$$dp/dt = f(p), \qquad \operatorname{sign} f_h = \operatorname{sign} \zeta_h, \; h = 1, \ldots, l.$$

Making the substitution $q := p - \bar{p}$ we may instead consider the process

$$dq/dt = Z(q), \qquad Z(q) := f(p), q := p - \bar{p}, \tag{2}$$

where $\bar{q} = 0$ is a locally stable point.

To get an idea of the kind of result we can obtain, let us first look at a linear adjustment process which has 0 as a stable equilibrium. This process has the form

$$dq/dt = A \cdot q,$$

where A is a $(l \times l)$-matrix.

For this differential equation we can explicitly state the solution, which has the form

$$a(t) = K \cdot e^{tA},$$

where $K \in \mathbb{R}^l$ is a constant determined by an initial value condition.

For a linear process local stability is equal to global stability. Because the real parts λ_i of the eigenvalues of A are negative, we get the existence of a positive constant C such that

$$\forall t \; \| q(t) \| / e^{at} < C, \qquad \text{where } \max_i \lambda_i < a < 0,$$

$$\| q(t) \| := | q_1(t) | + \cdots + | q_l(t) |$$

Since $e^{at} \to 0$, $\| q(t) \| \to 0$ for $t \to \infty$, it is reasonable to say that the speed of adjustment of the q-vector is, in the limit, as least as fast as the convergence of the exponential map e^{at}.

The result is not astonishing because of the form of the solution of the linear differential equation. However, this estimate of the speed of adjustment (in the limit) extends to the general case.

THEOREM. *Suppose the map* $f : P \to \mathbb{R}^l$ *is of class* C^2 *and* (1), (2), *and* (3) *of the stability theorem are satisfied.*

Then the solutions $p(t)$ *to the tatonnement process*

$$dp/dt = f(p)$$

fulfill the following:

(1) $\| p(t) - \bar{p} \| / e^{at} \to 0$ *as* $t \to \infty$ *with* $\lambda_i < a < 0, \forall i$.

(2) $\log \| p(t) - \bar{p} \| / t \to \lambda_i$ *for some* i,

where $\bar{p}$ *is the equilibrium price vector and the* λ_i*'s are the negative real parts of the eigenvalues of* $Df(\bar{p})$.

Proof. The theorem is a straightforward application of theorem 6.1, Chapter IX in Hartman [1]: Using the Taylor expansion around $\bar{q} = 0$ we have the tatonnement process written in the form

$$dq/dt = DZ(0) \cdot q + R(q),$$

where $R(q)$ is of class C^1 and $R(q)/\| q \| \to 0$ as $q \to 0$ (i.e., $R(0) = 0$ and $D_q R(0) = 0$), as $\bar{q} = 0$ is a sink.

This implies that the conditions in Hartman's theorem are satisfied, so (1) and (2) follow immediately.

Remark. The following example shows that we cannot generally obtain a better estimate of the speed of adjustment, i.e., choose $a = \max_i \lambda_i$. Suppose $f(p) = Df(\bar{p})(p - \bar{p}) + R(p - \bar{p})$ with $Df(\bar{p}) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, $R(p - \bar{p}) = ((p_2 - \bar{p}_2)^{2k+1}, 0)$ (k large), and $\bar{p} \gg 0$ the equilibrium price vector.

In a small neighborhood of $\bar{p}$, the solution to the adjustment process

$$dp/dt = f(p)$$

will behave as the solution to the process

$$dp/dt = Df(\bar{p})(p - \bar{p}).$$

As $Df(\bar{p})$ has -1 as an eigenvalue with multiplicity 2, the solution to this process is of the form

$$(p_1(t), p_2(t)) = (\bar{p}_1 + K_1e^{-t}, \bar{p}_2 + K_2e^{-t} + K_1te^{-t}); \qquad K_1, K_2 \in \mathbb{R}.$$

Finally we notice that the theorem extends to cover the case with a globally stable system without any further restrictions, as the proof only uses the behavior of the adjustment process in a neighborhood of the equilibrium.

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REFERENCES

1. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.
2. T. NEGISHI, The stability of a competitive economy: A survey article, *Econometrica* **30** (1962), 635–669.
3. H. UZAWA, The stability of dynamic processes, *Econometrica* **29** (1961), 617–631.

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A Criterion for the Global Stability of a Price Adjustment Process

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1. INTRODUCTION

For a market in which n commodities are offered for sale, consider the well-known type of price adjustment process [1, 2]

$$\frac{dp}{dt} = f(p), \qquad t \geq 0, p(0) = p_0 \tag{1}$$

in which p is the n-vector of prices, p_0 is the initial price vector, and f takes into account the pertinent economics. Here n is an arbitrary positive integer, and, to fix ideas, the function f can be viewed as usual as being basically an excess-demand function. However, our assumptions concerning f are in some respects nonstandard; the reader is asked to keep in mind that no assumptions are made that are not stated.

Our main purpose is to give, and to discuss, a necessary and sufficient condition under which, in a certain setting of economic interest, there is an equilibrium point p^* such that all solutions of (1) approach p^* as $t \to \infty$.

Also considered, in Sections 5 and 6, are related results concerning weak gross substitute demand functions Q. Necessary and sufficient conditions are given for the existence of Q^{-1} (when Q satisfies a certain reasonable condition) and for the existence and antitonicity of Q^{-1}. In addition, under the hypotheses that Q^{-1} exists and is antitone, and that a certain mild regularity condition is met, it is proved that Q is dynamically stable in the strong sense that a familiar associated price adjustment process is globally stable.

2. NOTATION AND ASSUMPTIONS CONCERNING (1)

The following notation is used throughout the paper. The set of all real n-vectors is denoted by R^n. For each $x \in R^n$, x_i denotes the ith component of x

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for each i, and $|x|$ denotes $\sum_{i=1}^{n} |x_i|$. For $x \in R^n$ and $y \in R^n$, the inequality $x \geq y$ $(x > y)$ means that $x_i \geq y_i$ $(x_i > y_i)$ for all i. The zero vector of R^n is denoted by θ.

We suppose that there are two elements of R^n, β and α, with $\beta > \alpha$ such that the domain of f is the n-dimensional price interval $S = \{z \in R^n: \alpha \leq z \leq \beta\}$. Roughly speaking, S is the set of all conceivable price vectors of interest to us.

It is assumed that f is a continuous mapping from S into R^n such that for each i, $f_i(x) \leq f_i(y)$ whenever $x \in S$, $y \in S$, $x \leq y$, and $x_i = y_i$ (i.e., f has the weak gross substitutes property). It is also supposed that f satisfies the Lipschitz condition $|f(x) - f(y)| \leq \rho |x - y|$ for all $x \in S$ and $y \in S$, in which ρ is a positive constant.

It is further assumed that S contains two price vectors v and u such that

$$v \geq u \quad \text{and} \quad f(u) \geq \theta \geq f(v). \tag{2}$$

It is unnecessary to introduce additional assumptions to prove the results we wish to describe.[1] However, insofar as the interpretation of the results is concerned, we think of u and v as having sufficiently small and sufficiently large components, respectively, that (2) holds and D defined by $D = \{z \in S: u \leq z \leq v\}$ contains all initial price vectors of interest to us.

By a solution of (1) is meant a function p that takes values in S, is defined and differentiable with respect to t on $[0, \infty)$, and satisfies (1) with $p_0 \in D$. The following result is proved in [3].

THEOREM 1. *For each $p_0 \in D$, there is a unique solution of* (1).

3. THE MAIN STABILITY RESULT AND RELATED DISCUSSION

We refer to the following conditions.

Condition 1. There exists an element h of D such that

$$(v_i - h_i) f_i(v) < 0 \qquad \text{for some } i$$

for each $v \in D$ with the property that $v \neq h$.

Condition 2. For each x and y in D such that $x \geq y$ and $x \neq y$, we have $f_i(x) < f_i(y)$ for some i.

Our principal result, which is proved in Section 4, is

THEOREM 2. *There is a point $p^* \in D$ such that $p(t)$ of* (1) *satisfies $p(t) \to p^*$ as $t \to \infty$ for every $p_0 \in D$ if and only if Condition* 1 *is satisfied.*

---

[1] In particular, it is not necessary to make the natural assumption that α (and therefore u) have components that are all nonnegative.

Since f is continuous, it follows that p^* of Theorem 2 satisfies $f(p^*) = \theta$ (i.e., p^* is an equilibrium point).

When $n = 1$, Condition 1 has the simple interpretation that there is a point h such that the values of f are negative to the right of h (i.e., are negative to the right of h for all arguments in D that lie to the right of h) and are positive to the left of h. Condition 1 is not only an interesting extension of the condition for $n = 1$, it is also a useful condition. For example, the following proposition is proved in Section 4.

PROPOSITION 1. *If Condition 2 is satisfied, then Condition 1 holds.*

Therefore Condition 1 is met if f is an excess-demand function with the very reasonable property that the excess demand for at least one commodity is increased whenever some prices are lowered and all others are unchanged.

We now consider the case in which f is *differentiable*, by which we mean that each $f_i(x)$ has a total differential $\sum_{j=1}^{n} f_{ij}(x)\, dx_j$ at each $x \in D$ in the sense that we have

$$f_i(y) = f_i(x) + \sum_{j=1}^{n} f_{ij}(x)(y_j - x_j) + o(\|y - x\|)$$

for y and x in D. The matrix of order n whose elements are the $f_{ij}(x)$ is denoted by $f'(x)$ for $x \in D$.

Since Condition 1 is satisfied whenever Condition 2 is met, the following result (which is proved in Section 4) shows, roughly speaking, that the price adjustment process is globally stable if every linearization of the process is stable.

PROPOSITION 2. *Condition 2 is satisfied if f is differentiable and the eigenvalues of $f'(x)$ have negative real parts for every $x \in D$.*[2]

It is known that certain very similar assertions concerning global stability and more general systems of nonlinear differential equations are false. (For an excellent and detailed discussion of that topic, see [4].)

4. PROOF OF THE RESULTS IN SECTION 3

Proof of Theorem 2

We shall use the following two lemmas, the first of which is proved in [3]. Lemma 2 is a direct modification of a result proved in [5] (where the function that plays the role of our f is defined on a different domain and satisfies a condition obtained from basically our weak gross substitutes condition by reversing the inequality signs).

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[2] The example: $n = 1$, $S = D = [0, 2]$, and $f(p) = 1 - p^2$ for $p \in S$ (for which $f'(0) = 0$) shows that the converse of Proposition 2 is false for a more serious reason than the fact that f of Section 2 need not be differentiable in order to satisfy Condition 2.

LEMMA 1. *There is an equilibrium point*[3] *in D, and if there is exactly one equilibrium point p^* in D, then $p(t)$ of* (1) *satisfies $p(t) \to p^*$ as $t \to \infty$ for each $p_0 \in D$.*

LEMMA 2. *The element e of D is the unique equilibrium point in D if and only if $(v_i - e_i) f_i(v) < 0$ for some i for each $v \in D$ such that $v \neq e$.*

Suppose Condition 1 is satisfied. Then, using Lemma 2, there is exactly one equilibrium point in D, and, by Lemma 1, there is a $p^* \in D$ such that $p(t) \to p^*$ as $t \to \infty$ for every $p_0 \in D$.

Now suppose that Condition 1 is not satisfied. Since, according to Lemma 1, there exists at least one equilibrium point in D, we see, using Lemma 2, that there are at least two equilibrium points in D. Of course, it follows that there cannot be a $p^* \in D$ such that $p(t) \to p^*$ as $t \to \infty$ for every $p_0 \in D$. This completes the proof of Theorem 2.

Proof of Proposition 1. As indicated in the proof of Theorem 2, there is at least one equilibrium point in D. In [6], it is shown[4] that Condition 2 is equivalent to the condition that $(x_i - y_i)[f_i(x) - f_i(y)] < 0$ for some i for any x and y in D such that $x \neq y$. Thus, if Condition 2 is met, and h is an equilibrium point in D, we have $(v_i - h_i) f_i(v) < 0$ for some i for every $v \in D$ such that $v \neq h$.

Proof of Proposition 2. Suppose that f is differentiable and that all eigenvalues of $f'(x)$ have negative real parts throughout D. Let P_M denote the class of real matrices of order n whose principal minors are positive. According to a result proved in [7], we have $-f'(x) \in P_M$ for every $x \in D$. Therefore, by [8, Theorem 3], the only solution in D of the system of inequalities $x \geq a$ and $f(x) \geq f(a)$ with $a \in D$ is the trivial solution $x = a$. Thus, for any x and y in D such that $x \geq y$ and $x \neq y$, we have $f_i(x) < f_i(y)$ for some i, which completes the proof.

5. PROPERTIES OF INVERTIBLE W.G.S. PRICE-DEMAND RELATIONS

5.1. *Preliminaries*

In Sections 5 and 6, P denotes $\{x \in R^n : x > \theta\}$.

We say that a function A from P into P has an *inverse* A^{-1} if $A(x) = y$ has a unique solution x (which we denote by $A^{-1}(y)$) in P for each $y \in P$, and A^{-1} is said to be *antitone* if it exists and we have $A^{-1}(a) \leqslant A^{-1}(b)$ whenever a and b are elements of P such that $a \geqslant b$.

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[3] As usual, by an equilibrium point is meant an $x \in S$ such that $f(x) = \theta$.

[4] In this connection, observe that x and y in D such that $x \geq y$ and $f_i(x) < f_i(y)$ implies that $x_i > y_i$.

5.2. *The Price-Demand Relation and Theorem* 3

Let Q denote a continuous demand function that maps price vectors in P into demand vectors in P such that the weak-gross-substitutes condition is satisfied (i.e., such that for each i, $Q_i(a) \geqslant Q_i(b)$ for a and b in P such that $a \geqslant b$ and $a_i = b_i$).

Conditions under which Q^{-1} exists, as well as properties of Q^{-1} such as its antitonicity or lack of antitonicity, are often of interest in applications of economic theory. One very reasonable and easily interpreted assumption concerning Q is that the following condition is satisfied.

Condition 3. For each q and r in P, there is a $p \in P$ such that $p \geqslant r$ and $Q(p) \leqslant q$.

A corollary of Theorem 3, the principal result given in Section 5, is that Q^{-1}, assuming it exists, is antitone if and only if Condition 3 is met. (An $n = 2$ example of a Q for which Q^{-1} exists, Condition 3 is violated, *and* Q satisfies the reasonable condition that each $Q_i(p)$ is a nonincreasing function of p_i is given by $Q_1(p) = p_2$ and $Q_2(p) = p_1$. For an example along the same lines in which each $Q_i(p)$ is a strictly decreasing function of p_i, consider $q_1 = p_2(p_1 + 2)(p_1 + 1)^{-1}$ and $q_2 = p_1(p_2)^{-1}$.)

To state the theorem, which also gives necessary and sufficient conditions under which Q^{-1} exists and is antitone, it is necessary to introduce the following two conditions.

Condition 4. For each q and r in P, there is a $p \in P$ such that $p \leqslant r$ and $Q(p) \geqslant q$.

Condition 5. For each a and b in P such that $a \geqslant b$ and $a \neq b$, we have $Q_k(a) < Q_k(b)$ for some index k.

THEOREM 3. *The following three statements are equivalent.*

(i) Q^{-1} *exists and is antitone.*

(ii) Q^{-1} *exists and Condition* 3 *holds.*

(iii) *Conditions* 3, 4, *and* 5 *are satisfied.*

The theorem is proved in Section 6.

In the next section we describe a result to the effect that if Q^{-1} exists and Condition 3 is met, and if a certain mild regularity condition is satisfied, then the price-demand function is dynamically stable in the sense that a familiar associated price adjustment process[5] is globally stable.

5.3. *Strong Stability of an Associated Price Adjustment Process*

We assume in this section that Q satisfies the Lipschitz condition that for each r and s in P such that $r \leq s$, there is a positive constant $\rho(r, s)$ with the property that $|Q(a) - Q(b)| \leq \rho(r, s)|a - b|$ for a and b in $\{x \in P: r \leq x \leq s\}$.

Consider the adjustment process

$$\frac{dp}{dt} = Q(p) - q, \qquad t \geq 0, p(0) = p_0 \tag{3}$$

in which q and p_0 are elements of P. As usual, by a solution of (3) is meant a differentiable P-valued function $p(\cdot)$ defined on $[0, \infty)$ such that (3) is satisfied. Assuming that Q^{-1} exists, we say that (3) is *globally stable* if for each q and p_0, (3) has a unique solution $p(\cdot)$ and $p(t) - Q^{-1}(q) \to \theta$ as $t \to \infty$.

THEOREM 4. *The process* (3) *is globally stable if Q^{-1} exists, Condition* 3 *is met, and the Lipschitz condition is satisfied.*

The proof is given in Section 6.

Using Theorem 3 and the main result given in Section 3, a similar theorem can be proved concerning the more general case in which $Q(p) - q$ in (3) is replaced with $H[Q(p) - q]$, where, for example, H is a diagonal matrix of order n with positive diagonal elements.

6. PROOFS

6.1. *Proof of Theorem* 3

We use the following two lemmas.

LEMMA 3. *If q, x, and y are elements of P such that $Q(x) - q \geq \theta \geq Q(y) - q$ and $x \leq y$, then there is a z in P such that $Q(z) = q$ and $x \leq z \leq y$.*

LEMMA 4. *If Condition 5 is met, then a and b in P such that $Q(a) \geq Q(b)$ implies that $a \leq b$.*

Lemma 1 is proved in [9], and Lemma 2 is a special case of Lemma 1 in [10].

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[5] See, for example, [1]. Section 5.3 considers the case of a market in which in the short run the supply of each commodity (such as a housing unit of a certain desirability) can be considered fixed.

Suppose that (iii) holds, and let $q \in P$ be given. According to Conditions 3 and 4, there are elements x and y of P such that $Q(x) - q \leq \theta \leq Q(y) - q$ and $y \leq x$. By Lemma 3 there is a $z \in P$ such that $Q(z) = q$, which shows that Q maps P onto itself. From Lemma 4 it follows that Q is one to one on P, and that Q^{-1} is antitone. Therefore (iii) implies (i).

Now suppose that (i) holds, and let q and r be elements of P. Let $p \in P$ satisfy $Q(p) = \min[q, Q(r)]$, in which $\min[q, Q(r)]$ denotes the element of P whose ith component is $\min[q_i, Q_i(r)]$ for each i. Clearly, $Q(p) \leq q$ and $Q(p) \leq Q(r)$. From $Q(p) \leq Q(r)$ and the antitonicity of Q^{-1}, we have $p \geq r$. This shows that (i) implies (ii).

Finally, assume that (ii) holds. Suppose that Condition 5 is not satisfied, in which case there are r and u in P such that $r \leq u$, $r \neq u$, and $Q(r) \geq Q(u)$. Let q denote $Q(u)$. We have $Q(r) - q \geq \theta$, and (using Condition 3) there is a $p \in P$ such that $p \geq r$ and $Q(p) - q \leq \theta$. By Lemma 3, there is a $z \in P$ such that $Q(z) = q$ and $z \geq r$. Since $Q(u) = q$ and $z_i > u_i$ for some i, we have a contradiction to the hypothesis that Q^{-1} exists. This proves that Condition 5 is met. By Lemma 4, Q^{-1} is antitone, and a proof similar to the one given in the immediately preceding paragraph shows that Condition 4 is satisfied. This shows that (ii) implies (iii), and it completes the proof.

6.2. *Proof of Theorem* 4

Referring to (3), let q and p_0 be given. According to Theorem 3, there are r and u in P such that $r \leq p_0 \leq u$ and $Q(u) - q \leq \theta \leq Q(r) - q$. Of course $|Q(a) - Q(b)| \leq \rho(u, r)|a - b|$ for a and b in the set D defined in Section 2. By Theorem 1, there is a solution $p(\cdot)$ of (3). By Theorem 2, Proposition 1, and the fact that Condition 5 is met (see Theorem 3), there is a p^* in D such that $p(t) - p^* \to \theta$ as $t \to \infty$. Since Q^{-1} exists, $p^* = Q^{-1}(q)$. (For an alternative proof of $p(t) - Q^{-1}(q) \to \theta$ as $t \to \infty$, see Lemma 1.) In view of the hypothesis that the Lipschitz condition is satisfied, $p(\cdot)$ is the unique solution of (3). This completes the proof.

REFERENCES

1. P. A. SAMUELSON, "Foundations of Economic Analysis," p. 263, Harvard Univ. Press, Cambridge, Mass., 1966.
2. K. J. ARROW AND L. HURWICZ, Competitive stability under weak gross substitutability: The "Euclidean distance" approach, *Internat. Econ. Rev.* **1** (1960), 38–49.
3. I. W. SANDBERG, A theorem concerning properties of an arms-race model, *IEEE Trans. Systems Man. Cybernetics* **1** (1978), 29-31.
4. J. C. WILLEMS, "Analysis of Feedback Systems," Research Monograph No. 62, pp. 161–182, M.I.T. Press, Cambridge, Mass., 1971.
5. I. W. SANDBERG, On the mathematical theory of social processes characterized by weak reciprocity, *J. Peace Sci.* **1** (1978), 1–30.

6. I. W. SANDBERG, Some theorems concerning multifirm alternatives to the single regulated multiservice firm, *IEEE Trans. Systems Man Cybernetics* **2** (1975), 201–208.
7. L. A. METZLER, Stability of multiple markets: The Hicks conditions, *Econometrica* **13** (1945), 272–292.
8. D. GALE AND H. NIKADO, The Jacobian matrix and global univalence of mappings, *Math. Ann.* **159** (1965), 81–93.
9. I. W. SANDBERG, Proof of convergence for a nonlinear SOR-type process, modeling and simulation, *in* "Proceedings of the Sixth Annual Pittsburgh Conference, University of Pittsburgh, 1975," Vol. 6, pp. 997–999.
10. I. W. SANDBERG, Some qualitative properties of nonlinear Richardson-type arms race models, *in* "Mathematical Systems in International Relations Research," (J. V. Gillespie and D. A. Zinnes, Eds.) Praeger, New York, 1977.

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Barriers to Trade and Disadvantageous Middlemen: Nonmonotonicity of the Core*

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In applying cooperative game theory to economic problems of exchange, it is standard to assume that all logically possible coalitions may form. However, because of institutional, legal, or physical barriers, it may in fact be impossible for certain sets of agents to communicate or trade with one another directly. It would seem worthwhile to recognize this and to analyze the impact of such barriers to trade. In [3], Myerson introduced the concept of a game with a communications graph,[1] which is a graph whose nodes correspond to the players in a game and within which the presence or absence of a particular link indicates the possibility or impossibility of communication

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[1] Let $N = \{1, 2,\dots, n\}$. A graph on the set of nodes N is a collection of links $\mathbf{ij}$ where i and j are distinct elements of N and $\mathbf{ij} = \mathbf{ji}$. A graph is complete if it contains all the possible links. A path from node i to node j is a collection of nodes $i_1, i_2,\dots, i_k$ with $i_1 = i$, $i_k = j$, and $\mathbf{i}_r\mathbf{i}_{r+1}$ is a link in the graph for $r = 1, 2,\dots, k - 1$. A graph is connected if there is a path from any one node to any other node.

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between the relevant pair of players. Here we apply this approach to the problem of exchange. If a particular pair of traders are linked in the graph, direct trade between them is possible; if they are not linked, they cannot trade directly with one another, although it may be possible to trade indirectly through other linked traders.

The use of the graph structure on the set of traders permits the modeling of differing forms of market organization. For example, if the graph g on the set of traders is not connected, we are effectively looking at a system of autarkies, while if g is connected but not necessarily complete one obtains models of different forms of economic systems. For example, the complete graph in part (a) of Fig. 1 represents a system under which all agents are free to exchange directly with one another, while the graph in part (b) represents the existence of a middleman through which all trade flows. Figure 2 can be considered as representing two economies, where all foreign trade in one economy must flow through an export import agency.

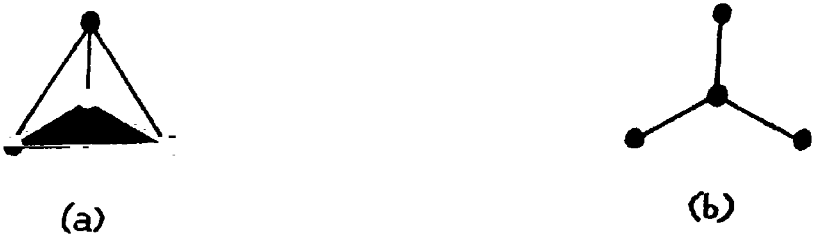

FIGURE 1

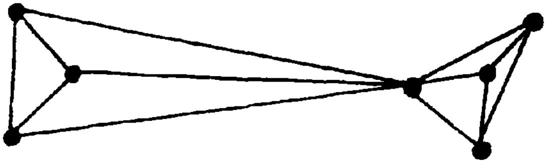

FIGURE 2

A particularly interesting aspect of the question of comparing different systems involves comparative statics analysis on the equilibria as the communications graph is changed, with a view to answering such questions as precisely who gains or loses when a link is introduced or deleted from g. A specific example of this involves the role of the middleman. Suppose that initially g is complete and then all links except those with one particular agent, say 1, are broken (as in Fig. 1). Since the graph is still connected, the set of Pareto optima is unchanged. However, no multiplayer coalitions not involving player 1 are connected in the second situation. Thus, the core allocations of the second economy will include those of the first. Intuitively, one might expect that the player in the middle would not do worse in this situation: all trade must flow through him, which ought to improve his position. Somewhat more formally, all coalitions including this player remain as before, while those excluding him are now powerless. The question is then whether he may in fact lose from having his special position.

The definition of "may lose" we adopt here involves a comparison of the core allocations to the middleman. Clearly, the core in the game with a middleman will contain that in the game where all coalitions are possible. We say then that the middleman may lose if there is a core allocation x in the middleman economy which is strictly worse for him than every core allocation of the unrestricted economy. Theorem 1 shows that in three-person games arising from situations of exchange, the middleman cannot lose. However, if we consider general three-person games, this result is not true: for some three-person games, if we do not allow the coalition of players 2 and 3 to form, every point entering the core may be no better or even strictly worse for player 1 than every point in the core of the original game. Moreover, in general n-person games arising from markets, there may be points entering the core which are worse for the middleman than any point in the core of the original game. However, in such cases, if preferences are strictly monotonic and the trade through the middleman is beneficial to the grand coalition, then some of the new core points will be better for the middleman.

An economic agent or trader is defined by his characteristic $a_i = (u^i, w^i)$, where u^i is a continuous, quasi-concave utility function over R_+^m and where $w^i \in R_+^m$. Here, m is the number of commodities and w^i is his initial endowment. Given n such traders a_i, $i \in N = \{1,\dots, n\}$, a communications graph is a graph g whose nodes correspond to the n traders. An economy E is then a pair (a, g), where $a = (a_1, \dots, a_n)$ and g is a communications graph. A coalition is a nonempty subset of N. A coalition S is connected if the restriction of g to S is connected. We assume that N is connected and that $\{i\}$ is connected for every $i \in N$. An allocation is a vector $x = (x^1, \dots, x^n)$, $x^i \in R_+^m$. An allocation x is feasible for the coalition S (S-feasible) if $\sum_{i\in S} x^i = \sum_{i\in S} w^i$ and S is connected. If x is feasible for N, we simply say that it is feasible. A coalition S can improve upon (block) an allocation x if S is connected and there exists an allocation y which is feasible for S and such that $u^i(y^i) > u^i(x^i)$ for every $i \in S$. The core of E consists of those feasible allocations which cannot be improved upon by any coalition.

Let $g_c = \{\mathbf{ij}: i, j \in N\}$ be the complete graph with n traders and let $g_m = \{\mathbf{1i}: i \in N, i \neq 1\}$. Then g_c represents free trade and g_m represents trading through a middleman (trader 1). Let $E_c = (a, g_c)$ and $E_m = (a, g_m)$ be the economies corresponding to these two structures. We are interested in comparing 1's payoffs at core allocations in E_c and E_m.

THEOREM 1. *Let $n \geq 3$ and suppose u^i is strictly monotonic for all i. For every $x \in$* Core(E_m) *there exists a $y \in$* Core(E_r) *with $u^1(x^1) \geq u^1(y^1)$.*

Thus, for every solution allocation x which arises when he is a middleman there is a solution allocation y in the economy with unrestricted trading which is no better than x: any point becoming a solution when he becomes a middleman is better than some solution under free trade.

To prove this theorem we use the methods of the theory of balanced cooperative games.

An n-person cooperative game is defined to be a collection $V = \{V_S\}_{\emptyset \neq S \subseteq N}$, where, for every nonempty $S \subseteq N$, V_S is a nonempty, strict subset of R^n which is closed and comprehensive (i.e., if $x \in V_S$, $y \in R^n$ and $y^i \leq x^i$ for every $i \in S$ then $y \in V_S$). We let $0_S = \{x \in V_S$: for some $y \in V_S$, $y^i > x^i$ for every $i \in S\}$. Then Core$(V) = V_N \cap (\bigcap_{S \neq \emptyset} 0_S^c)$, where the superscript c denotes the complement of a set.

With an economy E we can associate an n-person cooperative game $V(E)$ as follows. We let u^i be a utility function for trader i normalized so that $u^i(w^i) = 0$. For $\emptyset \neq S \subseteq N$, define

$$V_S = \{v \in R^n : \text{there exists some } S\text{-feasible allocation } x \text{ with } v^i \leq u^i(x^i)$$
$$\text{for every } i \in S\} \text{ if } S \text{ is connected,}$$

and

$$V_S = -R^n_+ = \{v \in R^n : v^i \leq 0 \text{ for } i = 1, 2, \ldots, n\} \text{ if } S \text{ is not connected.}$$

A collection of coalitions $\{S_r\}_{r \in R}$ is called balanced (see [4]) if there is a collection of nonnegative real numbers $\{\delta_r\}_{r \in R}$ such that for every $i \in N$, $\sum_{r: i \in S_r} \delta_r = 1$.

LEMMA. *Let $E_r = (a, g_r)$ be an economy with strictly monotonic preferences and let V be an n-person game associated with it.*

1. *For every nonempty coalition S and for every $x, y \in V_S$, if $0 \leq y^j \leq x^j$ for every $j \in S$ and $y \neq x$, then $y \in 0_S$. Thus the Pareto surface of V_S contains no segments which are parallel to the axes of the players in S.*

2. *For every $x \in R^n$, $x \in$* Core(V) *if and only if for some $y \in$* Core(E_r) *$x = (u^1(y^1), u^2(y^2), \ldots, u^n(y^n))$.*

3. *For $x \in R^n$ let $T_x = \{S \subseteq N: x \in V_S\}$. If T_x is a balanced collection of sets with weights $\{\delta_S\}_{S \in T_x}$, and if $x \in 0_S$ for some $S \in T_x$ with $\delta_S > 0$, then there exists $y \in V_N$ with $y^i > x^i$ for every $i \in N$.*

Proof of the Lemma. Part 1 follows immediately from the monotonicity of the utility functions. Part 2 follows from the definitions of the cores and from part 1.

To prove part 3 we use a method due to Scarf [7]. Assume x satisfies the hypotheses of the lemma. For every $S \in T$, there exists an S-feasible allocation z_S with $x^i \le u^i(z_S^i)$ for every $i \in S$. Also for some $S' \in T$, $z_{S'}$ can be chosen so that $x^i < u^i(z_{S'}^i)$ for every $i \in S'$. Consider the allocation z defined by $z^i = \sum \delta_S z_S^i$, where the summation is over those $S \in T$, with $i \in S$. Observe that z^i is a convex combination of the z_S^i's and therefore (by the quasi-concavity of the utility functions) $u^i(z^i) \ge x^i$ for every $i \in N$. Also, $u^i(z^i) > x^i$ for every $i \in S$. Let $y^i = u^i(z^i)$, $i \in N$. Since

$$\sum_{i \in N} z^i = \sum_{i \in N} \sum_{S \in T, i \in S} \delta_S z_S^i = \sum_{S \in T} \delta_S \sum_{i \in S} z_S^i \le \sum_{S \in T} \delta_S \sum_{i \in S} \omega^i = \sum_{i \in N} \omega^i,$$

z is feasible and thus $y = (y^1, \ldots, y^n) \in V_N$. By the monotonicity of the utility functions we can slightly perturb y to have $y^i > x^i$ for every $i \in N$.

Proof of Theorem 1. Let V be the game associated with E_c, and V' be that associated with E_m. We wish to show that for any $z \in$ core V' there exists $w \in$ core V with $z^1 \ge w^1$. To this end, suppose $z^1 \le v^1$ for all $v \in B = V'_{\{123\}} \cap O^c_{\{12\}} \cap O^c_{\{13\}} \cap O^c_{\{1\}} \cap O^c_{\{2\}} \cap O^c_{\{3\}}$, $z \in B$, and $z \notin O_{\{123\}}$. (Note that the core of V' is the set of Pareto optima in B.) If we can show that $z \notin O_{\{23\}}$, then z must also be in the core of V. Then any point in the core V' is at least as good for 1 as some point in core V, namely z, and we are done.

Thus, to obtain a contradiction, we will suppose that $z \in O_{\{23\}}$. Let $T = \{S \subseteq N: S \ne \emptyset, S \ne \{23\}$, and $z \in V_S\}$. Then $1 \in \bigcup_{S \in T} S$, since otherwise the comprehensiveness of the V_S sets would imply that z does not minimize z^1 over B. If $\{1\} \in T$ then the proof of is completed because part 3 of the lemma would contradict the Pareto optimality of z ($\{1\}, \{23\}$ is a balanced collection). Then suppose $\{1\} \notin T$. We claim $i \in \bigcup_T S$, $i = 1, 2$. For example if $2 \notin \bigcup_T S$, i.e., $z \notin V_{\{2\}} \cup V_{\{12\}}$, then for some small enough $\epsilon > 0$, $z' = z - (0, \epsilon, 0)$ also minimizes z^1, while $z' \in B$, $z' \notin O_{\{123\}}$, and $z' \notin V_S$ for any $S \ne N$ with $2 \in S$. Therefore for some small enough $\delta > 0$, $z'' = z' + (0, 0, \delta)$ we have $z'' \notin O_{\{123\}}$ and $z'' \notin V_S$ for $\emptyset \ne S \ne N$ and $S \ne \{2, 3\}$. Thus z^1 is minimized at an interior point of B, which is impossible. So we assume without loss of generality that $\{12\} \in T$ and either $\{3\} \in T$ or $\{13\} \in T$. If $\{13\} \in T$ then we obtain a contradiction by the lemma ($\{12\}, \{13\}, \{23\}$ is a balanced collection). So we are left with the cases $T = \{\{12\}, \{3\}\}$ or $T = \{\{12\}, \{3\}, \{2\}\}$. In either of these two cases it is possible to transfer a small amount of utility from 1 to 2 while still staying in B, which again yields a contradiction. Thus, $z \notin O_{\{23\}}$, so z is unblocked in V, and thus all points in the core of V' are at least as good for the middleman as some point in the core of V.

This result seems intuitive, and it is surprising that the proof is so involved. However, the following examples show that the analogue of this theorem is not true for general three-person cooperative games, or for market games with a larger number of players.

We first show an example of a three-person nonmarket, cooperative game in which the middleman may lose. Let ϵ be a small nonnegative real number and define the game V ($= V(\epsilon)$) as follows:

$$\begin{aligned}
V_{\{i\}} &= \{v \in R^3 : v^i \leq 0\} \quad \text{for } i = 1, 2, 3;\\
V_{\{123\}} &= \{v \in R^3 : \text{for some } x \in \text{convex hull } \{A, B, C, F\},\ v \leq x\};\\
V_{\{12\}} &= \{v \in R^3 : v^1 + v^2 \leq 0.01\};\\
V_{\{13\}} &= \{v \in R^3 : v^1 + v^3 \leq 1\};\\
V_{\{23\}} &= \{v \in R^3 : \text{for some } x \in \text{convex hull } \{C, D, E\},\ v \leq x\},
\end{aligned}$$

where

$$\begin{aligned}
A &= (1 + \epsilon, 0, 0), & B &= (1, 1, 0), & C &= (0, 0, 1),\\
D &= (0, 1, 0), & E &= (0, \tfrac{1}{2}, \tfrac{3}{4}), & F &= (0, 2, 0),\\
G &= (1, 0, 0). & & & &
\end{aligned}$$

The core of $V(\epsilon)$ is $B = \{(1, 1, 0)\}$. When the coalition $\{2, 3\}$ is no longer allowed to block, the core becomes all of the triangle ABC with the exception of a small set of points near C. Thus the middleman is worse off at most of the points of the new core. Moreover, by letting $\epsilon \to 0$ we can make the proportion of the points where the middleman loses go to one. When $\epsilon = 0$

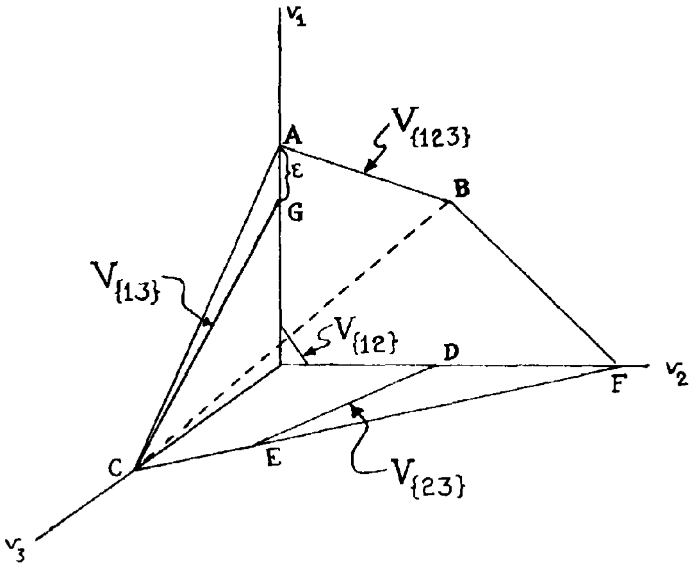

FIGURE 3

the middleman is never better off, and he is worse off at almost all the new points. Note, however, that when $\epsilon \to 0$, V_N contains segments which are parallel to the v^2 axis. Note too that none of these games are balanced in the sense of Billera and Bixby [2], and so they could not come from markets with concave utility functions.

Thus, the condition that the game be balanced is crucial to the middleman not losing in three-person games. One might still hypothesize that Theorem 1 would continue to hold for market games with any finite number of players. However, the following example indicates that this is not the case.

Consider the economy with five goods and five people, A, B, C, D, and E, whose utility functions and endowments are as follows:

$$
\begin{array}{llll}
U^A(x_1, x_2, x_3, x_4, x_5) = x_1 x_2, & & \omega^A = (1, 0, 0, 0, 0), \\
U^B(x_1, x_2, x_3, x_4, x_5) = x_1 x_2, & & \omega^B = (0, 1, 0, 0, 0), \\
U^C(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_5)\, x_3, & & \omega^C = (0, 0, 1, 0, 0), \\
U^D(x_1, x_2, x_3, x_4, x_5) = (x_2 + x_5)\, x_4, & & \omega^D = (0, 0, 0, 1, 0), \\
U^E(x_1, x_2, x_3, x_4, x_5) = (x_3 + x_4)(x_5 + 8), & & \omega^E = (0, 0, 0, 0, 1).
\end{array}
$$

With all coalitions allowed, the coalition $\{A\ B\}$ can block any allocation unless $U^A + U^B \geq 1$. There are no feasible allocations for which this is possible except those in which A and B end up with all of the goods x_1 and x_2. Hence core allocations will be those in which the goods x_1 and x_2 are efficiently distributed among A and B and the goods x_3, x_4, and x_5 are divided among C, D, and E in such a way that no coalition can block the allocation.

Consider now the coalition $\{C\ E\}$. If C gets the bundle $(0, 0, \alpha, 0, 1)$ and E gets $(0, 0, 1 - \alpha, 0, 0)$, $U^C = \alpha$ and $U^E = 8(1 - \alpha)$. If $\alpha < 1/9$ the marginal rate of substitution for C of good 5 for good 3 is $1/\alpha > 9$. For E the marginal rate of substitution between goods 5 and 3 is $8/(1 - \alpha) < 9$, and hence this allocation of the goods of C and E is efficient with respect to them. Thus among the allocations which $\{C, E\}$ can block are any which give $U^C = \alpha < 1/9$ and $U^E < 8(1 - \alpha)$. Similarly $\{D, E\}$ can block any allocation which gives D, $U^D = \alpha < 1/9$ and $U^E < 8(1 - \alpha)$.

Now consider the coalition $\{C\ D\ E\}$. If C gets $(0, 0, 2\alpha, 0, 1/2)$ and D gets $(0, 0, 0, 2\alpha, 1/2)$, E gets $(0, 0, 2 - 4\alpha, 0)$. The utilities are $U^C = U^D = \alpha$ and $U^E = 8(2 - 4\alpha)$. If $\alpha < 1/18$, $MRS^C_{5,3} = (1/2)/2\alpha > 9/2$ while $MRS^E_{5,3} = 8/(2 - 4\alpha) < 9/2$. Thus the allocation is efficient with respect to C and E. The same argument shows that it is efficient with respect to D and E. It is trivially efficient with respect to C and D as well, and no reallocation among the three agents can make all of them better off. Thus this allocation between C, D, and E is efficient. Thus any allocation which gives utilities $U^C = U^D = \alpha < 1/18$ and $U^E = 8(2 - 4\alpha)$ cannot be blocked by $\{C\ D\ E\}$. However, if $\alpha < 1/3$ the coalition $\{C\ E\}$ (or $\{D\ E\}$) can block since the utility which E can be guaranteed while C gets $U^C = \alpha$ is $U^E = 8(1 - \alpha) > 8(2 - 4\alpha)$.

We now claim that at any core allocation, we must have $U^E \geq 16/3$. To see this, note that if $U^E = \beta$, then $\{C\ D\ E\}$ can achieve $U^C = U^D = (16 - \beta)/32 = \alpha$ via the allocation given above, with $\alpha = 16 - \beta/32$. But if $\beta < 16/3$, this allocation is blocked by either $\{C\ E\}$ or $\{D\ E\}$, since in either E can get $8(1 - \alpha) = 8 - 8(16 - \beta)/32 = 4 + \beta/4 > \beta$, while the other member gets α, and it is then possible to redistribute so as to make both members strictly better off.

Now suppose we restrict blocking to only those coalitions containing E. Consider the allocation and associated utilities below:

$$
\begin{array}{lll}
x^A = (0, 0, 0, 0, 0), & & U^A = 0, \\
x^B = (0, 0, 0, 0, 0), & & U^B = 0, \\
x^C = (1, 0, \tfrac{2}{3} + \epsilon, 0, \tfrac{1}{2}), & & U^C = 1 + \tfrac{3}{2}\epsilon, \\
x^D = (0, 1, 0, \tfrac{2}{3} + \epsilon, \tfrac{1}{2}), & & U^D = 1 + \tfrac{3}{2}\epsilon, \\
x^E = (0, 0, \tfrac{1}{3} - \epsilon, \tfrac{1}{3} - \epsilon, 0), & & U^E = \tfrac{16}{3} - 16\epsilon.
\end{array}
$$

We will show that for suitably small ϵ, this allocation is in the core as defined above of the corresponding game V'.

The coalitions $\{A\ E\}$, $\{B\ E\}$, and $\{A\ B\ E\}$ clearly cannot block this allocation. The coalitions $\{C\ E\}$ and $\{D\ E\}$ cannot block since these coalitions cannot give C or D utility greater than 1. The coalition $\{C\ D\ E\}$ similarly cannot block since they cannot simultaneously guarantee C and D utility greater than 1/2. We will now examine the marginal rates of substitution of C, D, and E to show that this distribution is efficient among them. $MRS^C_{5,3}$ is $(3/2)/(2/3 + \epsilon) < 9/4$, while $MRS^E_{5,3} = 8/(2/3 - 2\epsilon) > 12$. Thus this allocation is efficient with respect to C and E. Similarly it is efficient with respect to D and E and trivially so with respect to C and D. Further, no reallocation between the three agents dominates this allocation. Adding A and B to $\{C\ D\ E\}$ cannot yield a blocking coalition either. The only coalitions left to examine are those which contain C or D (but not both), E, and A and/or B.

Consider the coalition $\{A\ C\ E\}$. Can this coalition block? It will be enough to show that with their combined resources C and E cannot both improve upon their utilities in the proposed allocation. Consider the distribution $x^C = (1, 0, \alpha/2, 0, 1)$, $x^E = (0, 0, 1 - \alpha/2, 0, 0)$. This gives $U^C = \alpha$ and $U^E = 8(1 - \alpha/2)$. If $\alpha \geq 2/5$, $MRS^C_{5,3} = 2/(\alpha/2) = 1/\alpha \leq 5/2$ and $MRS^E_{5,3} = 8(1 - \alpha/2) \geq 5/2$, hence the distribution is efficient between them. In the proposed allocation we have $U^E = 16/3 - 16\epsilon > 8[1 - (1 + 3/2\epsilon)/2] = 8(1/2 - 3/4\epsilon)$ for $0 < \epsilon < 2/15$. Hence even with A's endowment, C and E cannot both improve upon the utilities in the proposed allocation. Clearly adding B to the coalition $\{A, C, E\}$ will not change this. Also it is clear that the same argument shows that the coalitions $\{B, D, E\}$ and $\{A, B, D, E\}$ cannot block the proposed allocation.

Thus the proposed allocation is in the core if only coalitions containing E are allowed to block. Yet this allocation is clearly worse for E than any previous core allocation (when all coalitions were allowed to block). In fact, his minimal utility over core allocations has gone down from 16/3 to $16/3 - 16(4/30) = 3.2$.

The preferences in this example are not strictly monotonic, as was assumed in Theorem 1. However, it is possible to modify the example so that this condition holds and yet there are still core allocations when E is a middleman which are worse for him than any point in the core of the original game. Specifically, consider modifying the above example by adding a term $r(\Sigma x_j)$ to each player's utility function, which then becomes strictly monotone for all $r > 0$. We show that for r sufficiently small, there are still core points when E is a middleman yielding $U^E < 16/3$.

Consider the allocation

$$\begin{aligned} x^A &= (\delta, \delta, 0, 0, 0), \\ x^B &= (\delta, \delta, 0, 0, 0), \\ x^C &= (1 - 2\delta, 0, 21/30, 0, 1/2), \\ x^D &= (0, 1 - 2\delta, 0, 21/30, 0, 1/2), \\ x^E &= (0, 0, 9/30, 9/30, 0). \end{aligned}$$

This corresponds to $\epsilon = 1/30$ in the example above, except that a small amount of x_1 and x_2 has been taken from C and D and given to A and B. Arguments similar to those used above show that for $0 < \delta < 1/84$, this allocation yields a utility vector that not only is in the core of the game V' with E as a middleman (and the original preference orderings), but also has the property that, for all coalitions $S \neq N$, it lies outside V_S. Note that $U^E < 16/3$ at any such point.

Now, if we consider the games $V(r)$ and $V'(r)$ obtained by adding $r\Sigma x_j$ to each player's utility function, it is clear that for any point u^r in $V_S'(r)$ (resp., $V_S(r)$) there is some point $u \in V_S$ (resp., V_S') with $u_i^r \leq u_i + r\,|S|$, $i \in S$. Now consider sequences $\{V(r_k)\}$ and $\{V'(r_k)\}$, and let $u_k \in \text{core } V(r_k)$, $u_k \to u$. Then it is simple to show that $u \in \text{core } V$, so that for any $\epsilon > 0$ there exists K such that core $V(r_k)$ is contained in an ϵ neighborhood of core V for $k > K$. This in turn means that for any ϵ, the minimum of U^E on the core of $V(r_k)$ is at least $16/3 - \epsilon$ for all k large enough. If we can now show that for large enough k there are points in the core of $V'(r_k)$ which are arbitrarily close to the utility image of the allocation given above, we are done. To show this, take the utility image $\bar{u}$ of this allocation, and note that for k large enough (r_k small enough), $\bar{u}$ is not blocked by any $S \neq N$, since $V_S'(r_k)$ is within an ϵ-ball of V_S'. Now consider any Pareto optimal point u_k in $V'(r_k)$ with $u \leq u_k$. (Such a point must exist since $U^i(x) \leq U^i(x) + r\Sigma x_j$ for any

$x = (x_1, \ldots, x_5)$, with strict inequality if $x \neq 0$.) Since $V'_N(r_k)$ is within an ϵ-ball of V'_N, this sequence converges to u, and since u_k is unblocked in $V'(r_k)$, we are done.

It is worth noting, however, that although core $V'(r_k)$ contains points strictly worse for E than any in core $V(r_k)$, in contrast to the situation with V and V' it also contains points which are better for him than any in core $V(r_k)$. This is, in fact, generally true with strict monotonicity.

THEOREM 2. *Consider an economy where, for some trader i with strictly monotonic preferences, the initial endowment is not Pareto optimal for* $S = N \sim \{i\}$. *Then restriction of blocking to coalitions containing i must result in the addition of points to the core which i strictly prefers to any point in the core with unrestricted coalition formation.*

Proof. Consider a core allocation in the unrestricted game which gives the central trader utility $\bar{u}^i$ which is at least as high as any other core allocation At least one trader $j \neq i$ must have utility strictly higher than his initial endowment yields because ω is not Pareto optimal for $N \sim \{i\}$. Now consider a Pareto efficient allocation which gives i the maximal utility possible subject to the condition that the noncentral traders' utilities at least as big as their initial endowment utilities. Compactness of the feasible set and continuity of the utility functions guarantee that there is such an allocation and strict monotonicity implies that it yields $u^i > \bar{u}^i$. By construction, no coalition containing i can block this allocation; hence it is in the core when i is a middleman.

These examples of disadvantageous middlemen are reminiscient of the examples of disadvantageous monopolies or syndicates (see, e.g., [1]), although the phenomena do differ somewhat. Exploring the nature of the relationships between these two phenomena and, more generally, the monotonicity of the core would seem to be an important but difficult open problem. In any case, the rather surprising (to us, at least) nature of our results here would seem to indicate the value of further explorations of the communications graph approach.

REFERENCES

1. R. AUMANN, Disadvantageous monopolies, *J. Econ. Theory*, **6** (1973), 1–11.
2. L. BILLERA AND R. BIXBY, Market representations of n-person games, *Bull. Amer. Math. Soc.* **80** (1973).
3. R. MYERSON, Graphs and cooperation in games, *Math. Operations Res.*, **2** (1977), 225–229.
4. H. SCARF, The core of an n-person games, *Econometrica* **35** (1967).

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On the Concept of Fairness*

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1. While the concept of Pareto Optimality forms the basis for much of modern Welfare Economics, it has long been recognized that, alone, it is an insufficient criterion for social choice. The basic reason is that the set of Pareto Optimal allocations is simply too large. Foley [2] has attempted to narrow the set of allocations to be considered through the introduction of the additional concept of equity. Loosely, an allocation is equitable if each individual prefers his bundle to that received by any other individual. The imposition of such a condition is undoubtedly appealing and, though difficult to extend to cases of production and non-tradeable attributes, has generated substantial attention for problems of distributing a fixed bundle of goods.

Now, an equitable allocation need not be Pareto Optimal. For example, if individuals differ in their tastes, then the mere equal division of the total endowment will generally fail to be efficient. Consequently, various authors have investigated the existence of allocations which are both equitable and efficient (e.g., Kolm [3] and Varian [7]).

Feldman and Kirman [1] have indicated some potential problems with the joint use of equity and efficiency. Specifically, they have offered an example of a 2-person, 2-good exchange economy where an equitable endowment leads to a non-equitable competitive equilibrium. We shall show below (Theorem 1) that this result can be substantially strengthened. With great generality there will exist an equitable allocation in such economies which is neither efficient nor dominated (in the Pareto sense) by any equitable allocation. Consequently, mutually profitable trade from this endowment (including all fair net trades (see Schmeidler and Vind [5])) will result in an inequitable allocation. Further, with greater generality, there is an in-

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equitable allocation which is always Pareto dominated by some inequitable allocation.

After stating these results, their implications for the fairness concept are discussed. The main suggestion is that the concern for both equity and efficiency may be incompatible.

2. A 2-person, 2-good economy (henceforth simply an economy) is a tuple $e = (w, U)$ such that[1]:

(i) $0 \ll w = (w_1, w_2) \in E_+^2$,

(ii) $U = (U^1, U^2)$, where, for each individual i, U^i is a function from E^2 to the reals. (1)

Intuitively, w indicates the total amounts of the two goods available and U^i is individual i's ordinal utility function.

For each i, let $X^i = \{x \in E_+^2 \mid x \leqq w\}$ and let $X = \{x \in X^1 \times X^2 \mid$ for each good, $j = 1, 2\ \sum_{i=1}^{2} x_j^i = w_j\}$. An element $x \in X$ is called an allocation.

Given e, the set of Pareto efficient allocations is denoted by $P(e)$, i.e., $P(e) = \{x \in X \mid \forall y \in X,$ s.t. $y \neq x, U^i(y^i) \geqq U^i(x^i)$ for $i = 1, 2$ implies $U^i(x^i) = U^i(y^i)$ for $i = 1, 2\}$. An allocation x is called *equitable* iff $U^i(x^i) \geqq U^i(x^k)$ for all $i, k = 1, 2$, $i \neq k$. If x is both Pareto optimal and equitable, then it is called *fair* (see Varian [7]). $F(e)$ will denote the set of all fair allocations for the economy e. Also, let $\hat{w} = (w/2, w/2)$, i.e., the equal division allocation.

An economy e will be called an *A-type* economy if the following are true:

(i) $\forall i = 1, 2$, U^i is continuous, strictly monotonic and strictly quasi-concave,

(ii) $\forall i$, $\forall j$ and $\forall x \in \operatorname{int} X^i$, $\delta U^i/\delta x_j^i$ exists and is continuous, (2)

(iii) $\forall x \in P(e)$ with $x^i \neq 0$ or w, $i = 1, 2$, $x \in \operatorname{int} X$.

We have the following generalization of the Feldman and Kirman example:

THEOREM 1. *Let $e = (w, U)$ be any A-type economy. If $\hat{w}$ is not the unique fair allocation, then there exists an equitable allocation $x \in X/F(e)$ with no equitable $y \in X$, $y \neq x$ such that $(U^1(y^1), U^2(y^2)) \geq (U^1(x^1), U^2(x^2))$.*

Thus, if such an equitable and inefficient x is the endowment point, then *any* mutually advantageous trade must lead to an inequitable outcome. This suggests a conflict between the desire for mutual gain and that for equity.

A similar result which is true for *every* A-type economy is:

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[1] For $x, y \in E^2$, $x \geqq y$ iff $x_j \geqq y_j$, $j = 1, 2$; $x \geq y$ iff $x_j \geqq y_j$, $j = 1, 2$, and $x_j > y_j$ for some j; and $x \gg y$ iff $x_j > y_j$, $j = 1, 2$. For sets A and B, the notation A/B defines the set of elements of A which are not members of B.

THEOREM 2. *Let* $e = (w, U)$ *be any A-type economy, Then there exists an equitable* $x \in X$ *and an inequitable* $y \in X$ *such that* $U^i(y^i) > U^i(x^i)$ *for* $i = 1, 2$.

The proofs of Theorems 1 and 2 are presented in the appendix.

The above results can be interpreted as follows: Let us suppose that, given an economy e, we wish to select a subset $M \subset X$ as being those allocations which are socially desirable in some sense. Such an M can be viewed as the maximal elements of some Social Preference Relation (SPR), R_e, on the elements of X (where $xR_e y$ means that x is at least as socially desirable as y). Now if one is to be concerned with both efficiency and equity, then it is reasonable to require that the SPR reflect the Pareto ranking and some equitable ranking of the allocations. A simple consequence of Theorem 2 is that no SPR can adequately reflect both efficiency and equity considerations (see Theorem 3).

To make the above precise, given an economy e and some SPR, R_e, let P_e and I_e denote the strict preference and indifference relations generated by R_e. The Pareto SPR, R_e^p, is defined by $xR_e^p y$ iff $U^i(x^i) \geq U^i(y^i)$, $i = 1, 2$ and the equitable SPR, R_e^E, by $xR_e^E y$ iff x is equitable and y is not. An SPR, R_e, is called *Pareto inclusive*[2] iff $xR_e^p y$ implies $xR_e y$ and $xP_e^p y$ implies $xP_e y$. Similarly, R_e is called *equitable inclusive* iff $xR_e^E y$ implies $xR_e y$ and $xP_e^E y$ implies $xP_e y$.

THEOREM 3. *Let e be any A-type economy, then there exists no SPR which is both Pareto inclusive and equitable inclusive.*

Proof. By Theorem 2 $\exists x, y \in X$ with x equitable, y not equitable and $yP_e^p x$. But also, $xP_e^E y$. If R_e is to be both Pareto and equitable inclusive then $xP_e y$ and $yP_e x$. A contradiction. Q.E.D.

Now an SPR that is both Pareto and equitable inclusive reflects strong concerns for both efficiency and equity. Theorem 3 means that these are simultaneously incompatible. Thus, to arrive at an SPR which reflects concerns for equity and the Pareto ranking, we shall need a weaker criterion than "inclusiveness".

We shall next show that it is possible to find such an SPR whose maximal elements are exactly the fair allocations (Theorem 4).

Consider the SPR, R_e', defined by $xR_e' y$ iff either $xR_e^p y$ or $xR_e^E y$. Two properties of R_e' are of interest. Firstly, R_e is still concerned with efficiency and equity, though very weakly. For example, it is possible that $xP_e^p y$ and yet $xI_e' y$. Secondly, R_e' involves no additional considerations apart from efficiency and equity, because if x and y are incomparable under both R_e^p and R_e^E, they remain incomparable under R_e'. Now, generally, $F(e)$ will not contain all the maximal elements under R'_e.[3] However, define R''_e as the transitive extension of R'_e, i.e., $xR''_e y$ iff there is a positive integer n and a sequence $x(l)$, $l = 1, 2, \ldots, n$ with $x(l) \in X$, $\forall l$, $x(1) = x$, $x(n) = y$ and $x(l) R'_e x(l+1)$, $1 \leq l \leq n-1$. Then the set of fair allocations, $F(e)$ will be exactly the maximal elements of X under R''_e.

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[2] This is similar to the definition of a pareto inclusive collective choice rule by Sen [6, p. 30].

Let us denote the set of maximal elements of X under any relation R_e by $M(R_e, X) = \{x \in X \mid \forall y \in X, \text{ if } yR_e x \text{ then } xR_e y\}$.

THEOREM 4. *Given any A-type economy*, $F(e) = M(R''_e, X)$.

(The proof is contained in the appendix.)

Therefore, $F(e)$ can be selected using only considerations of equity, efficiency and transitivity.

Now, given e, for $S \subset X$, let

$$M(R''_e, S) = \{x \in S \mid \forall y \in S, yR''_e x \text{ implies } xR''_e y\}.$$

We can then regard R''_e as generating a social choice function, which selects $M(R''_e, S)$ from any $S \subset X$. As a consequence of theorem 1, this social choice function will not generally satisfy the independence of irrelevant alternatives condition of Nash [4].

THEOREM 5. *Let e be any A-type economy. If $\hat{w}$ is not the unique fair allocation, then $\exists S \subset X$ such that $F(e) \subset S$ and $M(R''_e, S) \neq F(e)$.*

Proof. From theorem 1, there is an equitable $x \in X/F(e)$ with no equitable $y \in X$, $y \neq x$, such that $(U^1(y^1), U^2(y^2)) \geq (U^1(x^1), U^2(x^2))$. Let $S = F(e) \cup \{x\}$. It is easy to verify that $M(R''_e, S) = S$. Q.E.D.

Thus, if one believes that independence is important, then additional considerations apart from efficiency, equity and transitivity must be incorporated into the SPR to arrive at $F(e)$ as its maximal elements over X. However, even with additional considerations, if one should require the SPR to select exactly the fair allocations from *every* subset of X, then independence will not be satisfied.[4] Specifically, for a given e and $S \subset X$, let

$$F^S(e) = \{x \in S \mid x \text{ is equitable and } \forall y \in S, y \neq x, \text{ if } U^i(y^i) \geq U^i(x^i), i = 1, 2, \text{ then } U^i(y^i) = U^i(x^i), i = 1, 2\},$$

denote the set of fair allocations in S. Then, from the proof of theorem 5, it is clear that there is an $S \subset X$ such that $F(e) \subset S$ and $F^S(e) \neq F(e)$.

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[3] For example, if x is the equitable allocation in the statement of theorem 1, then x is a maximum element under R'_e.

[4] This requirement is not met by the social choice function $M(R''_e, S)$, i.e., there are $S \subset X$ where $M(R''_e, S)$ is not the set of fair allocations in S.

APPENDIX

First, given an economy e, let $E(e)$ denote the set of all equitable allocations. Before giving the proofs of theorems 1, 2, and 4, we establish three lemmas.

LEMMA 1. *Let e be any A-type economy, then both $E(e)$ and $F(e)$ are nonempty and closed.*

Proof. That $F(e) \neq \emptyset$ follows from Varian [7, theorem 2.3]. As $E(e) \supset F(e)$, $E(e) \neq \emptyset$. For $i = 1, 2$, by the continuity of U^i, $N^i = \{x \in X \mid U^i(x^i) \geq U^i(x^k), k \neq i\}$ is closed. Hence, $E(e) = N^1 \cap N^2$ is closed. $P(e)$ is closed by Varian [7, corollary to theorem 2.4]. Hence, $F(e) = E(e) \cap P(e)$ is also closed. Q.E.D.

LEMMA 2. *Let $e = (w, U)$ be any A-type economy. If $\hat{w}$ is not the* unique *fair allocation, then $\exists i \in \{1, 2\}$ and $x \in E(e)/F(e)$ with $U^i(x^i) > U^i(y^i)$, $\forall y \in F(e)$.*

Proof. From lemma 1 and strict quasi-concavity of the U^i's, $\exists \bar{x}, \bar{y} \in F(e)$ with $U^1(\bar{x}^1) > U^1(x^1)$, $\forall x \in F(e), x \neq \bar{x}$, and $U^2(\bar{y}^2) > U^2(x^2), \forall x \in F(e)$, $x \neq \bar{y}$. If $\hat{w}$ is not the unique fair allocation, then either $\bar{x} \neq \hat{w}$ or $\bar{y} \neq \hat{w}$. W.l.o.g., let $\bar{x} \neq \hat{w}$. Note first that $\bar{x}^i \not\geq \hat{w}^i$ and $\bar{x}^i \not\leq \hat{w}^i$, $i = 1, 2$, otherwise, by strict monotonicity, either $U^1(\bar{x}^1) < U^1(\bar{x}^2)$ or $U^2(\bar{x}^2) < U^2(\bar{x}^1)$, contradicting $\bar{x} \in F(e)$.

The proof now proceeds by construction. There are two cases.

Case (i) $U^2(\bar{x}^2) > U^2(\bar{x}^1)$.

Clearly, there is some small number $\epsilon > 0$ such that $U^2(\bar{x}^2 - \underline{\epsilon}) > U^2(\bar{x}^1 + \underline{\epsilon})$, where $\underline{\epsilon} = (\epsilon, \epsilon)$. From (2)(iii), such an ϵ can be chosen so that $x = (\bar{x}^1 + \epsilon, \bar{x}^2 - \epsilon) \in X$. Then, from strict monotonicity and the fact that $U^1(\bar{x}^1) \geq U^1(\bar{x}^2)$, we have $U^1(x^1) > U^1(x^2)$. Therefore x is equitable and clearly, as $U^1(x^1) > U^1(\bar{x}^1)$, it follows from the construction of $\bar{x}$ that $x \notin F(e)$.

Case (ii) $U^2(\bar{x}^2) = U^2(\bar{x}^1)$.

For this case, first note that:

$$U^1(\bar{x}^1) > U^1(\bar{x}^2), \tag{3}$$

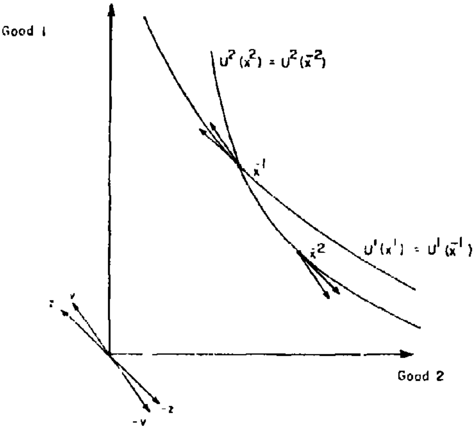
otherwise, if $U^1(\bar{x}^1) = U^1(\bar{x}^2)$, then, as $\bar{x}^1 \neq \bar{x}^2$, by strict quasi-concavity $U^i((\bar{x}^1 + \bar{x}^2)/2) > U^i(\bar{x}^i)$, $i = 1, 2$, contradicting $\bar{x} \in P(e)$.

Now, w.l.o.g. let $\bar{x}_1{}^1 > \hat{w}_1{}^1$ and $\bar{x}_2{}^1 < \hat{w}_2{}^1$. This implies that:

$$\bar{x}_1{}^1 > \bar{x}_1{}^2 \text{ and } \bar{x}_2{}^1 < \bar{x}_2{}^2. \tag{4}$$

Let Z be the set of all unit vectors in E^2. For $i = 1, 2$, $z \in Z$ and $x^i \in \text{int. } X^i$, let $D_z^i(x^i)$ be the directional derivative of U^i at x^i in the direction z. From (2)(ii), these exist, and $D_z^i(x^i) = \nabla U^i(x^i) \cdot z$, where $\nabla U^i(x^i)$ is the gradient of U^i at x^i. Also, from (2) and the construction of $\bar{x}$, there exists a unique $z \in Z$ with $z_1 > 0$ and $z_2 < 0$, and:

$$D_z^1(\bar{x}^1) = 0 = D^2_z(\bar{x}^2). \text{ (See diagram 1.)} \tag{5}$$



Next, consider $D_z^2(\bar{x}^1)$. From (4), (5) and strict quasi-concavity, $D_z^2(\bar{x}^1) < 0$. Thus $\exists \epsilon > 0$ such that $\forall v \in Z$ with $| z - v | < \epsilon$;

$$D_v^2(\bar{x}^1) = \nabla U^2(\bar{x}^1)\, v < \nabla U^2(\bar{x}^2)(-v) = D^2_{-v}(\bar{x}^2). \tag{6}$$

Choose such a v such that $D_v^1(\bar{x}^1) > 0$. Then, from (3), (6) and the facts that $U^2(\bar{x}^2) = U^2(\bar{x}^1)$ and $x \in \text{int. } X$, there exists a $\delta > 0$ such that;

(i) $\tilde{x} = (\bar{x}^1 + \delta v, \bar{x}^2 - \delta v) \in X$,
(ii) $U^1(\tilde{x}^1) > U^1(\bar{x}^1)$,
(iii) $U^1(\tilde{x}^1) > U^1(\tilde{x}^2)$,
(iv) $U^2(\tilde{x}^1) = U^2(\bar{x}^1 + \delta v) < U^2(\bar{x}^2 - \delta v) = U^2(\tilde{x}^2)$.

$\tilde{x}$ is thus the desired equitable allocation. Q.E.D.

LEMMA 3. *Let $e = (w, U)$ be an A-type economy with $\hat{w}$ as the unique fair allocation. Then $\exists x \in E(e)$, $x \neq \hat{w}$.*

Proof. Let $y, z \in X$ be defined by:

(i) $y^1 = (w_1/2, 0), y^2 = (w_1/2, w_2)$,
(ii) $z^1 = (w_1, w_2/2), z^2 = (0, w_2/2)$.

By strict monotonicity, $U^1(y^1) < U^1(y^2)$ and $U^1(z^1) > U^1(z^2)$. As N^1 (as defined in the proof of lemma 1) is closed, $\exists\lambda$, $0 < \lambda < 1$, such that $U^1(x^1) = U^1(x^2)$, where $x = \lambda y + (1 - \lambda) z$. Clearly, $x \neq \hat{w}$. Thus, if $U^2(x^2) \geq U^2(x^1)$, then x is the desired allocation. If $U^2(x^2) < U^2(x^1)$, then $\tilde{x} = (x^2, x^1)$ is the desired allocation. Q.E.D.

We can now give the proofs of the theorems.

Proof of theorem 1.

For $i = 1, 2$, let $S^i = \{x \in E(e)\colon U^i(x^i) \geq U^i(y^i), \forall y \in E(e)\}$. By lemma 1, these are non-empty. By lemma 2, $\exists i \in \{1, 2\}$, with $S^i \cap F(e) \neq \varnothing$. W.l.o.g., let $S^1 \cap F(e) \neq \varnothing$. By continuity, S^1 is closed, so $\exists x \in S^1$ such that $U^2(x^2) \geq U^2(y^2)$, $\forall y \in S^1$. It is easy to check that x is the desired equitable allocation. Q.E.D.

Proof of theorem 2.

If $\hat{w}$ is not the unique fair allocation, the result follows directly from theorem 1, so let $\hat{w}$ be the unique fair allocation. By lemma 3, $\exists x \in E(e)$, $x \neq \hat{w}$. Thus, $x \notin P(e)$. Now, by continuity and strict quasi-concavity of the U^i's, $\forall \bar{x} \in X$, $\exists \tilde{x} \in P(e)$ with $U^1(\bar{x}^1) = U^1(\tilde{x}^1)$. Hence, as $x \notin P(e)$, $\exists z \in P(e)$ with $U^1(z^1) = U^1(x^1)$ and $U^2(z^2) > U^2(x^2)$. Also, from the above, there is an allocation y near z such that $y \in P(e)$, $y \neq \hat{w}$ and $U^i(y^i) > U^i(x^i)$, $i = 1, 2$. x and y are thus the desired allocations. Q.E.D.

Proof of theorem 4.

First, $\forall y \in F(e)$, $\forall x \in X$ $x \neq y$, clearly not $xR''_e y$. Thus, $F(e) \subset M(R''_e, X)$. Next, let $x \in X/F(e)$. If x is inequitable, then, for a $y \in F(e)$, $yR_e{}^E x$, so that $yR''x$($F(e)$ is non-empty by lemma 1). From the above, not $xR''_e y$, so $x \notin M(R''_e, X)$. Lastly, let x be equitable. Then $x \notin P(e)$, so $\exists z \in P(e)$ with $zR''_e x$. If $z \in F(e)$, then not $xR''_e z$ and so $x \notin M(R''_e, x)$. If $z \notin F(e)$, then $xR_e{}^E z$, so $xI''_e z$. But $\exists y \in F(e)$ with $yR'_e z$. Thus, by transitivity, $yR''_e x$ and so $x \notin M(R''_e, X)$. Thus, $M(R''_e, X) \subset F(e)$. Q.E.D.

REFERENCES

1. A. FELDMAN AND A. KIRMAN, Fairness and envy, *Amer. Econ. Rev.* (December 1974).
2. D. FOLEY, Resource allocation and the public sector, *Yale Econ. Essays* (Spring 1967).
3. S. C. KOLM, Justice et Équité, mimeo., Centre d'Études Prospective d'Économie Mathématique Appliquées à la Planification, Paris, 1971.
4. J. F. NASH, The bargaining problem, *Econometrica*, (1950).
5. D. SCHMEIDLER AND K. VIND, Fair net trades, *Econometrica*, (July 1972).
6. A. K. SEN, "Collective Choice and Social Welfare," Holden–Day, San Francisco, 1970.
7. H. R. VARIAN, Equity, envy, and efficiency, *J. Econ. Theory* **9** (1974), 63–91.

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A Comment on "Decentralized Planning and Increasing Returns"*

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In a recent article in this journal [1] Xavier Calsamiglia (C) shows that decentralized resource allocation processes which can compute efficient production plans in economies with nondecreasing returns to scale must use messages of infinite dimensionality. On the other hand, I have proposed in [2] a planning procedure for those economies which is informationally decentralized, computes an optimum as seen by a central planning board (hence an efficient production plan), and uses a finite-dimensional message space.

There are two differences between C's set up and mine. First the class of environments he requires his processes to perform in is wider: the quantity-quantity algorithm (q.q.a.) requires the production possibility sets to show free disposal. An examination of C's proof shows that this limitation would not affect his result.

The q.q.a. also requires the C.P.B. to have some a priori estimate of the optimal production plan. This might be a requirement which cannot be dispensed with without cost; and actually we would be led to this conclusion if it were not for more fundamental questions in the economics of the problem.

Indeed, I believe the main difference between the two papers comes from conflicting uses of the terms "process" and "message space." In this note I try to throw light on the use of those terms in C's article and to sharpen the economic interpretation of his mathematical results. Both papers use the standard terminology of their respective traditions, i.e., the "informational size" and the planning literatures, hence, the comments made below should have a broader significance than explaining the paradox exposed at the beginning of this note.

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I. The Problem

C uses a setup first introduced by Hurwicz [3] and refined by Mount and Reiter [5]. Let $\mathcal{M}$ be the message space which can be used by each agent; $\mathcal{M}^{(n)} = \prod_{i=1}^{n} \mathcal{M}$ is the collective message they send. The allocation process is an iteration in $\mathcal{M}^{(n)}$ such that

$$m^i_{t+1} = f^i(m_t^1, m_t^2, \ldots, m_t^n; e^i),$$

where $m_t = \{m_t^i\}$ represent the messages sent at the tth iteration and e^i is a description of the ith agent. In production planning, to which we restrict ourselves, e^i is the production possibility set of the firm. The privacy requirement is expressed by the fact that m^i_{t+1} depends only on e^i, at the exclusion of e^j, $j \neq i$. When the process comes to an equilibrium we have, for some $\bar{m} = \{\bar{m}^i\}$: $\bar{m}^i = f^i(\bar{m}; e^i)$ for all i. Let $\mu^i(e^i)$ be the set of $m \in \mathcal{M}^{(n)}$ such that $f^i(m^1, \ldots, m^i, \ldots, m^n; e^i) = m^i$; if $e = (e^1, e^2, \ldots, e^n)$ describes the production possibility of the economy we have $\mu(e) = \bigcap_{i=1}^{n} \mu^i(e^i)$; $\mu(e)$ is the set of $m \in \mathcal{M}^{(n)}$ such that $f^i(m; e^i) = m^i$ for all i. The equilibrium messages must still be translated into action; this is done through an "outcome function" h which associates to $\bar{m} \in \mathcal{M}^{(n)}$ a production plan. The allocation process is satisfactory if $h \circ \mu$ associates with every e a satisfactory plan. C defines satisfactoriness as efficiency. He shows that there exist no finite-dimensional $\mathcal{M}$ such that the allocation process $(\mu, \mathcal{M}, h)$ is satisfactory.

It is important to note that the formulation puts a dynamic notion (the planning process) into a static framework as we focus only on equilibrium messages. In the following paragraphs I present what I believe to be a correct economic interpretation of C's results, first in a static and then in a dynamic framework.

II. A Static Interpretation

Let $g^i(m; e^i) = f^i(m; e^i) - m^i$. We an studying the solutions of $\{ g^i(\bar{m}; e^i) = 0, i = 1, 2, \ldots, n; h(\bar{m}) = x\}$, where x is efficient and $\bar{m} \in \mathcal{M}^{(n)}$. C shows that there exist no $\{g^i\}$ and h such that this system of equations has a solution for all environments with increasing returns to scale. Economically, this translates into the following problem. The g's represent questions asked to the firms; given the m^j's ($j \neq i$) and g^i, firm i's answer, m^i, is computed only from data internal to the firm. C proves that there is no set of questions with manageable answers which can in all environments with increasing returns to scale describe at least one efficient point.

If such a set of questions existed, the interpretation of the equilibrium messages would be similar to that of a price equilibrium in Walrasian systems: We know it exists but finding it requires a separate analysis.

III. DYNAMIC ANALYSIS

This leads us to the explicit consideration of the dynamic elements of an allocation process. A planning procedure could be described as follows. The economy is composed of $n + 1$ agents, n firms, and the planning board. At stage t, the center asks questions q_t to the firms. Firms send an answer r_t,

$$r_t = \rho(q_t\,; e; w_{t-1}),$$

where e represent the characteristics of the firms and w_{t-1} their accumulated memory of past questions and answers,

$$w_t = \xi(w_{t-1}\,, q_t\,, r_t\,, e).$$

From those responses the planning agency asks the questions of the next stage

$$q_{t+1} = \gamma(r_t\,; u; v_t),$$

where u is its utility function and v_t its own memory of past questions and answers;

$$v_t = \nu(v_{t-1}\,; q_t\,; r_{t-1}\,; n).$$

Quite a few planning algorithms have been built using elaborate computations and memories (v_t) from the center. Much less effort has been devoted to the possibility for firms to remember something about past iterations, although it could guide them in their choices of answers as they derive a better understanding of the center's utility function and production possibility sets of other firms. (Note that this is only a question of interpretation: q_t can always be made to include all q_θ and r_θ, $\theta < t$; the firms could then recompute w_t in the process of formulating r_t, without the use of a memory.)

In this framework there are two interpretations of the Mount and Reiter framework:

(a) There is no center so that $q^{t+1} = r^t$. Each firm takes the answers of other firms as given and computes its response. This is clearly not very satisfactory if we are discussing a theory of planning. Other aspects are discussed in (b) of which this interpretation is a specific case.

(b) Define $m_t = (r_t\,, v_t\,, w_t)$; we have $r_{t+1} = \rho(\gamma(r_t\,; u; v_t); e; w_t)$, $v_{t+1} = \nu(v_t\,; \gamma(r_t\,; u; v_t); r_t\,; u)$, and $w_{t+1} = \xi(w_t\,; \gamma(r_t\,; u; v_t); r_{t+1}\,; e)$. It is therefore possible to define f, in the sense of C's framework, from ρ, ξ, γ, and ν. However, m_t is not really a message but a combination of messages and memories.

This remark points to the solution of the paradox exposed at the beginning of this note: The planning literature defines the size of messages from the

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Cournot and Walras Equilibrium

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I. INTRODUCTION

This paper unifies the two leading classical concepts of equilibrium for an economy: Walras equilibrium and Cournot equilibrium. The theory provides a fresh setting for the study of competitive markets, and leads to a description of economic equilibrium which differs in substance from the one offered by modern formal competitive theory (see, e.g., Debreu [3]).

Modern formal competitive theory does not permit free entry, hence the number of firms is fixed. Further, it is posited that firms behave in one of two ways. In one case, perfect competition is assumed: Firms act as if they have no effect on price; they maximize profit taking prices as given. In the other case, imperfect competition is assumed: Firms recognize and act on their ability to influence price. In the model we present, price and market power are determined by free entry and the size of efficient scale relative to demand. Several distinctive features of classical economic analysis occupy a central role in the development.

Specifically, we study a model in which the number of firms is determined endogenously; firms enter when it is profitable to enter, and the entry of firms is a driving force in the explanation of value. The presence of fixed costs (more precisely, the fact that the efficient scale of firms is bounded away from zero) places a limit on the number of firms which are "active" in an equilibrium. When efficient scale is small relative to the size of the market, equilibrium will be characterized by a large number of firms and small profits for each firm. In this case, firms will have very little effect on price when confined to the range in which they make nonnegative profit. (We do not study the case of large efficient scale; however, in that situation, firms

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may make large profits and their actions may have a substantial effect on price. The resulting allocations will typically lack the efficiency properties associated with perfect competition.) In contrast to modern formal theory, a change in the specification of tastes (and/or technology) will change not only the actions of firms present in the market, but typically it will alter as well the number of firms in each industry.

Price taking producer behavior obtains only as an idealization; the primitive solution concept employed is Cournot–Nash equilibrium with quantity choosing firms and entry. Firms choose a quantity action given the actions of other firms; they evaluate the profitability of their actions according to the demand function of the price-taking (consumer) sector. More precisely, prices are identified by the condition that consumer excess demand matches the aggregate action of firms. Cournot–Nash equilibrium is defined by the following two conditions: No firm in the market can increase profit by altering production, and no firm absent from the market can enter and achieve a positive profit.

There are a countable infinity of firms of a variety of basic types available; however, since efficient scale is bounded away from zero, only a finite number of them are active in an equilibrium. Since there are only a finite number of active firms, each firm is significant and affects price: Thus, isoprofit manifolds are not linear. Furthermore, profit functions are not concave. The assumption that efficient scale is bounded away from zero could be defended on grounds of realism alone. However, the need for this assumption goes deeper, and is in fact dictated by the requirement that the number of firms be endogenous and determined by the opportunities for profit. For if production becomes increasingly efficient as output converges to zero, then the size of firms will be indeterminate (arbitrarily small) and the number of firms likewise indeterminate (arbitrarily large). Nonconvex technology and free entry are intimately related. No theory of economic equilibrium which requires nonincreasing returns to scale throughout can provide a satisfactory explanation of the number of firms in a market.

Now suppose that efficient scale is very small relative to the size of the market. Assume that there are a countable infinity of firms of a variety of types available. Profit functions may not be concave and production sets are not convex; this suggests the possible nonexistence of Cournot–Nash equilibrium with entry (see, e.g., Roberts and Sonnenschein [12]). But, if an equilibrium with entry exists, it is highly likely that it will approximate a Walras equilibrium of the economy obtained by viewing there to be available, for each firm type, a continuum of price taking infinitesimal sized firms. (Formally, one replaces the given technology by a cone technology which includes all multiples of possible productions and convex combinations of such points.) Although generally loosely presented, reasoning to support this belief is not uncommon in the partial equilibrium analysis of a single market.

The availability of a continuum of price taking infinitesimal sized firms is formalized by a horizontal supply curve the height of which is given by minimum average cost. The perfectly competitive (Walras) output is determined by the intersection of that supply curve with the demand curve. Cournot–Nash equilibrium output cannot fall short of the perfectly competitive output by more than the cost minimizing output (efficient scale) of a single firm; otherwise, there would be entry. Thus, if the scale at which firms are most efficient becomes small relative to demand, then the deviation of Cournot–Nash output from perfectly competitive equilibrium is small. (Similar reasoning is no doubt behind a conjecture of Samuelson [14]. "As the size of the market grows relative to the size of the minimum scale at which unit costs are at their lowest, the system approaches the perfectly competitive equilibrium.") But the above argument is possibly vacuous unless one can show that the notion of Cournot–Nash equilibrium with entry is viable when efficient scale is small relative to the market. Novshek [10] provided the required argument for the case of the partial equilibrium analysis of a single market. The central achievement of this paper is to extend the analysis to the case of general economic equilibrium. We next make the preceding argument more explicit.

Consider an Arrow–Debreu private ownership economy $\mathscr{E}$ which satisfies sufficient conditions for the existence of Walras equilibrium—with one exception: Production sets are not convex. (As a leading case, consider the zero vector union a "standard" and strictly convex production possibility set which is displaced by a vector (of fixed costs) $w \leq 0$ ($\neq 0$).) If firms are infinitesimal relative to the market, and if a continuum of each type of firm is available, it is natural to consider the economy $\mathscr{E}$ in which each firm is replaced by the smallest convex cone which contains the production possibility set of that firm. The cone so defined is viewed as an industry; and the implied industry supply curve is the counterpart of the perfectly elastic supply curve of partial equilibrium analysis. (Ownership shares are most conveniently translated into industry ownership shares; however, this is not essential.) Since $\mathscr{E}$ has convex production sets, Walras equilibrium exists, and an equilibrium of $\hat{\mathscr{E}}$ is viewed as a Walras equilibrium of $\mathscr{E}$ in which firms both must and do take prices as given. Associated with every equilibrium is a measure of firms in each industry.

Alternatively, for each α ($0 < \alpha \leq 1$), consider the economy $\mathscr{E}(\alpha)$ obtained from the original economy $\mathscr{E}$ by replacing each firm Y by a sequence of firms, referred to as an industry, each with production set $Y(\alpha) = \alpha Y$. When $1/\alpha$ is an integer, $Y(\alpha)$ is viewed as a representation of Y in per capita terms in an economy in which each consumer has been replicated $1/\alpha$ times. As α becomes small, efficient scale becomes small relative to the size of the market. As small becomes infinitesimal, i.e., as α approaches zero, it is natural to think of $\mathscr{E}(\alpha)$ as converging to $\hat{\mathscr{E}}$. This is because (in per capita terms) the sum of the production sets of each industry in $\mathscr{E}(\alpha)$ converges to the production set of the corresponding industry in $\mathscr{E}$.

We prove that it is a generic property of economies $\mathscr{E}$ that there exists $\bar{\alpha} > 0$ such that Cournot–Nash equilibrium with free entry exists in $\mathscr{E}(\alpha)$ for all $\alpha < \bar{\alpha}$. The number of firms in the market is determined endogenously. Firms who choose to produce, typically have positive profit, and all firms with positive expected profit adopt pure strategies. Pure strategies are not required as a condition for equilibrium, rather they arise as a characteristic of equilibrium. Firms are permitted mixed strategies. Marginal firms are firms which maximize profit by choosing strategies of the form "produce y with probability π and stay out of the market with probability $(1 - \pi)$"; by profit maximization, they must make zero expected profit. For $\alpha \leq \bar{\alpha}$ only pure and marginal firm strategies are profit maximizing in the equilibria which are exhibited; furthermore, the number of marginal firms is uniformly bounded in α. Thus, as $\alpha \rightarrow 0$, the proportion of firms which are marginal, and the aggregate output of marginal firms (per capita) converge to zero. Marginal firms arise naturally in the theory and are a proper analog of the entity which bears their name in less formal analysis. Without marginal firms, Cournot–Nash equilibrium will in general not exist and nonexistence will be "robust."

The existence theorem requires as one of its hypotheses the presence of a Walras equilibrium for $\mathscr{E}$ at which no industry can increase profit by increasing the scale of its operation. From the present point of view, equilibria of $\mathscr{E}$ which fail to have this property are not proper Walras equilibria. Without this property, firms of arbitrarily small size will profit from entering the industry, and upset equilibrium. Thus, a notion of downward sloping demand (DSD) arises naturally as a requirement for Walras equilibrium with entry, and equilibria which do not satisfy this condition are artifacts of a specification of the competitive model which requires that firms be infinitesimal. While it is natural to think of these equilibria as failing a "stability test," the considerations involved are in fact more basic. If α is sufficiently small, but not zero (i.e., if efficient scale is small, but not infinitesimal), then Cournot–Nash equilibria of $\mathscr{E}(\alpha)$ *cannot* exist close to a point where DSD fails. For at any proposed equilibrium of $\mathscr{E}(\alpha)$ near such a point, profits in every industry must be nonnegative, and an inactive firm can make positive profit by entering an industry in which profit increases with scale.

Having settled the problem of existence, it becomes meaningful to inquire whether the set of Cournot–Nash equilibria approaches the set of (DSD) Walras equilibria as efficient scale (measured per capita) approaches zero. Two questions are distinguished. First, whether an arbitrary (DSD) Walras equilibrium of $\mathscr{E}$ can be obtained as the limit (as $\alpha \rightarrow 0$) of a sequence of Cournot–Nash equilibria, one point in the sequence for each $\mathscr{E}(\alpha)$. Second, whether every limit of Cournot–Nash equilibria (as $\alpha \rightarrow 0$) is a DSD Walras equilibrium of $\mathscr{E}$. Both questions are answered in the affirmative. Since the Walras equilibria of $\mathscr{E}$ coincide with the Cournot–Nash equilibria of $\mathscr{E}$, the preceding questions might be rephrased as pertaining to the lower and upper hemicontinuity of the Cournot–Nash correspondence at the point of infinitesimal (per capita) efficient scale. We note here that Cournot's approach of replicating firms with a fixed "demand sector" does not lead to prices which approach Walras equilibrium prices. With efficient scale bounded away from zero and nonnegative profit, prices diverge from those associated with efficient scale (competitive prices) as the number of active firms is increased. This is because the output of some firms will necessarily approach zero, and this will require prices higher than minimum average cost so that profits do not become negative. And, of course, equilibrium does not in general exist (Roberts and Sonnenschein [12]). For a partial equilibrium analysis, see Novshek, [10] and Ruffin [13].

Taken together, the existence and convergence results unify the concepts of Cournot and Walras equilibrium. For economies in which efficient scale is per capita small and entry is free, the assumption of price taking behavior (in $\mathscr{E}$) leads to an outcome which approximates the result obtained with strategic behavior. Free entry, absent from the current formal analysis of perfect competition, is a driving force in the analysis. With free entry, as efficient scale becomes small in per capita terms, the difference between Walras equilibrium in $\mathscr{E}$ and Cournot equilibrium becomes small. The concept of Walras equilibrium is derived from a primitive concept of strategic behavior; it is appropriate when the data of the economy indicate small (per capita) efficient scale and there is free entry. Finally, only the equilibria of $\mathscr{E}$ which satisfy DSD are true economic equilibria. Most of this is of course very classical; what we offer is an adequate formal setting.

A word about the way in which "free entry" is to be understood. There are an infinite number of basic technologies, which for simplicity are identical in the present model. Once one technology is used, it is no longer available to others, but there are always additional (identical) technologies available. With little additional complication, we could have considered a situation in which there are an infinite number of distinct basic technologies. Free entry would not mean that the technologies remain available to others once employed, but would require them to be available when not employed. This naturally leads to strictly convex production sets in $\mathscr{E}$ and "scarce factor" rents.

Novshek's Theorem [10] on the existence of Cournot–Nash equilibrium with entry in a single market suggested the possibility of the general equilibrium analysis presented here. That work is surveyed in Appendix I. Prior to the completion of our paper, we had access to a manuscript by O. Hart [6]. While Hart's analysis does not include the main concern of this paper—the existence of Cournot—Nash equilibrium with entry—he does provide a convergence result which is similar in spirit to our Theorem 3, but for an economy in which firms choose the commodity they will produce from an infinite set of possible commodities. We heartily recommend this paper to the attention of the reader. Finally, we note that the interplay between significant fixed costs, the number of firms, and the variety of products actually produced, has for many years been a central ingredient of the theory of monopolistic competition. For a particularly interesting modern treatment in partial equilibrium, we recommend a paper by M. Spence [16].

The next section introduces the formal model. This is followed by Section III which presents the theorems and two sections (IV and V) devoted to their proof. Section IV is expository and is designed to introduce some of the concepts used. It is an important part of our presentation, as the formal proof involves an unusually large amount of computation. Remarks and conclusions are contained in Section VI and Section VII.

II. THE MODEL

The notion of a basic private ownership economy $\mathscr{E}$ is standard.

a. An *economy* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ is:

(a1) For each consumer $i = 1, 2, \ldots, m$, a consumption set $X_i \subset R^l$, an initial endowment vector $\omega_i \in R^l$, and a preference ordering $\succsim_i \subset X_i \times X_i$. [Whenever we speak of a collection of consumers $(X_i, \omega_i, \succsim_i)$ the following hypothesis is maintained. Desirability: for any sequence $\{p^t\} \subset \Delta_l = \{(p_1, p_2, \ldots, p_l) \gg 0 : \Sigma p_r = 1\}$, if $p^t \to \bar{p} \in \bar{\Delta}_l \setminus \Delta_l$, and if for each i and t, $h^i(p^t)$ is $\succsim_i$ maximal subject to $p^{t\prime} x_i = p^{t\prime} \omega_i$,[1] then $\| \sum_i h^i(p^t) \| \to \infty$];

(a2) for each firm j or $k = 1, 2, \ldots, n$, a production set $Y_j \subset R^l$, [we maintain the hypothesis that for all j, $0 \in Y_j$ and the asymptotic cone (see Debreu [3, p. 22]) of Y_j contains no vectors with positive coordinates];

(a3) for each i and j a nonnegative number θ_{ij} which indicates the fraction of firm j owned by individual i. For each j, $\sum_i \theta_{ij} = 1$.

The economies $\mathscr{E}(\alpha)$ are derived from the basic economy $\mathscr{E}$ by replicating the consumer sector ($1/\alpha$ times), viewing production in per capita terms, and for each j, positing the availability of a countable infinity of each of the Y_j of $\mathscr{E}$. In the economy $\mathscr{E}(\alpha)$, we speak of the jth industry;[2] for simplicity, ownership shares in $\mathscr{E}$ are translated into industry ownership shares in $\mathscr{E}(\alpha)$.

b. For any economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ and any number $\alpha > 0$, the economy $\mathscr{E}(\alpha) = (X_i, \omega_i, \succsim_i, Y_{jt}(\alpha), \theta_{ijt})$ is defined by:

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[1] Prime (′) denotes transpose.

[2] By industry, we mean the set of firms with identical technology. Several industries may use the same inputs to produce the same outputs.

(b1) the m consumers are those of $\mathscr{E}$;

(b2) for each $j \leq n$ and each positive integer t,

$$Y_{jt} \in Y_{jt}(\alpha) = Y_j(\alpha) \text{ if } Y_{jt}/\alpha \subset Y_j \text{ ; and}$$

(b3) $\theta_{ijt} = \theta_{ij}$ for all i, j, and t.

The economy $\hat{\mathscr{E}}$ is derived from the basic economy $\mathscr{E}$ by replacing each production set of $\mathscr{E}$ with the smallest convex cone with vertex at the origin which contains it. The economy $\hat{\mathscr{E}}$ is interpreted as the limit of $\mathscr{E}(\alpha)$ as α approaches zero, and corresponds to a view in which the actions of firms in $\hat{\mathscr{E}}$ are infinitesimal in per capita terms.

c. For any economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$, the *natural limit of* $\mathscr{E}$ is the economy $\hat{\mathscr{E}} = (X_i, \omega_i, \succsim_i, \hat{Y}_j, \theta_{ij})$ defined by:

(c1) the m consumers are exactly those of $\mathscr{E}$;

(c2) for each firm $j = 1, 2, \ldots, n$, $\hat{Y}_j$ is the intersection of all closed convex cones with vertex at the origin which contain Y_j; and

(c3) ownership shares are exactly those of $\mathscr{E}$.

The definition of a Walras equilibrium and a Walras allocation for economies and pure exchange economies is standard. The Walras correspondence indicates how equilibrium prices vary with initial endowments in a pure exchange economy.

d. For an economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$, the triple $(p^*, x^*, y^*) \in R_+^l \times \Pi X_i \times \Pi Y_j$ is a *Walras equilibrium* if[3]

(d1) $p^{*\prime}x_i^* = p^{*\prime}\omega_i + \sum_j \theta_{ij} p^{*\prime} y_j$, $i = 1, 2, \ldots, m$;

(d2) $\sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i$;

(d3) $x_i \succ_i x_i^*$ implies $p^{*\prime}x_i > p^{*\prime}\omega_i + \sum_j \theta_{ij} p^{*\prime} y_j$; and

(d4) $p^{*\prime}y_j > p^{*\prime}y_j^*$ implies $y_j \notin Y_j$.

If (p^*, x^*, y^*) is a Walras equilibrium for $\mathscr{E}$, then (x^*, y^*) is called a *Walras allocation*.

An exchange economy $\mathscr{E} = (X_i, \omega_i, \succsim_i)$ is an economy minus firms and profit shares (a2) and (a3) of (a). The pair $(p^*, x^*) \in R_+^l \times \Pi X_i$ is a *Walras equilibrium* for $\mathscr{E}$ if

(d5) $p^{*\prime}x_i^* = p^{*\prime}\omega_i$, $i = 1, 2, \ldots, m$;

(d6) $\sum x_i^* = \sum \omega_i$; and

(d7) $x_i \succ_i x_i^*$ implies $p^{*\prime}x_i > p^{*\prime}\omega_i$.

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[3] $x^{*\prime} = (x_1^{*\prime}, x_2^{*\prime}, \ldots, x_m^{*\prime}) \in R^{lm}$, and $y^{*\prime} = (y_1^{*\prime}, y_2^{*\prime}, \ldots, y_n^{*\prime}) \in R^{ln}$.

The vector p^* is called an equilibrium price and x^* is called an equilibrium allocation. For (X_i) and $(\succsim_i)$ fixed, the *Walras correspondence* W assigns to each vector $(\omega_1, \omega_2, ..., \omega_m)$ the equilibrium prices of $\mathscr{E} = (X_i, \omega_i, \succsim_i)$.

As mentioned in the introduction, the study of equilibrium with entry dictates the requirement that efficient scale occur away from zero output. Since we are ultimately interested in Cournot–Nash equilibria which approximate Walras equilibria of $\mathscr{E}$, it is natural to view efficient scale relative to Walras equilibrium prices, p^* of $\mathscr{E}$. Not only is it assumed that efficient scale occurs away from zero, but in addition that such efficiency cannot be approximated in a neighborhood of zero output. For simplicity, we add the requirement that efficient scale (relative to p^*) is uniquely attained.

e. Let (p^*, x^*, y^*) be a Walras equilibrium of $\mathscr{E}$. We say that $\mathscr{E}$ has *efficient outputs bounded away from zero relative to* p^* if for all $j \in \{1, 2, ..., n\}$:

(e1) $(Y_j \cap \{y : p^{*\prime}y = 0\})\setminus\{0\}$ is a singleton $\gamma_j Y_j^*$ (this defines the scalar γ_j); and

(e2) for all $K, \delta \in (0, \infty)$, there exists $\epsilon > 0$, such that $p^{*\prime}y/\|y\| < -\epsilon$ for all $y \in (Y_j \cap N_l(0, K))\setminus(N_l(\gamma_j y_j^*, \delta) \cup \{0\})$, where $N_l(a, b)$ is the ball in R^l around a with radius b.[4]

It is generic that no more than $l - 1$ industries satisfy (e1),[5] and the remaining industries are such that for all prices near p^*, all nonzero outputs in the cones $\hat{Y}_j$ yield strictly negative profit. Since the equilibria we construct for Theorems 1 and 2 will have prices converging to p^*, these remaining industries must eventually be entirely inactive, with no profit incentive for entry. Without loss of generality, we ignore these remaining industries, and assume that the number of industries which must be considered, n, is less than l. Of course, the set of industries which must be considered may vary as we look at different equilibria of $\mathscr{E}$, but it is a generic property that the number of industries which must be considered is less than l.

Production sets are required to have one of two forms in a neighborhood of efficient scale. Either they are finite polyhedra, or their boundaries are smooth manifolds. While we do not require that the manifolds have maximal dimension $(l - 1)$, we consider a formulation which rules out (mixed) cases in which, for example, inputs can only be combined in fixed proportion but yield varying marginal returns. While cases of this type could be included in the analysis, their treatment increases notational complexity.

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[4] In an abuse of notation, we use y_j to indicate a vector in R^l and a vector in R^{ln} (with all components zero except possible $l(j - 1) + 1$, $l(j - 1) + 2, ...$ $l(j - 1) + l$). For example, in $y + \{y_j + e_1\}$, $e_1 = (1\ 0\ 0 \cdots 0)' \in R^l$ is added to $y_j \in R^l$, which is then imbedded in R^{ln}, and added to $y \in R^{ln}$.

[5] Let C be the set of closed convex cones in R^l with vertex the origin and dimension greater than or equal to one, and define the metric d on C by $d(c_1, c_2) = h(c_1 \cap N_l(0, 1), c_2 \cap N_l(0, 1))$, where h is the Hausdorff metric. For each $c \in C$ let $P(c) = \{p \in R^l\setminus\{0\} \mid p'y \leq 0$ for all $y \in c$ and $p'x = 0$ for some $0 \neq x \in c\}$. Let $(c_1, c_2, ..., c_n) \in C^n$. Then the property that $\cap P(c_i) = \varnothing$, $i \in I \subset \{1, 2, ..., n\}$, whenever $\#I > l - 1$ is generic (i.e., the property holds on an open dense subset of C^n).

f. If there exists a cube C with center at $\gamma_j y_j^*$ such that $C \cap Y_j$ is a finite polyhedron, then Y_j is called *locally polyhedral*, at $\gamma_j y_j^*$.

Let $S_j(y)$ be the largest subset of $\{1, 2, \ldots, l\}$ for which there exists a neighborhood $N(y)$ such that the projection of the set of efficient productions of Y_j in $N(y)$ onto the $S_j(y)$ coordinate subspace is a singleton. $L \backslash S_j(y)$ gives the co-ordinates in which production is locally variable by the jth firm at y. If there exists a neighborhood $N(\gamma_j y_j^*)$ such that the set of efficient productions of Y_j in $N(\gamma_j y_j^*)$ is a smooth manifold of dimension $\#(L \backslash S_j(\gamma_j y_j^*)) - 1$, then Y_j is called *smooth* at $\gamma_j y_j^*$. (We maintain the hypothesis that $1 \in L \backslash \bigcup S_j(\gamma_j y_j^*)$, where the union is taken over j such that Y_j is smooth at $\gamma_j y_j^*$.) Y_j is *regular smooth* if $D^2 \hat{g}_j(\gamma_j y_j^*)$ is negative definite where $\hat{g}_j$ gives the first coordinate of efficient points in $N(\gamma_j y_j^*)$ as a function of the remaining $\#(L \backslash S_j(\gamma_j y_j^*)) - 1$ smoothly variable coordinates.[6]

A conceptual problem must be addressed prior to the definition of Cournot-Nash equilibrium in $\mathscr{E}(\alpha)$. Suppose that several firms assert a quantity action. In general, there will be more than one price consistent with their actions; i.e., there may be more than one price which generates a demand that matches the aggregate action of firms. To see this, consider the exchange economy $\tilde{\mathscr{E}}$ obtained by distributing the actions of firms in $\mathscr{E}(\alpha)$ according to profit shares (and adding these amounts to initial commodity endowments). It is clear that each equilibrium price of $\tilde{\mathscr{E}}$ generates in $\mathscr{E}(\alpha)$ a demand which balances the asserted quantity actions of the firms (Rader [11]). Unless one price is singled out, it is impossible to evaluate the actions of each firm, and so a Cournot–Nash game is not well defined. Since our methods demand that price varies smoothly with the quantity actions of firms, we develop a terminology to apply to situations in which there exists locally a twice continuously differentiable selection from an appropriate Walras correspondence. Balasko [2] has demonstrated that the existence of such a selection is generic in an appropriate space.

Prices are expressed relative to the price of the first commodity; i.e., $p_1 = 1$. We note that in noncompetitive theory the choice of numeraire typically affects the profit maximizing action of firms.[7] Our formalization requires that there is a salient commodity in which all firms measure profit.

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[6] If (e1) holds, $D^2 \hat{g}(\gamma_j y_j^*)$ must be negative semidefinite.

[7] Peter Hammond pointed this fact out to us. See, for example, Gabszewicz and Vial [5, pp. 398–400]. Aside from the requirement that the numeraire is a commodity which is smoothly variable for each firm, the existence theorem remains valid independent of the choice of numeraire. This is because there are only a finite number of commodities and an α which will work for each one of them.

g. Let p^* be an equilibrium price for the exchange economy $\mathscr{E} = (X_i, \omega_i, \succsim_i)$. The pair (F, p^*) is a p^* *based inverse demand function* if

(g1) F is a selection from the Walras correspondence of the the exchange economy $\mathscr{E}$ with $p_1 = 1$; and

(g2) F is twice continuously differentiable in a neighborhood of $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ and has value p^* at ω.[8]

Two notions of equilibrium for an economy $\mathscr{E}(\alpha)$ are defined. The first corresponds to a Cournot Nash equilibrium in mixed (quantity) strategies, where payoffs are defined relative to some selection from the appropriate Walras correspondence. The second, called Cournot equilibrium, corresponds to equilibria of the above type in which optimal mixed strategies are either pure quantity strategies (typically indicating positive expected profit) or "produce y with probability π and stay out with probability $(1 - \pi)$" (indicating zero expected profit). Equilibria of the latter form, Cournot equilibria, are economically more meaningful and their existence is demonstrated in Theorem 1. Despite the fact that arbitrary mixed strategies are available in a Cournot equilibrium, optimizing action leads to only marginal firms and pure strategy firms.

h. Let α be a positive number. Consider an economy $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ and the associated economy $\mathscr{E}(\alpha)$. A Cournot equilibrium in mixed strategies for $\mathscr{E}(\alpha)$ is a nonnegative n vector η, with integer components (η_j is the number of active firms in industry j), a function μ from $\{(j, r) : j \leq n$ and $r \leq \eta_j\}$ into the set of probability measure on R^l ($\mu(j, r)(A)$ is the probability that the rth firm of type j has a production in $A \subset R^l$), and a p based inverse demand function (F, p) for the exchange economy $\tilde{\mathscr{E}} = (X_i, \omega_i + \sum_j \theta_{ij} (\sum_{r \leq \eta_j} I(y, d\mu(j, r))), \succsim_i)$[9] ($\tilde{\mathscr{E}}$ is the exchange economy with endowments determined by expected output) such that

(h1) for all (j, r) in the domain of μ, the support of $\mu(j, r) \subset Y_j(\alpha)$ (i.e., all of a firm's actions must be in its production set);

(h2) for all $(\bar{j}, \bar{r})$ in the domain of μ, and all probability distributions ν with support in $Y_{\bar{j}}(\alpha)$,

$$I\left(F\left(\sum_{\substack{j,r \\ r \leq \eta_j}} y_{jr}\right)' y_{\bar{j}\bar{r}}, d\mu\right) \geq I\left(F\left(\sum_{\substack{j,r \\ r \leq \eta_j}} y_{jr}\right)' y_{\bar{j}\bar{r}}, d\mu(\nu \text{ in } \bar{j}\bar{r})\right).$$

---

[8] F is a function of endowments $\omega \in R^{lm}$. To determine prices we consider exchange economies where production has been distributed according to ownership shares. For convenience we will write $F(y)$ for $F((\omega_i + \sum_j \theta_{ij} y_j)_i)$, where $y = (y_1', y_2', \ldots, y_n') \in R^{ln}$ is the vector of productions, and $\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in R^{lm}$ is the vector of original endowments. Throughout the analysis ω will remain fixed, while y varies.

[9] $I(x, d\mu)$ is the Lebesgue-Stieltjes integral.

where μ is the product measure of the $\mu(j, r)$, $(r = 1, \ldots, \eta_j, j = 1, \ldots, n)$, and $\mu(\nu \text{ in } j, \bar{r})$ is μ with the $j, \bar{r}$ coordinate replaced by ν (i.e., each active firm is maximizing profit); and

(h3) for all $k = 1, \ldots, n$, and all ν with support in $Y_k(\alpha)$,

$$\int \left(F\left(\sum_{\substack{j, r \\ r \le \eta_j + \delta_{jk}}} y_{jr} \right)' y_{k\eta_k+1}\, d(\mu \times \nu) \right) \le 0,$$

where $\mu \times \nu$ is μ with the additional coordinate (the $k, \eta_k + 1$ coordinate) ν, and $\delta_{kk} = 1$, $\delta_{jk} = 0$ if $j \neq k$ (i.e., it is not profitable for any additional firm to enter). The probability measure ρ on R^{ln} is the *Cournot production* for $\mathscr{E}(\alpha)$ corresponding to the above Cournot equilibrium in mixed strategies if for all "μ-measurable" $A \subset R^{ln}$, $\rho(A)$ is the "μ probability" that for all j, $\sum_{r \le \eta(j)} y_{jr}$ belongs to the projection of A onto the $(l(j-1)+1, \ldots, l(j-1)+l)$ coordinates subspace of R^{ln}. The measure ρ on R^{ln} induces a measure ξ on R^{l-1} by $\xi(P) = \rho\{y \in R^{ln} : F(y) \in \{1\} \times P\}$. The measure ξ is the *Cournot price distribution* corresponding to the above Cournot equilibrium in mixed strategies.

A *Cournot equilibrium* for $\mathscr{E}(\alpha)$ is a Cournot equilibrium in mixed strategies $(\mu, \eta, (F, p))$ such that for all j and r, either

(h4) $\mu(j, r)$ is degenerate, or

(h5) $\mu(j, r)$ is of the form "0 with probability π, $(\pi \in (0, 1))$, $y_j \in Y_j(\alpha)$ with probability $(1 - \pi)$".

Firms of the type described in (h5) are referred to as *marginal firms*.[10] Since $\pi > 0$, profit maximization implies zero expected profit.

We now formalize the requirement of downward sloping demand.[11] The economy exhibits downward sloping demand at (p^*, x^*, y^*, F) if F has the property that $\lambda y_j^{*\prime} F(y^* + \lambda y_j^*) < 0$ for $\lambda \neq 0$, but small; i.e., profit of the action y_j^* decreases (from zero) as λ increases from zero, and increases (from zero) as λ decreases from zero. It is convenient at the same time to introduce a measure of the effect on the price of commodity r of a "one unit" increase in the use (output) of commodity s by industry j; for this we introduce the matrices B_j.[12] Recall that in the introduction we argued that equilibria of

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[10] A probing question by Kenneth Arrow led us to consider the possibility of marginal firms.

[11] The *DSD* condition only applies to the industries which are "active" in the target Walras equilibrium (p^*, x^*, y^*), i.e., those j such that there exists $0 \neq y_j \in \hat{Y}_j$ such that $p^{*\prime} y_j = 0$. As noted in the remarks after (e), all other industries are completely inactive, with no incentive for entry, for all prices near p^*, and therefore they can be ignored. (It is possible that there is a $0 \neq y_j \in \hat{Y}_j$ such that $p^{*\prime} y_j = 0$ but $y_j^* = 0$. In this case, we use y_j rather than y_j^* in the *DSD* condition.)

[12] The definition of B_j implicitly assumes that F is defined for certain points near y^*;

$\mathscr{E}$ which fail to satisfy DSD cannot be achieved as a limit of Cournot–Nash equilibria.

i. Let $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ be given. Assume (p^*, x^*, y^*) is a Walras equilibrium of $\mathscr{E}$ and (F, p^*) is a p^* based inverse demand function for the exchange economy

$$\mathscr{E}(y^*) = (X_i, \omega_i + \textstyle\sum_j \theta_{ij} y_j^*, \succsim_i).$$

For $j \in \{1, 2, ..., n\}$, let B_j be the $l \times l$ matrix $(\partial F / \partial y_j)(y^*)$[13].

The economy ϵ exhibits *downward sloping demand* (*DSD*) at (p^*, x^*, y^*, F) if $y_j^{*\prime} B_j y_j^* < 0$ for all j.[14]

It is now easy to see why a Cournot equilibrium for $\mathscr{E}(\alpha)$ (α small) cannot exist near a Walras equilibrium of $\mathscr{E}$ at which DSD fails. At any such potential Cournot equilibrium, prices p are near p^*, so active firms of type j must have actions near $\alpha\gamma_j y_j^*$, and these firms earn nonnegative profit. If an additional firm of type j enters with action $\alpha\gamma_j y_j^*$, then the new prices are approximately $p + \alpha\gamma_j B_j y_j^*$ and the entrant earns profit (approximately) $\alpha\gamma_j y_j^{*\prime}(p + \alpha\gamma_j B_j y_j^*)$ which is strictly positive if $y_j^{*\prime} B_j y_j^* > 0$. Thus additional entry will occur, and no Cournot–Nash equilibrium with entry can exist near the Walras equilibrium at which DSD fails.[15]

We give a geometric argument (see Fig. 1) that DSD is satisfied when the economy acts as a single consumer. Under the above hypothesis, there is a single Walras equilibrium production y^*, which is given by maximizing a utility function u on the production set $\hat{Y}$ (we assume strict quasi-concavity of u). Since $\hat{Y}$ exhibits constant returns to scale, $p^* y^* = 0$, where p^* is the

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later we will require that F is twice continuously differentiable in a neighborhood of y^*. If in the Walras equilibrium (p^*, x^*, y^*), all consumers do not consume a bundle in the interior of their consumption set, then the definition of $F(y)$ for some y near y^* may depend on a notion of equilibrium in which some consumers are not in their consumption sets. See remark J for a related discussion.

[13] When F is viewed as a function of endowments ω rather than production (i.e., F is written in full form $F((\omega_i + \sum_j \theta_{ij} y_j)_i)$ rather than as $F(y)$ -- see footnote 8) then the rs entry of B_j is

$$\sum_{k=1}^{m} \theta_{kj} \frac{\partial F_r}{\partial \omega_{ks}} \left(\left(\omega_i + \sum_{t=1}^{n} \theta_{it} y_t^* \right)_i \right).$$

[14] This has a natural cost–benefit interpretation. At any Pareto optimum of $\hat{\mathscr{E}}$ which does not satisfy *DSD*, a planner (just as a firm) could buy resources at market prices and sell their product at a price which would more than cover cost. As such, they fail a cost–benefit test.

[15] This is formalized in Theorem 3. Note that we do not treat the case $y_j^{*\prime} B_j y_j^* = 0$, where higher order terms must be considered. Similarly, Theorem 3 shows (weak) *DSD* must hold (i.e., $y_j^{*\prime} B_j y_j^* \leq 0$).

equilibrium price vector. If the measure of firms changes, then output also changes to $y^* + \Delta y$; and so prices must change to $F(y^* + \Delta y)$ in order that this new amount will be demanded. (Both the case of more firms and less firms are considered simultaneously in the diagram.) Clearly $F(y^* + \Delta y)\Delta y \leq 0$ for all Δy, and this is the finitistic equivalent of DSD. Also, the geometry is inessential as $F(y^* + \Delta y)\,\Delta y \leq 0$ follows directly from the weak axiom of revealed preference. In fact, it is clear that DSD will be satisfied if the Walras correspondence satisfies the weak axiom of revealed preference locally and in certain directions.

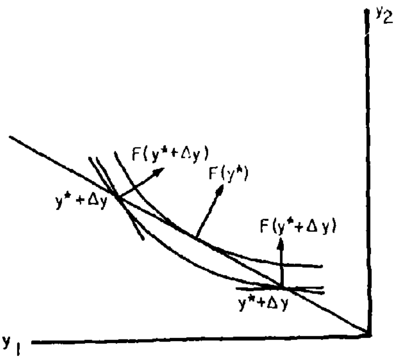

FIGURE 1

Several nondegeneracy conditions are necessary to the analysis. Appendix II establishes the genericity of these assumptions in an appropriate space.

j. Let $\mathscr{E} := (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ be given, and let (p^*, x^*, y^*) be a Walras equilibrium of $\mathscr{E}$. Suppose that (F, p^*) is a p^* based inverse demand function for the exchange economy $\mathscr{E}(y^*) = (X_i, \omega_i + \sum_j \theta_{ij} y_j^*, \succsim_i)$ and that for all j, Y_j is either locally polyhedral or regular smooth at $\gamma_j y_j^*$. If Y_j is locally polyhedral at $\gamma_j y_j^*$ define G_j to be the $l \times l$ matrix of zeros. If Y_j is regular smooth at $\gamma_j y_j^*$, then $R^{L\setminus(S_j(\gamma_j y_j^*)\cup\{1\})}$ is the $\#(L\setminus S_j(\gamma_j y_j^*)) - 1$ dimensional subspace of R^l corresponding to the components other than one which are smoothly variable along the efficient production surface of Y_j near $\gamma_j y_j^*$ (see (f)). Let $\hat{g}_j$ be as in (f)) (i.e., $\hat{g}_j$ gives the values of the first component of efficient points as a function of the other smoothly variable components). Let $\hat{G}_j := (D^2 \hat{g}_j(\gamma_j y_j^*))^{-1}$ (which is negative definite) and $\hat{P}_j := I$ (the identity matrix) for $R^{L\setminus(S_j(\gamma_j y_j^*)\cup\{1\})}$. Let G_j and P_j be the corresponding matrices (filled out with zeros) for the full dimensional space R^l.

Also define $\bar{p} := p^*_{\setminus\{1\}}$; i.e., $\bar{p}' = (p_2^* p_3^* \cdots p_l^*) = (F_2(y^*)\, F_3(y^*) \cdots F_l(y^*))$. The economy $\mathscr{E}$ satisfies the condition ND (nondegeneracy) at (p^*, x^*, y^*, F) provided:

(j1)

$$\begin{bmatrix} \frac{1}{\gamma_1}\begin{bmatrix} 0 & \bar{p}' \\ 0 & I \end{bmatrix} G_1 & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\gamma_n}\begin{bmatrix} 0 & \bar{p}' \\ 0 & I \end{bmatrix} G_n \end{bmatrix}_{ln \times ln} \begin{bmatrix} B_1 & B_2 & \cdots & B_n \\ \vdots & & & \vdots \\ B_1 & B_2 & \cdots & B_n \end{bmatrix}_{ln \times ln} \quad H_1$$

does not have eigenvalue -1 (G_i and B_i are $l \times l$, I is $(l-1) \times (l-1)$).

(j2)

$$\begin{bmatrix} \frac{1}{\gamma_1}\begin{bmatrix} 0 & \bar{p}' \\ 0 & I \end{bmatrix} G_1 \quad m_s\left(\frac{y_1^* y_1^{*\prime}}{y_1^{*\prime} B_1 y_1^*}\right) & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\gamma_n}\begin{bmatrix} 0 & \bar{p}' \\ 0 & I \end{bmatrix} G_n - m_s\left(\frac{y_n^* y_n^{*\prime}}{y_n^{*\prime} B_n y_n^*}\right) \end{bmatrix}_{ln \times ln} \begin{bmatrix} B_1 & B_2 & \cdots & B_n \\ \vdots & & & \vdots \\ B_1 & B_2 & \cdots & B_n \end{bmatrix}_{ln \times ln} \cdot H_2(m_s)$$

does not have eigenvalue -1, for a sequence of integers $m_s \to \infty$. (Only one m value is used in the proof, but its magnitude must be greater than $\bar{m}$, where $\bar{m}$ is determined by the data of the economy.)

(j3)

$$\begin{bmatrix} y_1^{*\prime} \\ y_2^{*\prime} \\ \vdots \\ y_n^{*\prime} \end{bmatrix}_{n \times l} [B_1 \; B_2 \cdots B_n]_{l \times ln} \{I + H_1\}^{-1}_{ln \times ln} \begin{bmatrix} \gamma_1 y_1^* & & & 0 \\ & \gamma_2 y_2^* & & \\ & & \ddots & \\ 0 & & & \gamma_n y_n^* \end{bmatrix}_{ln \times n}$$

has rank n.

III. THEOREMS

Theorem 1 is the main result of this paper. For sufficiently small α, we prove the existence of Cournot equilibrium with free entry. The number of firms in the market is determined endogenously. Firms who choose to produce typically have positive profit, and all firms with positive expected profit adopt pure strategies. The number of marginal firms is uniformly bounded.

THEOREM 1. *Let $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ be given. Assume that $\mathscr{E}$ has a Walras equilibrium (p^*, x^*, y^*), and that there exists a p^* based inverse* *demand function* (F, p^*) [*for the exchange economy* $\mathscr{E}(y^*) = (X_i, \omega_i + \sum_j \theta_{ij} y_j^*, \succsim_i)$], *which exhibits downward sloping demand at* (p^*, x^*, y^*, F). *Assume* $\mathscr{E}$ *has efficient outputs bounded away from zero relative to* p^*, *and for all* j, Y_j *is either locally polyhedral or regular smooth at* $\gamma_j Y_j^*$. *Assume in addition that the nondegeneracy conditions ND are satisfied. Then, there exists* $\bar{\alpha} > 0$ *such that for all* $\alpha \leq \bar{\alpha}$ *there is a Cournot equilibrium for the economy* $\mathscr{E}(\alpha)$.

Theorem 2 is a corollary to the proof of Theorem 1. It establishes that an arbitrary (DSD) Walras equilibrium of $\mathscr{E}$ can be obtained as the limit (as $\alpha \to 0$) of a sequence of Cournot equilibria, one point in the sequence for each $\mathscr{E}(\alpha)$. Furthermore, there exists a fixed number N such that these equilibria may be chosen so that for all $\alpha \leq \bar{\alpha}$, the number of marginal firms is less than N. It follows that as α approaches zero, the maximum of the sum of the outputs of the marginal firms in each industry becomes arbitrarily small (in euclidean norm) relative to the sum of the outputs of all firms in that industry.

This is important from a descriptive point of view because we do not want existence to rely very much on the presence of firms who move in and out of the market. No such firms are necessary in the limit economy $\mathscr{E}$. This leads us to the notion of a proper sequence of Cournot–Nash equilibria. A sequence of Cournot equilibria in mixed strategies, one for each $\mathscr{E}(\alpha_k)$, is called *proper* if the product of α_k and the number of active firms that use zero with positive probability, converges to zero. This formalizes the requirement that the firms which use zero with positive probability become a negligible part of the economy. Theorem 2 establishes that when the conditions of Theorem 1 are satisfied, there exists a proper sequence of Cournot Nash equilibria which converge to the Walras equilibrium (p^*, x^*, y^*).

THEOREM 2. *Let the conditions of Theorem* 1 *be satisfied. Then there exists* ρ *from* $(0, 1]$ *to the set of probability measures on* R^{ln} *with the following property: There exists* $S_1, S_2, \bar{\alpha} \in (0, \infty)$ *and an integer* N *such that for all* $\alpha \leq \bar{\alpha}$, $\rho(\alpha)$ *is the Cournot production associated with some Cournot equilibrium* $(\mu(\alpha), \eta(\alpha), (F(\alpha), \rho(\alpha)))$ *of* $\mathscr{E}(\alpha)$, *and*

(2.1) *the number of marginal firms is less than* N;

(2.2) $Max_{y \in \rho\,\mathrm{support}(\alpha)} \| y^* - y \| \leq \alpha S_1$; *and*

(2.3) $Max_{p \in \mathrm{support}\,\xi(\alpha)} \| \bar{p} - p \| \leq \alpha S_2$, *where* $\xi(\alpha)$ *is the Cournot price distribution corresponding to* $(F(\alpha), \rho(\alpha))$, *and* $\bar{p}' = (p_2^*, p_3^*, \ldots, p_l^*)$.

Theorem 3 establishes upper hemicontinuity of the Cournot equilibrium correspondence at $\alpha = 0$: Any proper sequence of Cournot equilibria with mixed strategies, one for each $\alpha_k (\alpha_k \to 0)$, for which expected output converges, converges to a DSD Walras equilibrium of $\mathscr{E}$ (which is a Cournot equilibrium of $\mathscr{E}(0)$). This makes precise the sense in which DSD is a necessary condition for Cournot equilibrium. Any Walras equilibrium of $\mathscr{E}$ that does not satisfy DSD is, in the present interpretation, an artifact of the infinitesimal specification. Global analogs of (e1) and (e2) are introduced and guarantee that efficient outputs are bounded away from zero at all relevant prices.

(e1′) For all j, if $t \in$ the interior of Ω is such that $t'y \leq 0$ for all $y \in Y_j$ then there is at most one $0 \neq y(t) \in Y_j$ such that $t'y(t) = 0$.

(e2′) For all j, with t and $y(t)$ as in (e1′), given $K, \delta \in (0, \infty)$, there exists $\epsilon > 0$ such that for all $s \in N(t, \epsilon)$,

$$[\{y \in Y_j \mid s'y \geq 0\} \cap N(0, K)] \subset [\{0\} \cup N(y(t), \delta)].$$

(e3′) The cones $\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n, -\Omega$ are positively semi-independent (see Debreu [3, p. 22]), and for each j, 0 is an exposed point of $\hat{Y}_j$ (i.e., there exists a hyperplane of support H to $\hat{Y}_j$ through 0 for which $H \cap Y_j = \{0\}$; see Valentine [17, p. 52]).

THEOREM 3. *Let* $\mathscr{E} = (X_i, \omega_i, \succsim_i, Y_j, \theta_{ij})$ *be given. Suppose that the* $Y_1, Y_2, \ldots, Y_n$ *satisfy* (e1′), (e2′), *and* (e3′) *and that there exist:*

(i) $F : R^{lm} \to \{1\} \times R^{l-1}$, *a selection from the Walras correspondence for exchange economies with* $X_i, \succsim_i$ *fixed;*

(ii) *a sequence* $(\alpha_k)_{k=1}^{\infty}$ *of strictly positive reals; and*

(iii) *a function* ρ *from* $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ *to the set of probability measures on* R^{ln}, *such that*

(iv) $\alpha_k \to 0$ *as* $k \to \infty$,

(v) *for all* k, $\rho(\alpha_k)$ *is the Cournot production associated with some Cournot equilibrium in mixed strategies* $(\mu(\alpha_k), \eta(\alpha_k), (F, \rho(\alpha_k))$ *of* $\mathscr{E}(\alpha_k)$; *and*

(vi) *there exists* $y^* \in R^{ln}$ *such that* $I(y, d\rho(\alpha_k)) \to y^*$ *as* $k \to \infty$.[16] *Then*

(3.1) *for all* $\epsilon > 0$, $\lim_{k \to \infty} \rho(\alpha_k)\{N(y^*, \epsilon)\} = 1$.

Furthermore, if F *is continuous at* y^*, *and we let* $\xi(\alpha_k)$ *denote the Cournot price distribution corresponding to* $(\rho(\alpha_k), F)$,

(3.2) *there exists* $\bar{p} \in R^{l-1}$ *such that for all* $\epsilon > 0$, $\lim_{k \to \infty} \xi(\alpha_k)\{N(\bar{p}, \epsilon)\} = 1$; *and*

(3.3) *there exists* $x^* \in R^{lm}$ *such that* $((1, \bar{p}), x^*, y^*)$, *is a Walras equilibrium of* $\mathscr{E}$.

Finally, if the sequence of Cournot equilibria in mixed strategies is proper, if

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[16] The set of aggregate actions which are feasible is bounded, so there is always a subsequence for which $I(y, d\rho(\alpha_k))$ converges.

F is twice continuously differentiable in a neighborhood of y^*, [*more precisely in a neighborhood of* $(\omega_i + \sum_j \theta_{ij} y_j^*)_i$], *and if* B_j *is defined as in* (i) *for each* $j = 1, 2, \ldots, n$, *then*

$$y_j^{*\prime} B_j y_j^* \neq 0 \text{ for each } j = 1, 2, \ldots, n. \tag{3.4}$$

IV. INTRODUCTION TO THE PROOF OF THEOREM 1

The purpose of this section is to develop some of the concepts used in the proof of Theorem 1, in particular "backward mappings." Our framework is partial equilibrium: a single market with U-shaped average cost curve firms and downward sloping demand. The reader will profit from turning first to Appendix 1, where that framework is explicitly introduced. Let y^* be the competitive industry output, $\alpha\gamma y^*$ be the "competitive output" for a firm of size α, and EIS (α) be the largest y for which a firm of size α has a nonzero optimal response (it can be shown that for α sufficiently small, the nonzero response is unique, call it $y(\alpha)$). The following "residual demand" diagram (Fig. 2) is standard; the average cost function AC_α^1 and demand function F are drawn relative to the action $y^* - \alpha\gamma y^*$ by other firms. We note:

(1) The profit maximizing response of a single firm is less than $\alpha\gamma y^*$ and profit is strictly positive.

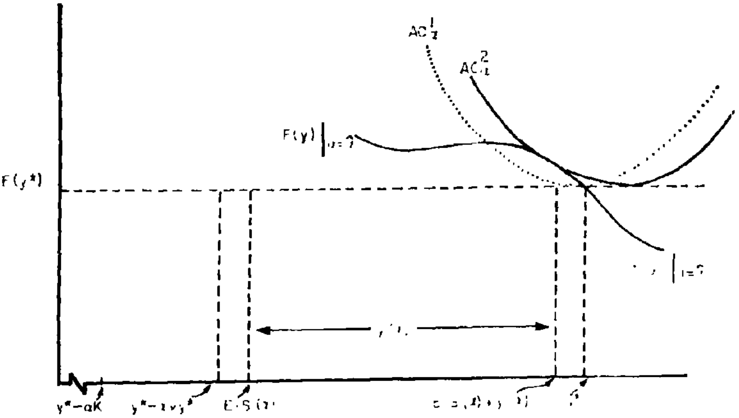

FIG. 2. AC_α^1 is the average cost curve positioned with respect to residual demand when the output of other firms is $y^* - \alpha\gamma y^*$. AC_α^2 is the average cost curve positioned with respect to residual demand when the output of other firms is $EIS(\alpha)$. (Note that αK converges to zero as α converges to zero. However, in this diagram $[y^* - \alpha K, y^*]$ is independent of α and F varies with α, because the horizontal axis is relabeled as α varies. K is chosen so that all relevant outputs lie in a ball around y^* with radius αK. In partial equilibrium, any $K > \gamma y^*$ will work.)

(2) As the origin of the average cost function (equal to the aggregate action of other firms) moves to the right, maximum profit declines, and at aggregate output of others, $EIS(\alpha)$, maximum profit is zero and is achieved at both zero output and output $y(\alpha)$.

(3) By adopting the convention that αK is a fixed distance on the horizontal axis, the shape of AC_α remains fixed in our diagram as α varies. For α sufficiently small, $[EIS(\alpha) - (y^+ - \alpha\gamma y^*)]/\alpha$ and $[\alpha\gamma y^* - y(\alpha)]/\alpha$ are nonnegative and converge to zero, since F becomes flatter in the diagram with standard unit $\alpha\gamma y^*$.

The backward map $b(y, \alpha)[b(y, \alpha) < y]$ indicates what the output of other firms in the industry must have been for the nonzero action of one maximizing firm to be $y - b(y, \alpha)$, yielding aggregate output y. For α small, and $y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]$, it can be shown that $b(y, \alpha)$ is a well-defined continuous function. For integers N, define the intervals $B(\alpha, N) = \{y - N(y - b(y, \alpha)) \mid y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]\}$, and note that $B(\alpha, N + 1)$ is left of $B(\alpha, N)$; however, they may intersect.

If $0 \in B(\alpha, N)$ for some N, then there exists $y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]$ with $y - N(y - b(y, \alpha)) = 0$, so there is a symmetric N firm equilibrium with aggregate output y, each operating firm acting optimally and earning nonnegative profit, and no incentive for entry (since 0 is an optimal response to $y = EIS(\alpha)$).

If $0 \notin B(\alpha, N)$ for all N, we must introduce the backward mapping for a marginal firm. First note that a marginal firm is also profit maximizing, so the aggregate action by other firms must be $EIS(\alpha)$, where the marginal firm is indifferent between action $y(\alpha)$ and 0. For any $y < EIS(\alpha)$, a pure strategy strictly dominates every mixed strategy. For expected output $y \in [EIS(\alpha), EIS(\alpha) + y(\alpha)]$ the aggregate action of other firms must be $EIS(\alpha)$, with the marginal firm producing $y(\alpha)$ with probability $[y - EIS(\alpha)]/y(\alpha)$ and producing 0 with probability $1 - [y - EIS(\alpha)]/y(\alpha)$. As y varies through $[EIS(\alpha), EIS(\alpha) + y(\alpha)]$, the expected output of the marginal firm varies through $[0, y(\alpha)]$.

Let N be such that $B(\alpha, N + 1) < 0 < B(\alpha, N)$. Using N regular firms and one marginal firm (and ignoring here a minor correction [which vanishes as $\alpha \to 0$] necessary because expected price is not equal to price of expected output), the total backward map from $EIS(\alpha) + y(\alpha)$ goes to $z \in B(\alpha, N + 1) < 0$ since the marginal firm has the same action as a regular firm, with probability equal to 1, while the total backward map from $EIS(\alpha)$ goes to $z \in B(\alpha, N) > 0$ since the marginal firm has action 0 with probability equal to 1. The map is continuous, so there exists an equilibrium with N regular firms and one marginal firm, with expected output $y \in (EIS(\alpha), EIS(\alpha) + y(\alpha))$. In the equilibrium, all firms are profit maximizing, regular firms earn strictly positive expected profit, the marginal firm earns zero expected profit, and there is no incentive for entry. (It should be noted that in the partial equilibrium model marginal firms are not needed for small α; i.e., $O \in B(\alpha, N)$ for some N (see Appendix I).)

The essential ingredients of the above argument are: first, to surround zero by the intervals $B(\alpha, N)$ and $B(\alpha, N + 1)$ [for some N], and then if necessary $(O \notin B(\alpha, N))$, to introduce marginal firms to "capture" zero. For situations in which there is more than one commodity (and/or more than one industry), the corresponding problem of surrounding and capturing zero (starting at a point of nonnegative profit and no entry) becomes more complex. The nondegeneracy assumptions (j1)-(j3) play a central role in the argument. Appendix II establishes the genericity of these assumptions.

V. PROOFS

The proof of Theorem I follows a sequence of lemmas. Let $z_j(Z, \alpha)$ denote the set of profit maximizing actions for a firm with production set $Y_j(\alpha)$ when the action of other firms is $y^* + \alpha Z$. Figure 3 is an analog of Fig. 2. We have represented the production possibilities of the jth firm with $\|\alpha K\|$ a fixed distance in the diagram. If the action of "other firms" is $y^* + \alpha Z$ (for some fixed α), isoprofit manifolds are defined for the firm. Since the axes are relabeled as α varies, the relation between y^* and $y^* + \alpha Z$ remains fixed in the diagram. As α approaches zero, the actions of "other firms" approach the competitive output, and in Fig. 2 (with $\|\alpha K\|$ a fixed distance), isoprofit manifolds "converge uniformly" to the isoprofit hyperplanes H_{p^*} associated with the competitive equilibrium price p^*. (The dependence of the isoprofit manifolds on α in Fig. 3 is analogous to the

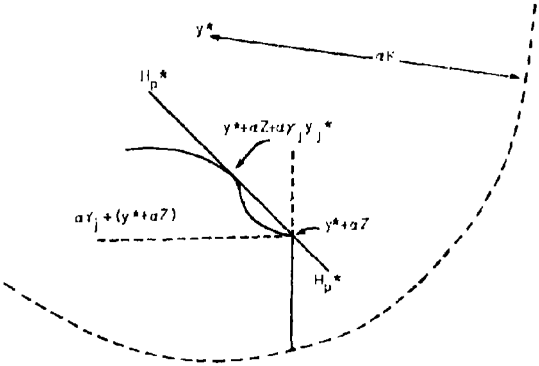

FIG. 3. H_p^* is the set of actions by a firm of type j which earn zero profit when the action of other firms is $y^* + \alpha Z$ and prices are fixed at p^*.

dependence of the inverse demand function F on α in Fig. 2.) In the limit, the profit maximizing actions for the jth firm are 0 and $\alpha\gamma_j y_j^*$. Lemmas 1–4 show that for α sufficiently small, the profit maximizing actions of a firm are either 0 or an efficient production in a neighborhood of $\alpha\gamma_j y_j^*$.

LEMMA 1. *Assume the hypotheses of Theorem 1 are satisfied. Given $K < \infty$, there exists $M < \infty$ such that $\|x\| \le M\alpha$ for all $x \in z_j(Z, \alpha)$ for all $(Z, \alpha) \in N(0, K) \times (0, 1] \subset R^{ln} \times R^1$.*

Proof of Lemma 1. The set of aggregate (over industries as well as firms) productions

$$S = \left[\bigcup_{\substack{\alpha \le 1\\ Z \in N(0,K)}} \left(Y_j(\alpha) + \left\{\sum_i (y_i^* + \alpha Z_i) + \sum_i \omega_i\right\}\right)\right] \cap \Omega^l$$

is bounded above. By desirability (see (a1)) and the upper hemicontinuity of the Walras correspondence, $\{F(y) \mid y = y_j(\alpha) + y^* + \alpha Z \in \text{Domain of } F$ for some $(y_j(\alpha), \alpha, Z) \in Y_j(\alpha) \times (0, 1] \times N(0, K)$ such that $y_j(\alpha) + \sum_i (y_i^* + \alpha Z_i) + \sum_i \omega_i \in S\}$ is contained in a closed cone C in the interior of Ω^l (this set contains all prices that could result from a feasible action in $Y_j(\alpha)$ when others' actions are $y^* + \alpha Z, (Z, \alpha) \in N(0, K) \times (0, 1]$). Since the asymptotic cone of Y_j is contained in $-\Omega^l$, there exists $M < \infty$ such that $px < 0$ for all $p \in C\backslash\{0\}$, $x \in Y_j(1)$, $\|x\| > M$. Noting $0 \in Y_j(\alpha) = \alpha Y_j(1)$ completes the proof. ▌

Let $EY_j(\alpha)$ be the set of technologically efficient productions in $Y_j(\alpha)$:

$$EY_j(\alpha) = \{y \in Y_j(\alpha) \mid \nexists\, x \in \Omega^l\backslash\{0\} \text{ such that } y + x \in Y_j(\alpha)\}.$$

LEMMA 2. *Assume that the hypotheses of Theorem 1 are satisfied. Given $K < \infty$ there exists $\bar{\alpha} > 0$ such that $x \in EY_j(\alpha)$ for all $x \in z_j(Z, \alpha)$, for all $(Z, \alpha) \in N(0, K) \times (0, \bar{\alpha}]$.*

Proof of Lemma 2. The profit of a firm in the jth industry which chooses x given that the actions of other firms are $y^* + \alpha Z$ is $\Pi(x \mid Z, \alpha) = x'F(y^* + \alpha Z + x)$. By the previous lemma $\|x\| \to 0$ as $\alpha \to 0$, and by (g2) F is continuously differentiable at $y^* + \alpha Z + x$ for all α sufficiently small. Thus, $\partial\Pi/\partial x_i = F_i(y^* + \alpha Z + x) + x'((\partial F/\partial y_{ji})(y^* + \alpha Z + x))$. Since $F_i(y^*) > 0$ for all i, it follows that $\partial\Pi/\partial x_i > 0$ for α sufficiently small, and $x \in EY_j(\alpha)$ for all $x \in z_j(Z, \alpha)$. ▌

LEMMA 3. *Assume that the hypotheses of Theorem 1 are satisfied. Given K, $\delta \in (0, \infty)$ there exists $\bar{\alpha} > 0$ such that $x \in N(\alpha\gamma_j y_j^*, \alpha\delta) \cup \{0\}$ for all $x \in z_j(Z, \alpha)$, $(Z, \alpha) \in N(0, K) \times (0, \bar{\alpha}]$.*

Proof of Lemma 3. By (e2) and the continuity of F in a neighborhood of y^*, there exists $\beta(M) > 0$ such that $y'F(W)/\|y\| < -\beta(M)$ for each $y \in [Y_j \cap N(0, M)]\backslash[N(\gamma_j y_j^*, \delta) \cup \{0\}]$ and W close to y^*. Setting M equal to the M value guaranteed by Lemma 1, for α small, $(1/\alpha)\, x'F(\alpha Z + y^* + x)/\|(1/\alpha)\, x\| < -\beta(M)$ for $(1/\alpha)\, x \in [Y_j \cap N(0, M)]\backslash[N(\gamma_j y_j^*, \delta) \cup \{0\}]$. Noting that $(1/\alpha)\, z_j(Z, \alpha) \subset N(0, M)$ for $(Z, \alpha) \in N(0, K) \times (0, 1]$ and that and zero action yields zero profit completes the proof. ▌

Lemma 4. *Assume that the hypotheses of Theorem* 1 *are satisfied and that* Y_j *is locally polyhedral at* $\gamma_j y_j^*$. *Given* $K < \infty$ *there exists* $\bar{\alpha} > 0$ *such that* $z_j(Z, \alpha) \subset \{0, \alpha\gamma_j y_j^*\}$ *for all* $(Z, \alpha) \in N(0, K) \times (0, \bar{\alpha}]$.

Proof of Lemma 4. Let $T_j = \{t \in R^l_- \mid \|t\| = 1$ and there exist $y_j \in Y_j$, $\lambda > 0$ such that $t = \lambda(\gamma_j y_j^* - y_j)$ and $\mu\gamma_j y_j^* + (1 - \mu)\, y_j \in Y_j$ for all $\mu \in [0, 1]\}$. If $T_j = \varnothing$, Lemma 3 yields the result. If not, there exists $\delta > 0$ such that $N(\gamma_j y_j^*, \delta) \cap Y_j = \{\gamma_j y_j^* - at \mid t \in T_j, a \in [0, \delta)\}$. By Lemma 3, $(1/\alpha)\, z_j(Z, \alpha) \subset (\{0\} \cup [N(\gamma_j y_j^*, \delta) \cap Y_j])$ for all $Z \in N(0, K)$, for all α sufficiently small. For any $x \in N(\gamma_j y_j^*, \delta) \cap Y_j$, $\Pi(\alpha x \mid Z, \alpha) = \alpha x'F(y^* + \alpha Z + \alpha x)$ is differentiable for α small, and $(\partial\Pi/\partial x)(\alpha x \mid Z, \alpha) = \alpha F(y^* + \alpha Z + \alpha x) + \alpha^2[(\partial F_i/\partial y_{jk})(y^* + \alpha Z + \alpha x)]'_{ik}\, x$. For each $t \in T_j$, $(1/\alpha)\, t'(\partial\Pi/\partial x)(\alpha x \mid Z, \alpha) = t'F(y^* + \alpha Z + \alpha x) + \alpha t'[\partial F_i/\partial y_{jk}]'_{ik}\, x$ which is strictly positive for α sufficiently small since $t'F(y^*) > 0$ for all $t \in T_j$, and $\alpha Z + \alpha x \to 0$. Thus, for α sufficiently small, for all $Z \in N(0, K)$, $x = \gamma_j y_j^*$ maximizes $\Pi(\alpha x \mid Z, \alpha)$ over the set $x \in N(\gamma_j y_j^*, \delta) \cap Y_j$, and $z_j(Z, \alpha) \subset \{0, \alpha\gamma_j y_j^*\}$. ▌

The technique of proof involves separating out the determination of how many firms of each type are in the market from what an active firm does. From Lemmas 1–4, the aggregate action of other firms is approximately y^*, then the action of a firm which is active in equilibrium (and thus making nonnegative profit), will be approximately $\alpha\gamma_j y_j^*$. We therefore study those actions of firms which maximize profit on a neighborhood of $\alpha\gamma_j y_j^*$. As α converges to zero, the number of firms present in an equilibrium grows without bound. Although the deviation of active firms from $\alpha\gamma_j y_j^*$ becomes insignificant, the total deviation of active firms may be significant from the viewpoint of entry of new firms, and is thus important for the analysis. Lemma 5 computes the deviation of an individual firm, and Lemma 6 makes use of the significant aggregate deviation.

For each j, let $\delta_j > 0$ be such that $\overline{N(\gamma_j y_j^*, \delta_j)}$ is a subset of the cube C (if Y_j is locally polyhedral at $\gamma_j y_j^*$) or the neighborhood $N(\gamma_j y_j^*)$ (if Y_j is locally regular smooth at $\gamma_j y_j^*$) which is described in (f).

Let $z_j^*(Z, \alpha)$ denote the set of profit maximizing actions in $Y_j(\alpha) \cap \overline{N(\alpha\gamma_j y_j^*, \alpha\delta_j)}$ when other firms' actions are $y^* + \alpha Z$. By Lemma 3, for α

sufficiently small, $z_j^*(Z, \alpha) = z_j(Z, \alpha)$ if $\Pi(x \mid Z, \alpha) > 0$ for some $x \in z_j^*(Z, \alpha)$.

LEMMA 5. *Assume the hypotheses of Theorem* 1 *are satisfied. Given* $K < \infty$ *there exists* $\bar{\alpha} > 0$ *such that for all* $(Z, \alpha) \in N(0, K) \times (0, \bar{\alpha}]$, $z_j^*(Z, \alpha)$ *is a singleton and*

$$z_j^*(Z, \alpha) = \alpha\gamma_j y_j^* - \alpha^2 \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} \left[\sum_t B_t Z_t + \gamma_j (B_j + B_j') y_j^*\right]$$
$$+ O(\alpha^3)^{17}$$
$$(= \alpha\gamma_j y_j^* \text{ if } Y_j \text{ is locally polyhedral at } \gamma_j y_j^*)$$

and

$$\frac{\partial z_j^*(Z, \alpha)}{\partial Z_t'} = -\alpha^2 \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j B_t + O(\alpha^3)$$
$$(= [0] \text{ if } Y_j \text{ is locally polyhedral at } \gamma_j y_j^*).$$

[Recall $\bar{p}' = (p_2^*, p_3^*, \ldots, p_l^*)$].

Proof of Lemma 5. The results for locally polyhedral Y_j follow from the proof of Lemma 3.

If Y_j is locally regular smooth at $\gamma_j y_j^*$, by Lemma 2 we can restrict attention to the smooth efficient manifold, so the profit maximization problem becomes

$$\max_{x \in P_j(R^l)} \Pi(\alpha\gamma_j y_j^* + \alpha x + \alpha g_j(x)\, e_1 \mid Z, \alpha),$$

where g_j gives the deviation of the first coordinate from $\gamma_j y_{j1}^*$ as a function of the deviations x of the other coordinates from $\gamma_j y_{j1}^*$ on the smooth efficient manifold of $Y_j \cap N(\gamma_j y_j^*, \delta_j)$. [$P_j$ is the projection to the space of components other than 1 which are smoothly variable].

Restricting attention to changes in x which are in $P_j(R^l)$, for α sufficiently small the profit function is differentiable, and

$$\left.\frac{\partial \Pi}{\partial x}\right|_{\substack{\text{restricted} \\ \text{to } P_j(R^l)}} = \alpha \left\{P_i + \left[\frac{\partial g_j(x)}{\partial x}_{l\times 1} \quad 0_{l\times(l-1)}\right]\right\} \left\{F(\ ) + \alpha \left[\frac{\partial F_i(\ )}{\partial y_{jk}}\right]_{ik}' \cdot\right.$$
$$\left.\times \{\gamma_j y_j^* + x + g_j(x)\, e_1\}\right\},$$

where F and $\partial F_i/\partial y_{jk}$ are evaluated at $y^* + \alpha Z + \alpha\{\gamma_j y_j^* + x + g_j(x)\, e_1\}$

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[17] A function $r(\alpha)$ is $O(\alpha^n)$ if there exists $M < \infty$ such that $|r(\alpha)/\alpha^n| < M$ for small α (i.e., $r(\alpha)/\alpha^n$ is bounded as $\alpha \to 0$).

and $\partial g_j(x)/\partial x$ is *defined* to be zero for those components not variable in $P_j(R^l)$.

Since $\| x \| \to 0$ as $\alpha \to 0$ for optimal choices, $(\partial^2 \Pi/\partial x\, \partial x')|_{\text{restricted to } P_j(R^l)} = \alpha(\partial^2 g_j(x)/\partial x\, \partial x') + O(\alpha^2)$ where the s, t entry of $(\partial^2 g_j(x)/\partial x\, \partial x')$ is *defined* to be zero if either x_s or x_t is not variable in $P_j(R^l)$. Y_j is regular smooth at $\gamma_j y_j^*$ so $x'(\partial^2 g_j(0)/\partial x\, \partial x')\, x < 0$ for all $0 \neq x \in P_j(R^l)$.

Since $(\partial^2 g_j(x)/\partial x\, \partial x')$ converges to $(\partial^2 g_j(0)/\partial x\, \partial x')$ as $x \to 0$, $z_j^*(Z, \alpha)$ is a singleton for α sufficiently small.

To find the unique profit maximizing x, set $(\partial \Pi/\partial x) = 0$, use the Taylor expansion for $F(\ )$ and $[\partial F_i(\ )/\partial y_{jk}]$(at y^*) and $(\partial g_j(x)/\partial x)$(at 0), note that, as defined, $(\partial g_j(0)/\partial x) = -P_j F(y^*)$ and solve to find

$$x = -\alpha G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} \left\{ \sum_t B_t Z_t + \gamma_j (B_j + B_j')\, y_j^* \right\} + O(\alpha^2).$$

(Recall $F_1 \equiv 1$ so the first row of each B_t is a zero vector.) With this evaluation of x, $g_j(x) = -F'(y^*)\, P_j x + O(\alpha^2)$ and the $z_j^*(Z, \alpha)$ result follows.

The $(\partial z_j^*(Z, \alpha)/\partial Z_t')$ result follows from implicit differentiation of the first order condition along with the evaluation of x above. ▌

Using the "reaction functions" of Lemma 5, it is possible to find a Cournot "equilibrium" when the number of firms of each type is fixed and allowed to be a non integer and α is small. Lemma 6 computes "equilibrium" aggregate output as a function of α and the number of operating firms of each type, which Lemma 7 shows that for α small, for certain numbers of operating firms, each firm earns positive profit (so, as noted in the discussion before Lemma 5, the "reaction function" gives the true profit maximizing action of the firm) and there is no incentive for entry.

Let $Z(v, \alpha)$, $v \in R^n$, be the normalized error from competitive output y^* in a symmetric (within industries) "equilibrium" in which there are $1/\alpha\gamma_j + v_j$ firms of type j operating and forced to maximize profit over the restricted set $Y_j(\alpha) \cap N(\alpha\gamma_j y_j^*, \alpha\delta_j)$ for $j = 1, 2, \ldots, n$ (no entry is allowed). The normalization is such that aggregate output in this "equilibrium" is $y^* + \alpha Z(v, \alpha)$. Note that $1/\alpha\gamma_j + v_j$ is not restricted to integral values.

LEMMA 6. *Assume the hypotheses of Theorem 1 are satisfied. Given $K < \infty$ there exists $\bar{\alpha} > 0$ such that for all $(v, \alpha) \in N(0, K) \times (0, \bar{\alpha}] \subset R^n \times R^1$, a $Z(v, \alpha)$ exists and*

$$Z(v, \alpha) = -\{I + H_1\}^{-1} \left\{ -v_j \gamma_j y_j^* + \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} B_j' y_j^* \right\}_j + O(\alpha)$$

where H_1 is the matrix in (j1) *(so $\{I + H_1\}$ is invertible).*

Proof of Lemma 6. Let $\varphi_j(Z, \alpha) = Z + (1/\alpha)\, z_j^*(Z, \alpha)$ and define $\psi_j(Z, \alpha)$ by $\varphi_j(\psi_j(Z, \alpha), \alpha) = Z$. By Lemma 5, given $K < \infty$, there exists $\bar{\alpha} > 0$ such that for all $(Z, \alpha) \in N(0, K) \times (0, \bar{\alpha}]$, $\psi_j(Z, \alpha)$ is a well-defined function and, by implicit differentiation,

$$\frac{\partial \psi_j(Z, \alpha)}{\partial Z_t'} = I + \alpha \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j B_t + O(\alpha^2).$$

Using $\varphi_j(\psi_j(0, \alpha), \alpha) = 0$ and $p_j(Z, \alpha) = Z + (1/\alpha)\, z_j^*(Z, \alpha)$, we find $\psi_j(0, \alpha) = -\gamma_j y_j^* + \alpha\gamma_j \left[\begin{smallmatrix} 0 & -\bar{p}' \\ 0 & I \end{smallmatrix}\right] G_j \left[\begin{smallmatrix} 0 & 0 \\ -\bar{p} & I \end{smallmatrix}\right] B_j' y_j^* + O(\alpha^2)$ and, for $Z \in N(0, K)$,

$$\psi_j(Z, \alpha) = Z - \gamma_j y_j^* + \alpha \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} \left\{ \sum_t B_t Z_t + \gamma_j B_j' y_j^* \right\} + O(\alpha^2).$$

For $v \in R^n$ the backward map $\Psi(Z, v, \alpha)$ is defined by

$$\Psi(Z, v, \alpha) = Z + \frac{1}{\alpha} y^* + \sum_j \left(\frac{1}{\alpha\gamma_j} + v_j \right) [\psi_j(Z, \alpha) - Z].$$

To understand the significance of this function, note that for aggregate output in a Cournot equilibrium in pure strategies of $\mathscr{E}(\alpha)$ to be $y^* + \alpha Z$, each operating firm of type j has action $\alpha(Z - \psi_j(Z, \alpha))$ [as a result of defining $\psi_j(\cdot, \alpha)$ as the inverse of $\varphi_j(\cdot, \alpha)$]. If there are $(1/\alpha\gamma_j) + v_j$ firms of type j operating with pure strategies, their aggregate production is $(1/\alpha\gamma_j + v_j)\, \alpha(Z - \psi_j(Z, \alpha))$. Hence if $\Psi(Z, v, \alpha) = 0$, then $y^* + \alpha Z = \sum_j (1/\alpha\gamma_j + v_j)\, \alpha(Z - \psi_j(Z, \alpha))$, and the aggregate output of the $1/\alpha\gamma_j + v_j$ firms, $j = 1, 2, \ldots, n$ is exactly the aggregate output needed. Thus, the "equilibrium" error $Z(v, \alpha)$ satisfies $\Psi(Z(v, \alpha), v, \alpha) = 0$.

$$\frac{\partial \Psi(Z, v, \alpha)}{\partial Z'} = I + \sum_j \left(\frac{1}{\alpha\gamma_j} + v_j \right) \left[\frac{\partial \psi_j(Z, \alpha)}{\partial Z'} - I \right] = I + H_1 + O(\alpha),$$

and for $(Z, v) \in N(0, K_1) \times N(0, K_2) \subset R^{ln} \times R^n$ converges uniformly to $I + H_1$ as α converges to 0 (where K_1, K_2 are any finite numbers). Also,

$$\Psi(0, v, \alpha) = \left(-v_j \gamma_j y_j^* + \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} B_j' y_j^* \right)_j + O(\alpha)$$

so $\Psi(Z, v, \alpha) = \Psi(0, v, \alpha) + \{I + H_1\} Z + O(\alpha)$ for $(Z, v) \in N(0, K_1) \times N(0, K_2)$, and $Z(v, \alpha) = -\{I + H_1\}^{-1} \Psi(0, v, \alpha) + O(\alpha)$. [Given the K in the statement of the lemma, an appropriate K_1 is determined by $\{Z(v, \alpha) \mid v \in N(0, K)\}$.] ||

LEMMA 7. *Assume the hypotheses of Theorem 1 are satisfied. There exist* $v^* \in R^n$, $\epsilon^* > 0$, $\bar{\alpha} > 0$ *such that* $\Pi(z_j^*(\psi_j(Z(v, \alpha), \alpha), \alpha) \mid \psi_j(Z(v, \alpha), \alpha), \alpha) > 0$ *and* $\Pi(z_j^*(Z(v, \alpha), \alpha) \mid Z(v, \alpha), \alpha) < 0$ *for all* $j = 1, 2, \ldots, n$, *for all* $(v, \alpha) \in \overline{N(v^*, \epsilon^*)} \times (0, \bar{\alpha}]$.

Proof of Lemma 7. Let

$$Z(v, 0) = -\{I + H_1\}^{-1}\left(-v_j \gamma_j y_j^* + \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ \bar{p} & I \end{bmatrix} B_j' y_j^*\right)_j.$$

Then for and $K < \infty$, $Z(\cdot, \alpha)$, coverges uniformly to $Z(\cdot, 0)$, $\alpha Z(\cdot, \alpha)$ converges uniformly to zero, and $(1/\alpha)\, z_j^*(\psi_j(Z(\cdot, \alpha), \alpha), \alpha)$ and $(1/\alpha)\, z_j^*(Z(\cdot, \alpha), \alpha)$ converge uniformly to $\gamma_j^* y_j$ for $v \in N(0, K)$.

The first expression for profit is

$$\begin{aligned} &z_j^{*\prime}(\psi_j(Z(v, \alpha), \alpha), \alpha)\, F(y^* + \alpha Z(v, \alpha)) \\ &\quad = \alpha\{Z(v, \alpha) - \psi_j(Z(v, \alpha), \alpha)\}'\, F(y^* + \alpha Z(v, \alpha)) \\ &\quad = \alpha^2 \gamma_j y_j^{*\prime} \sum_t B_t Z_t(v, \alpha) + O(\alpha^3) \left(\text{using } F'(y^*)\begin{bmatrix} 0 & \bar{p}' \\ 0 & I \end{bmatrix} = 0\right) \end{aligned}$$

so $(1/\alpha^2)$ times that profit converges uniformly to

$$\gamma_j y_j^{*\prime} \sum_t B_t Z_t(v, 0).$$

The second expression for profit is

$$\begin{aligned} &z_j^{*\prime}(Z(v, \alpha), \alpha)\, F(y^* + \alpha Z(v, \alpha) + z_j^*(Z(v, \alpha), \alpha)) \\ &\quad = \alpha^2 \gamma_j y_j^{*\prime} \left(\sum_t B_t Z_t(v, \alpha) + \gamma_j B_j y_j^*\right) + O(\alpha^3) \end{aligned}$$

so $(1/\alpha^2)$ times that profit converges uniformly to

$$\gamma_j y_j^{*\prime} \left(\sum_t B_t Z_t(v, 0) + \gamma_j B_j y_j^*\right).$$

Thus, it suffices to show that there exists a $v^* \in R^n$ such that

$$0 < \begin{bmatrix} y_1^{*\prime} \\ \vdots \\ y_n^{*\prime} \end{bmatrix} [B_1 \cdots B_n]\, Z(v^*, 0) < -\begin{pmatrix} \gamma_1 y_1^{*\prime} B_1 y_1^* \\ \vdots \\ \gamma_n y_n^{*\prime} B_n y_n^* \end{pmatrix}.$$

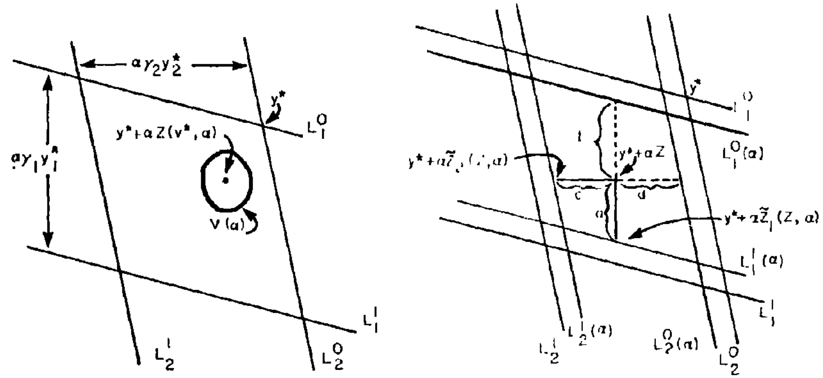
But by (j3)

$$\begin{bmatrix} y_1^{*\prime} \\ \vdots \\ y_n^{*\prime} \end{bmatrix} [B_1 \cdots B_n] \left[-\frac{\partial Z(v, 0)}{\partial v'} \right] = \begin{bmatrix} y_1^{*\prime} \\ \vdots \\ y_n^{*\prime} \end{bmatrix} [B_1 \cdots B_n] \{I + H_1\}^{-1}$$

$$= \begin{bmatrix} \gamma_1 y_1^* & 0 & & 0 \\ & \ddots & & \\ 0 & & \gamma_2 y_2^* & 0 \\ & & & \ddots \\ 0 & 0 & & \gamma_n y_n^* \end{bmatrix}$$

has full rank, so such a v^* does exist. By the uniform convergence on compact sets and the openness of the condition just above, the $\epsilon^* > 0$ and $\bar{\alpha} > 0$ also exist. ∎

There exists $\alpha^* > 0$ such that for all $(v, \alpha) \in \overline{N(v^*, \epsilon^*)} \times (0, \alpha^*]$, the $Z(v, \alpha)$ "equilibrium" has strictly positive profit for operating firms who are therefore, by the remark before Lemma 5, at *global profit maximizing positions* ($z_j^* = z_j$), and there is no incentive for entry *over the entire production set* $Y_j(\alpha)$. It only remains to show that there is an equilibrium with integral numbers of firms in each industry. To this end, we introduce marginal firms, and a backward map for marginal firms (see Figs. 4 and 5).



FIGS. 4 (left), 5 (right). $L_1^0 = \{y^* + \alpha Z \mid y_1^{*\prime}(B_1 Z_1 + B_2 Z_2) = 0\}$; $L_1^1 = \{y^* + \alpha Z \mid y_1^{*\prime}(B_1 Z_1 + B_2 Z_2) = -\gamma_1 y_1^{*\prime} B_1 y_1^*\} = \{y^* + \alpha Z \mid Z \in EIS_1(K)\}$; $L_2^0 = \{y^* + \alpha Z \mid y_2^{*\prime}(B_1 Z_1 + B_2 Z_2) = 0\}$; $L_2^1 = \{y^* + \alpha Z \mid y_2^{*\prime}(B_1 Z_1 + B_2 Z_2) = -\gamma_2 y_2^{*\prime} B_2 y_2^*\} = \{y^* + \alpha Z \mid Z \in EIS_2(K)\}$; $V(\alpha) = \{y^* + \alpha Z(v, \alpha) \mid v \in \overline{N(v^*, \delta^*)}\}$; $L_1^1(\alpha) = \{y^* + \alpha Z \mid Z \in EIS_1(K, \alpha)\}$; $L_1^0(\alpha) = \{y^* + \alpha Z + z_1^{\perp}(Z, \alpha) \mid Z \in EIS_1(K, \alpha)\}$; $L_2^1(\alpha) = \{y^* + \alpha Z \mid Z \in EIS_2(K, \alpha)\}$; $L_2^0(\alpha) = \{y^* + \alpha Z + z_2^{\perp}(Z, \alpha) \mid Z \in EIS_2(K, \alpha)\}$; $a/(a + b) = q_1(Z, \alpha)$; $c/(c + d) = q_2(Z, \alpha)$.

As in Section IV, a marginal firm must be faced with an aggregate output of other firms which makes it indifferent between entering and staying out of the market; i.e., the output of other firms must be on the marginal firm's entry indifference surface. Given an appropriate aggregate output y, the backward map for a marginal firm gives a point Z on the entry indifference surface for that firm, and a number q between zero and one such that when the firm is faced with output Z by other firms, and is active in the market (at its optimal nonzero output) with probability q, and inactive with probability $1 - q$, then expected aggregate output is y. Lemma 8 computes the backward map for marginal firms.

First, for each j, for each $K < \infty$ define the entry indifference surface $EIS_j(K, \alpha) = \{Z \in N(0, K) \mid \Pi(z_j^*(Z, \alpha) \mid Z, \alpha) = 0\}$ and $EIS_j(K) = \{Z \in N(0, K) \mid y_j^{*\prime} \sum_i B_i Z_i = -\gamma_j y_j^{*\prime} B_j y_j^*\}$. For α sufficiently small, $EIS_j(K, \alpha)$ is a C^1 manifold with boundary (since F and z_j^* are C^1) and there exists $M_1 < \infty$ such that $d(EIS_j(K, \alpha), EIS_j(K)) \le \alpha M_1$ (where d is the Hausdorff metric).

For $\alpha \in [0, \alpha^*]$ let $Q(\alpha) = \{Z(v, \alpha) \mid v \in \overline{N(v^*, \epsilon^*)}\}$. Since $Z(\cdot, \alpha)$ converges uniformly to $Z(\cdot, 0)$, there exists $M_2 < \infty$ such that $d(Q(\alpha), Q(0)) \le \alpha M_2$ for all small α. Thus for K sufficiently large (to include the entire area of interest) there exists $\alpha_1 > 0$ such that for all $\alpha \in (0, \alpha_1]$,

$$\min_j \min \{\| x - y \| \mid x \in EIS_j(K, \alpha), y \in Q(\alpha)\}$$

$$\ge \tfrac{1}{2} \min_j \min \{\| x - y \| \mid x \in EIS_j(K), y \in Q(0)\} > 0,$$

where the last inequality follows from Lemma 7 and the definitions of $Q(0)$ and $EIS_j(K)$.

For each j, for all $Z \in Q(\alpha)$, $\alpha \in [0, \alpha_1]$, the backward map for marginal firms is defined by $\psi_j^m(Z, \alpha) = (\hat{Z}_j(Z, \alpha), q_j(Z, \alpha)) \in EIS_j(K, \alpha) \times (0, 1)$ such that $\hat{Z}_j(Z, \alpha) + (1/\alpha) q_j(Z, \alpha) z_j^*(\tilde{Z}_j(Z, \alpha), \alpha) = Z$ (where K is sufficiently large to include all relevant points).

Because of the properties of ψ_j and the fact that $V(\alpha)$ is bounded away from the entry indifference surface and the zero profit surface ($L_j^1(\alpha)$ and $L_j^0(\alpha)$ in Fig. 5) for all j, the marginal backward map is approximately computed as follows: Find the unique point r in the intersection of the entry indifference surface $EIS_j(K, \alpha)$ and the line segment joining $y^* + \alpha Z$ and $y^* + \alpha\psi_j(Z(v, \alpha), \alpha)$, and set the probability q_j equal to the ratio of the length of the segment from $y^* + \alpha Z$ to r to the length of the (nonzero) optimal response to r, $\varphi_j((r - y^*)/\alpha, \alpha)$ (see Figs. 4 and 5).

LEMMA 8. *Assume the hypotheses of Theorem 1 are satisfied and v^*, ϵ^* are as in the statement of Lemma 7. Then there exists $\bar{\alpha} > 0$ such that*

$$\psi_j^m(Z(v, \alpha), \alpha) = (Z - q_j(Z(v, 0), 0)\, \gamma_j y_j^*,\ q_j(Z(v, 0), 0)) + O(\alpha)$$

for all $(v, \alpha) \in \overline{N(v^*, \epsilon^*)} \times [0, \bar{\alpha}]$, *where*

$$q_j(Z(v, 0), 0) = \frac{y_j^{*\prime} \sum_t B_t Z_t(v, 0)}{\gamma_j y_j^{*\prime} B_j y_j^*}.$$

Proof of Lemma 8. Existence and uniqueness of $\psi_j{}^m(Z(v, \alpha), \alpha)$ for small α follows from the existence and uniqueness of $\psi_j(Z, \alpha)$ and the fact that $\min_j \min\{\|x - y\| \mid x \in EIS_j(K, \alpha), y \in Q(\alpha)\} > 0$ for small α. This also guarantees $q_j(Z(v, \alpha), \alpha)$ is bounded away from 0 and away from 1 for α small and $v \in \overline{N(v^*, \epsilon^*)}$. The expression for $\psi_j{}^m(Z(v, \alpha), \alpha)$ follows from the properties of $z_j^*(Z, \alpha)$ and the fact that $d(EIS_j(K, \alpha), EIS_j(K)) < \alpha M_1$ for small α. The expression for $q_j(Z(v, 0), 0)$ follows from the form of $EIS_j(K)$ and the fact that $(1/\alpha)\, z_j^*(Z, \alpha)$ converges to $\gamma_j y_j^*$ as α converges to 0. ‖ Let

$$q(Z, \alpha) = \begin{pmatrix} q_1(Z, \alpha) \\ \vdots \\ q_n(Z, \alpha) \end{pmatrix}.$$

By (j3) and Lemma 8, $\{q(Z(v, \alpha), \alpha) \mid v \in \overline{N(v^*, \epsilon^*)}\}$ is the closure of a neighborhood of $q(Z(v^*, 0), 0)$ for α sufficiently small. Thus, there exists an integer $\bar{m} > 0$ and an $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha}]$,

$$\{\bar{m} q(Z(v, \alpha), \alpha) - v \mid v \in N(v^*, \epsilon^*)\} \supset ([-1, 1]^n + \{\bar{m} q(Z(v^*, 0), 0) - v^*\}).$$

We could use a different integer $\bar{m}_i$ for each component of q, which corresponds to having a different number of marginal firms in each industry.

Proof of Theorem 1. Let v^*, ϵ^* be as in Lemma 7, and m as in the remarks after Lemma 8. For all $\alpha > 0$, there exists a vector $\eta^*(\alpha)$ with integral compronents such that

$$|\eta_j^*(\alpha) - (1/\alpha\gamma_j + v_j^*) + \bar{m} q_j(Z(v^*, 0), 0)| < \tfrac{1}{2}$$

for all j, so there exists $\alpha^* > 0$ such that

$$\eta_j^*(\alpha) - (1/\alpha\gamma_j + \bar{v}_j(\alpha)) + \bar{m} q_j(Z(v(\alpha), \alpha), \alpha) = 0 \text{ for all } j,$$

for some $\bar{v}_j(\alpha) \in N(v^*, \epsilon^*)$, for all $\alpha \in (0, \alpha^*]$. (Note that $|\eta_j^*(\alpha) - (1/\alpha\gamma_j)| \leq |v_j^*| + \bar{m} + \tfrac{1}{2}$.)

Now consider a backward mapping from Z with $\bar{m}$ marginal firms of each type, and $\eta_j^*(\alpha)$ pure strategy firms of type j, $j = 1, 2, \ldots, n$.

If $F(y)$ is not linear in y near y^* then an error is introduced because expected price is not equal to price of expected output, so the backward maps, and the entry indifference surface, etc., must be redefined to take account of the effect of the mn firms using probability distributions. However, the effect of all these changes is only $O(\alpha)$ since the support of the aggregate distribution is contained in some ball with radius αK for some $K < \infty$, and for bounded Z, $F(y^* + \alpha Z)$ is essentially linear in Z for small α.

Thus the "contribution" of the pure strategy firms is $\Psi(Z, \eta^*(\alpha) - (1/\alpha)\Gamma, \alpha) + O(\alpha)$ where $\Gamma' = (1/\gamma_1, 1/\gamma_2, \ldots, 1/\gamma_n)$, while the "contribution" of the marginal firms is

$$-\bar{m}\begin{pmatrix} q_1\gamma_1 y_1^* \\ \vdots \\ q_n\gamma_n y_n^* \end{pmatrix} + O(\alpha).$$

The backward mapping from $Z(\bar{v}(\alpha), \alpha)$ with $\eta^*(\alpha) = (1/\alpha)\Gamma + \bar{v}(\alpha) - \bar{m}q(\bar{v}(\alpha), \alpha)$ is therefore

$$\Psi(Z(\bar{v}(\alpha), \alpha), \bar{v}(\alpha) - \bar{m}q(\bar{v}(\alpha), \alpha), \alpha) - \bar{m}\begin{pmatrix} q_1(\bar{v}(\alpha), \alpha)\,\gamma_1 y_1^* \\ \vdots \\ q_n(\bar{v}(\alpha), \alpha)\,\gamma_n y_n^* \end{pmatrix} + O(\alpha)$$

(where $q_i(v, \alpha) = q_i(Z(v, \alpha), \alpha)$) which is equal to

$$Z(\bar{v}(\alpha), \alpha) + \begin{pmatrix} \vdots \\ -(\bar{v}_j(\alpha) - \bar{m}q_j(\bar{v}(\alpha), \alpha))\,\gamma_j y_j^* + \dfrac{1}{\gamma_j}\begin{bmatrix} 0 & \bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} \\ \vdots \end{pmatrix}$$

$$\times \left.\left\{\sum_t B_t Z_t + \gamma_j B_j' y_j^*\right\}\right)_j - \bar{m}\begin{pmatrix} \vdots \\ (q_j(\bar{v}(\alpha), \alpha)\,\gamma_j y_j^*)_j \\ \vdots \end{pmatrix} + O(\alpha)$$

$$= \{I + H_1\}\, Z(\bar{v}(\alpha), \alpha) + \begin{pmatrix} \vdots \\ -\bar{v}_j(\alpha)\,\gamma_j y_j^* + \begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j \begin{bmatrix} 0 & 0 \\ -\bar{p} & I \end{bmatrix} B_j' y_j^* \\ \vdots \end{pmatrix}_j$$

$$+ O(\alpha)$$

$$= O(\alpha) \text{ by Lemma 6.}$$

If we call this backward map $\Psi_F(Z, \eta^*, \bar{m}, \alpha)$, then $\partial\Psi_F/\partial Z' = I + H_2 + O(\alpha)$, where H_2 is the matrix in (j2) (we choose $\bar{m}$ to be one of the m_s values). For α sufficiently small $\partial\Psi_F/\partial Z'$ is invertible by (j2). Since we are interested in a compact set of v values, and therefore a compact set of Z values, and the functions we consider converge uniformly on compact sets, there exists $\alpha_1 > 0$ such that the $\eta^*(\alpha)$ pure strategy and $\bar{m}n$ marginal firms are all acting optimally and generate aggregate expected output $y^* + \alpha Z(\bar{v}(\alpha), \alpha) + O(\alpha^2)$, and since $\bar{v}(\alpha) \in N(v^*, \epsilon^*)$, by Lemma 7 and the uniform convergence, there exists $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha}]$, there exists a true equilibrium with m marginal firms of each type and $\eta_j^*(\alpha)$ pure strategy firms of type j, $j = 1, 2, \ldots, n$. In this equilibrium

(a) there is an integral number of firms in each industry, $\bar{m} + \eta_j^*(\alpha)$;

(b) all firms, including marginal firms, are maximizing profit over all mixed strategies on their entire production set;

(c) all pure strategy firms earn strictly positive expected profit, while marginal firms earn zero expected profit,

(d) there is no incentive for entry (entrants earn strictly negative expected profit if existing firms maintain their actions);

(e) there exists $S < \infty$ such that the support of $\rho(\alpha)$ is contained in $N(y^*, \alpha S)$ where $\rho(\alpha)$ is the Cournot production associated with the constructed Cournot equilibrium of $\mathscr{E}(\alpha)$. ▮

Proof of Theorem 2. Let $\bar{\alpha}_0$ be the α value guaranteed by Theorem 1, and let ρ: $(0, \bar{\alpha}_0] \to$ probability measures on R^{ln} be defined by $\rho(\alpha) =$ the Cournot production associated with the Cournot equilibrium of $\mathscr{E}(\alpha)$ constructed in the proof of Theorem 1. Also, let $N = n\bar{m} + 1$ where $\bar{m}$ is as in the proof of Theorem 1. Then in each Cournot equilibrium, there are less than N marginal firms.

As noted in the proof of Theorem 1, there exists $S_1 < \infty$ such that supp $\rho(\alpha) \subset N(y^*, \alpha S_1)$ for all $\alpha \in (0, \bar{\alpha}_0]$. Since F is C^2 in a neighborhood of y^*, there exists $S_2 < \infty$ and $\bar{\alpha} \in (0, \bar{\alpha}_0]$ such that supp $\xi(\alpha) \subset N(\bar{p}, \alpha S_2)$ for all $\alpha \in (0, \bar{\alpha}]$, where $\xi(\alpha)$ is the Cournot price distribution corresponding to $(F(\alpha), \rho(\alpha))[F(\alpha) = F]$. ▮

Proof of Theorem 3. The set of feasible states is a subset of

$$\mathscr{F} = \left\{ y \in R^{ln} \mid y_j \in \hat{Y}_j, \sum_j y_j + \sum_i \omega_i \geq 0 \right\} \text{ for each economy } \mathscr{E}(\alpha).$$

By (e3′), there exists $\bar{K} < \infty$ such that $\mathscr{F} \subset N(0, \bar{K})$. Also by (e3′), 0 is an exposed point of $\hat{Y}_j$ so there exists x_j such that $y'x_j > 0$ for all $y \in \hat{Y}_j \backslash \{0\}$. By decomposing each $y \in \hat{Y}_j \backslash \{0\}$ into a multiple of x_j and a vector orthogonal to x_j, noting that

$$\min \left(\frac{y'x_j}{\| y \|} \,\middle|\, y \in \hat{Y}_j \backslash \{0\} \right) > 0$$

and using $\mathscr{F} \subset N(0, \bar{K})$ we see that there exists $K < \infty$ such that for any countable set A, and $\{ y_a \mid a \in A \}$ where $y_a \in \bigcup_{j=1}^{n} \hat{Y}_j$ for each $a \in A$, $\sum_{a \in A} \| y_a \| > K$ implies that the corresponding economy output $y \in R^{ln}$ is not feasible.

$\mathscr{F}$ is bounded, so by desirability and the upper hemicontinuity of the Walras correspondence, the set of feasible prices is a subset of a closed cone contained in the interior of Ω, and, as in Lemma 1, there exists $M < \infty$ such that (+) supp $\mu(\alpha_k)(j, v) \subset N(0, M\alpha_k)$ for all $0 < v \leqslant \eta_j(\alpha_k)$, $0 < j \leqslant n$, $k \geqslant 1$ (0 dominates any action outside $N(0, M\alpha_k)$, and the mixed strategies are noncooperative).

We now consider the actions of operating firms as independent (for each given k) random vectors with the same underlying probability space (with expectation operator E). Let $\bar{X}^k_{jv}$ be the random vector in R^l corresponding to the j, v firm in $\mathscr{E}(\alpha_k)$, and let $X^k_{jv} = \bar{X}^k_{jv} - E\bar{X}^k_{jv}$. Since $y^{*\prime} = (y_1^{*\prime}, y_2^{*\prime}, \ldots, y_n^{*\prime})$ and

$$E \sum_{v=1}^{\eta_j(\alpha_k)} \bar{X}^k_{jv} = \sum_{v=1}^{\eta_j(\alpha_k)} E\bar{X}^k_{jv} \to y_j^*$$

as $k \to \infty$, in order to prove (3.1) it is sufficient to show that for all $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, l$, and $\delta > 0$,

$$\lim_{k \to \infty} \text{prob} \left\{ \left| \sum_{v=1}^{\eta_j(\alpha_k)} X^k_{jvi} \right| < \delta \right\} = 1,$$

where X^k_{jvi} is the ith component of X^k_{jv}.

By definition, X^k_{jv} induces the measure $\mu(\alpha_k)(j, v)$ on R^l, so by $(\cdot)$ prob $\{X^k_{jvi} \in [-M\alpha_k, M\alpha_k]\} = 1$ and therefore prob $\{|X^k_{jvi}| \leq 2M\alpha_k\} = 1$.

Fix $j \in \{1, 2, \ldots, n\}$, $i \in \{1, 2, \ldots, l\}$, and $\delta > 0$, and for notational convenience let X_v^k denote X^k_{jvi}. Split the sequence $(|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k|)_k$ into two parts, depending on k: $\eta_j(\alpha_k) < (\alpha_k)^{-1/2}$; and, $\eta_j(\alpha_k) \geq (\alpha_k)^{-1/2}$. For the first part, prob $\{|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k| \leq \eta_j(\alpha_k)\, 2M\alpha_k < 2M(\alpha_k)^{1/2}\} = 1$. For k sufficiently large, $2M(\alpha_k)^{1/2} < \delta$, and prob $\{|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k| < \delta\} = 1$.

For the second part of the sequence, by the Markov inequality [8, p. 158],

$$\text{prob} \left\{ \left| \sum_{v=1}^{\eta_j(\alpha_k)} X_v^k \right| \geq \delta \right\} \leq E \left| \sum_{v=1}^{\eta_j(\alpha_k)} X_v^k \right|^r \Big/ \delta^r$$

for any $r > 0$. Let $r = 2$. The X_v^k are independent (for fixed k) with zero means, so $E|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k|^2 = \sum_{v=1}^{\eta_j(\alpha_k)} E|X_v^k|^2$. Also prob $\{|X_v^k| \leq 2M\alpha_k\} = 1$ and prob $\{\sum_{v=1}^{\eta_j(\alpha_k)} |X_v^k| > 2K\} = 0$ (otherwise a state which is not feasible occurs with positive probability). Thus,

$$\sum_{v=1}^{\eta_j(\alpha_k)} E|X_v^k|^2 \leq \sup \left\{ \sum_{v=1}^{\infty} |c_v|^2 : \sum_{v=1}^{\infty} |c_v| \leq 2K, |c_v| \leq 2M\alpha_k \text{ for all } v \right\}$$

$$< \left(\frac{2K}{2\alpha_k M} + 1 \right)(2M\alpha_k)^2 = \alpha_k[4KM + \alpha_k 4M^2],$$

where the second inequality follows from the fact that the sum $\sum_{v=1}^{\infty} |c_v|^2$ is maximized at a boundary solution with $|c_v| = 2M\alpha_k$ for as many v as possible. Thus, prob $\{|\sum_{v=1}^{\eta_j(\alpha_k)} X_v^k| \geq \delta\} = O(\alpha_k) \to 0$ as $k \to \infty$. This completes the proof of (3.1).

Let $\bar{p}$ be defined by $F(y^*) = (1, \bar{p})$. $\xi(\alpha_k)$ is completely determined by $p(\alpha_k)$ and F, and (3.2) follows from (3.1) and the continuity of F at y^*.

Let x^* be the allocation in the exchange economy $\mathscr{E}(y^*) = (X_i, \omega_i + \sum_j \theta_{ij} y_j^*, \gtrsim_i)$ that is an equilibrium allocation at $(1, \bar{p})$ (x^* exists by definition of F and $\bar{p}$). F is continuous at y^*, so $(1, \bar{p})$ must be supporting prices for each cone $\hat{Y}_j$ (otherwise entry would occur at some α_k, destroying the Cournot equilibrium in mixed strategies). Thus, $((1, \bar{p}), x^*, y^*)$ is a Walras equilibrium of $\mathscr{E}$, which establishes (3.3).

To prove (3.4), suppose $y_j^{*\prime} B_j y_j^* > 0$. By (e2′), the boundedness of the set of feasible prices and (3.2), given $\delta > 0$ there exists $k_1 < \infty$ such that

$$\operatorname{supp} \mu(\alpha_k)(j, v) \subset N(\alpha y_j((1, p^*)), \alpha\delta) \cup \{0\} \text{ for all } k \geq k_1 .$$

The sequence is proper, so if t_k is the number of active firms in the Cournot equilibrium which use zero with positive probability, then $\alpha_k t_k \to 0$ as $k \to \infty$. Thus, for any $\epsilon > 0$, there is an $S < \infty$ such that for large k, with probability one, the aggregate action of active firms which use zero with positive probability is in $N(0, t_k \alpha_k S) \subset N(0, \epsilon/2)$. If T_{jk} is the number of active firms of type j in the Cournot equilibrium which do not use zero, then $\alpha_k T_{jk}$ is bounded (by feasibility) and, for any $\delta > 0$, for large k, the aggregate action of these firms is in $N(T_{jk} \alpha_k y_j(1, \bar{p}), T_{jk} \alpha_k \delta)$ with probability one. Combining these properties with (vi) of the theorem, we see that for any $\epsilon > 0$, for all sufficiently large k, $\rho(\alpha_k)\, N(y^*, \epsilon) = 1$.

For all k sufficiently large, some firm of type j must be active (by (3.1) and $y_j^* \neq 0$). For each k, let $x_k \neq 0$ be an action with positive probability for some active firm of type j, and consider an entrant using pure strategy x_k. The active firm only has actions in $N(\alpha y_j((1, p^*)), \alpha\delta) \cup \{0\}$. Conditional on the action of the active firm being 0, the entrant has nonnegative expected profit (just as the active firm does at action x_k prior to entry). Conditional on the active firm having action in $N(\alpha y_j((1, p^*)), \alpha\delta)$, the entrant's profit increases for all states with aggregate output near y^* ($y_j^{*\prime} B_j y_j^* > 0$). For any $\epsilon > 0$, for k sufficiently large, $\rho(\alpha_k)\, N(y^*, \epsilon) = 1$ so for ϵ, α_k sufficiently small, the increase in profit is greater than $\frac{1}{2} \alpha_k \gamma_j y_j^{*\prime} B_j (\alpha_k \gamma_j y_j^*)$. For k sufficiently large (α_k sufficiently small), the net change in expected profit has the same sign as $y_j^{*\prime} B_j y_j^*$, and entry will occur, upsetting equilibrium. This completes the proof of (3.4).

VI. REMARKS

A. The analysis we have presented differs substantially from standard formal treatments of competitive theory: Nonconvexity is essential, market power is endogenous, downward sloping demand is a requirement for static equilibrium, there is free entry, etc. Yet our complete dependence on the results of modern competitive theory (as surveyed in Debreu [4]) should be apparent to the reader. To underscore this dependence, we list some of the ways in which the paper relates (even ties together) recent developments in competitive theory.

Theorem 1 on the existence of Cournot equilibrium with entry requires the existence of Walras equilibrium in $\hat{\mathscr{E}}$. As such the result rests on the existence of Walras equilibrium. The Pareto Optimality of Walras equilibrium translates to the approximate Pareto Optimality of Cournot Equilibrium when efficient scale is small (Theorem 3). Theorem 2, together with the result that Pareto Optima are Walras equilibria in $\hat{\mathscr{E}}$, translates to the theorem that provided efficient scale is small, any Pareto optimum which satisfies DSD is approximately a Cournot equilibrium (relative to the appropriate specification of private ownership). Also, the replication of consumers, the idea of a limit economy, convergence results, etc., borrow from a pattern well developed in the literature on limits of cores. (A natural way to obtain decreasing returns in $\hat{\mathscr{E}}$ is to exploit a measure theoretic specification of the kind used in that literature.) Theorem 2 contains a result on the "rate of convergence." The entire framework is equilibrium with endogenous uncertainty. A "differentiable" framework is for a great many reasons natural for the analysis; in addition, the possibility of a stability theory in $\hat{\mathscr{E}}$ based on the entry and exit of firms is an attractive possibility. Product differentiation with an infinity of conceivable commodities would require extensions to economies with infinite dimensional commodity spaces.

Finally, the notion of a regular economy is intimately related to the existence of a selection from the Walras correspondence, DSD is a requirement imposed on the Walras correspondence, and existence can only be guaranteed as a generic property of a class of economies.

B. The analysis could be enlarged to admit the case where industry production sets in $\hat{\mathscr{E}}$ exhibit (strictly) diminishing returns to scale. This could be modeled in one of two natural ways: diseconomies (external to the firm and internal to the industry); or a measure space of available firms in $\hat{\mathscr{E}}$ with differing efficiencies. Then, it would be possible for a firm to have a large positive profit in equilibrium (rents).

C. The limit economy $\hat{\mathscr{E}}$ is studied in greater detail in a sequel to this paper, titled "Walras Equilibrium as an Idealization of Small Efficient Scale and Free Entry." Since production in $\hat{\mathscr{E}}$ takes place in a continuum of firms, each of which reaches efficient scale at an infinitesimal level of output, the convexity of the aggregate production set $\hat{Y}$ is guaranteed by Richter's Theorem on the integral of correspondences. For any price vector p, firms may be active if they can make a nonnegative profit (and must be active if they can make a positive profit). The sum (integral) of their actions is both on the boundary of $\hat{Y}$ and is supported by p; furthermore, since some firms produce more efficiently than others, there are Ricardian rents. (Such rents disappear in the present model since all firms have access to the same technology.) Both the distribution of income among consumers as well as the amount of income which goes to each consumer depend on the ownership structure of the economy.

The existence of Walras equilibrium in $\hat{\epsilon}$ follows from two arguments: (1) the standard equilibrium existence theorem (Debreu, [3]), with some minor modification, shows that there is an equilibrium with $\hat{Y}$ as the production set. and (2) profit maximization on $\hat{Y}$ is equivalent to profit maximization on the (infinite collection of) sets which are added to form $\hat{Y}$. The classical results of welfare economics apply for the present model. They follow from (2), via the standard welfare theorems (Debreu, [3]); however, the "Second Theorem" (optimal are equilibria) does not require convexity as a basic hypothesis. That assumption is replaced by the hypothesis that firms reach efficient scale at an infinitesimal level of output.

D. The assumption that consumers exhibit price-taking behavior is not necessary; what is required is that every set of quantity actions by firms yield a well defined vector of prices. For suitably smooth "F," results analogous to Theorem 1 hold. For example, suppose that in each economy $\mathscr{E}(\alpha)$, the quantity actions of firms result in an allocation in the core of the associated pure exchange economy, and let price be the implied efficiency prices. Since the cores of the exchange economies converge to the Walras equilibria of the exchange economy associated with $\hat{\mathscr{E}}$, the modified (non-price-taking) Cournot equilibria of $\mathscr{E}(\alpha)$ converge to the Walras equilibria of $\hat{\mathscr{E}}$. In connection with this point, it is appropriate to underscore the asymmetry between our treatment of consumers and producers. The present analysis is designed to explain perfectly competitive *producer* behavior and has no role in the explanation of price taking consumer behavior. The key ingredients in our explanation of perfectly competitive producer behavior are small (per capita) efficient scale and the free entry of firms. Whatever the explanation of competitive consumer behavior, it is not free entry and small efficient scale! Firms and consumers are created and exist for different purposes; they perform a different function in the allocation of resources. In contrast with standard models of Walras equilibrium, we highlight the creation of firms; this dictates the asymmetric treatment afforded consumers and producers.

E. Since uncertainty disappears for the equilibria exhibited in Theorem 1 as α converges to zero, the results obtained are easily extended to firms that maximize a (smooth) expected utility of income function. This admits risk aversion into the analysis.

F. As long as there are only a finite number of conceivable commodities, our analysis includes the classical case of product differentiation. For any fixed value of α, there may be many commodities which are not produced. As

α converges to zero, the number of different commodities actually produced (from among the large but finite number of available commodities) may increase. For situations in which there are an infinite number of available commodities, it is necessary to consider economies $\mathscr{E}$ with infinite dimensional commodity spaces (see, e.g., Hart [6] and Mas-Colell [9]).

G. With only a countable number of firms, the introduction of mixed strategies does not remove all discontinuities. The only way to remove the discontinuity associated with entry is to have a continuum of firms. Here is an outline of a simple example in which equilibrium with entry does not exist in mixed strategies for all $\alpha > \tilde{\alpha} > 0$ (i.e., introduction of mixed strategies does not imply that equilibrium with entry exists for all α). Consider the partial equilibrium setting of Fig. 1 with linear inverse demand F. Assume constant average cost $\bar{p}$ (which is greater than the competitive price $F(y^*)$) up to some positive output level. For $\tilde{\alpha}$ sufficiently large, only the constant average cost portion of the cost curve is relevant. If equiilibrium were to exist with a finite number of firms, then expected price must exceed $\bar{p}$ in order to insure that active firms are maximizing profit. But then entry is profitable. This can easily be turned into a general equilibrium example.

More importantly, arbitrary mixed strategies are not consistent with observed behavior. Thus, the fact that only pure and marginal firm strategies are profit maximizing in the equilibria of Theorem 1, even though arbitrary mixed strategies are allowed, is significant.

H. The introduction of marginal firms is essential. Restrictive assumptions are necessary to guarantee the existence of equilibrium in pure strategies for all α sufficiently small. We have substantially explored such assumptions, and existence results will be reported elsewhere.

I. A natural test of the robustness of the equilibria considered here is given by the ability of a single firm with perfect information and perfect knowledge of the behavior of other agents to increase profit by departing from the prescribed behavior. We observe that the maximum profit such a firm could obtain is less than αK, where K is a constant. Thus, the benefit which accrues to strategies which are more sophisticated than those considered here (e.g., the strategy of a firm acting as a Stackelberg leader, knowing the responses of the rest of the economy) will eventually be swamped by the costs of obtaining information (which do not depend on α). Similarly, from the point of view of cooperative action, any cartel must be sensitive to the possibility of entry if it restricts output to increase profit. Threats aside, cartels must "limit price" to prevent entry, and so the profit which accrues to a cartel is less than αL, where L is a constant. (This is discussed by Novshek [10].)[18] For this reason, we believe that the results reported here have natural extensions beyond the case of quantity choosing Cournot equilibrium.

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[18] Since entrants can produce as efficiently as cartel members, each cartel member faces at least the same loss as an entrant when threats are carried out.

J. Firms that recognize their effect on price do not in general maximize a weighted sum of shareholders' utilities by profit maximization. Consider a sequence of Cournot equilibria, one for each economy $\mathscr{E}(1/k)$, $k = \bar{k}, \bar{k}+1, \bar{k}+2, \ldots$, as constructed in the proof of Theorem 1, where the $\mathscr{E}(1/k)$ are viewed as "per capita" economies for k replications of each consumer type. Let $v_i(p, w)$ be an indirect utility function for consumers of type i. For large k, the equilibrium prices and wealth (p^k, w_i^k) are approximately equal to the competitive prices and wealth $(p^*, w_i^*) = (p^*, p^{*\prime}\omega_i)$. For any single firm of type j for which the tth consumer of type i has ownership share θ^k_{itj} in $\mathscr{E}(1/k)$,

$$\frac{\partial v_i}{\partial y_j'} \approx \frac{1}{k}\frac{\partial v_i(p^*, w_i^*)}{\partial p'} B_j + \theta^k_{itj}\frac{\partial v_i(p^*, w_i^*)}{\partial w} p^{*\prime}.$$

For $\theta^k_{itj} = (1/k)\,\theta_{ij}$ both terms are of the same order of magnitude so either may dominate. Whenever $\theta^k_{itj} \to 0$ as $k \to \infty$, if the firm's actions are constrained to lie in a bounded set, then the effect of a single firm on any single consumer converges to zero.

If for some itj, θ^k_{itj} is bounded away from zero, then for k sufficiently large,

$$\frac{\partial v_i}{\partial y_j'} \approx \theta^k_{itj}\frac{\partial v_i(p^*, w_i^*)}{\partial w} p^{*\prime},$$

and the utility maximizing solution is approximately the competitive solution which is approximately the profit maximizing solution (as seen in the proof of Theorem 1). In either case, any cost (to the firm) of learning the preferences of a consumer/owner eventually outweighs the gain from knowledge of an exact "welfare function" to be maximized by the firm.[19] The second case is related to Hart [7], which deals with a stock market model.

K. Implicit in the manner in which firms determine how much they will have to pay for an input is the assumption that in equilibria of $\mathscr{E}$ the consumer sector consumes some of each commodity. Thus, there are neither pure inputs nor pure intermediate products. The producer sector can always obtain, from the price taking consumer sector, an additional amount of a commodity to use as an input by paying the appropriate price. This could be softened by introducing a sector of price taking firms, but such a device would be contrary to the spirit of the analysis.

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[19] We have assumed $\theta^k_{itj} = (1/k)\theta_{ij}$, so it is worthwhile for consumer owners to have whole industries act collusively, in which case equilibrium (if it exists) is not in general approximately competitive. This simplifying assumption can be eliminated, but the cost is increased notational complexity. The θ_{itj} must be specified for all i/t, and prices will depend on which firms of type j are active. However, near a Walras equilibrium of $\hat{\mathscr{E}}$, this will be only a low order effect, so under suitable conditions the proof of Theorem 1 can be modified to cover this generalization.

L. When α is small, changes in demand lead to new equilibria in which both firms' actions and the number of active firms have changed. As $\alpha \to 0$, the change in the number of firms is dominant. This suggests the possibility of a "real time" dynamics in which the adjustment of price is determined by the rate at which firms enter. In addition, since firms *recognize their effect on* price, there is no contradiction in having them change the amount which they offer to the market (or even enter!) when preferences, technology, or other data of the economy change. It is only in the limit that a tension arises between price taking behavior and the need for prices to adjust in order to clear markets (see Arrow [1]).

M. Under what conditions does the Arrow Debreu model apply? If one views industry constant returns to scale as being an idealization of small per capita efficient scale and free entry, as done here, the Arrow–Debreu model with constant returns to scale is applicable, but only if downward sloping demand is satisfied. If one views constant returns to scale for firms as empirically correct, then the Arrow–Debreu model captures formally the notion of perfect competition only in so far as we acknowledge that the technology is freely available to all users. In this case, our argument for downward sloping demand as a requirement for static equilibrium applies again, and perfectly competitive equilibria which do not satisfy the condition are simply the result of actions which are irrational (and in no approximate sense rational). If we understand the Arrow Debreu theory as appropriate for conditions of decreasing returns to scale for firms and no free entry, there is the "embarrassment" of possibly significant profit and possibly significant market power to explain. Free entry is not available to make these go away (and if it was available the model would collapse). Finally, the Arrow–Debreu model is not applicable under conditions of increasing returns to scale, since equilibrium will generally not exist.

VII. CONCLUSION

Theorem 1 proves the existence of Cournot equilibrium with entry. Theorems 2 and 3 unify the equilibrium concepts of Walras and Cournot by establishing that Cournot equilibria are approximated by certain Walras equilibria.

The Arrow–Debreu model is viewed as representing a "frictionless system," whose "frictions" are barriers to entry and noninfinitesimal efficient scale. When efficient scale is small (but significant) and entry is free, certain Walras equilibria serve as good approximations to Cournot equilibria with free entry. However, when there are barriers to entry, or if efficient scale is not attained at a level which is per capita small, the "frictionless" idealization is no longer appropriate. In the "frictionless system," demand determines the measure of firms which are active in each industry. Zero profit becomes a consequence of free entry, and firms take competitive prices as given because prices really are beyond their control. However, the analysis shows that the viewpoint of the Arrow–Debreu model as representing a frictionless system is only proper under conditions of downward sloping demand and when the efficient scale of firms is bounded away from zero. (Thus two cornerstones of the classical partial equilibrium diagram are introduced into formal general equilibrium analysis and play a central role.)

We present a simple proof that the perspective we have offered does "make a difference." Recall the standard line: Consider a perfectly competitive economy with one firm (two or three firms) and constant returns to scale. From the present viewpoint, it makes no sense: The number of firms is not endogenous, and perfectly competitive behavior is not a primitive solution concept; it applies if conditions are right. Finally, competitive behavior is unlikely to obtain with only one firm (two or three firms) active in equilibrium. In contrast, the formalism of the present analysis forces you to say this: Consider an economy with one industry (two or three industries), free entry, small efficient scale, and downward sloping demand. If efficient scale is small, then the production sets of each industry exhibit approximately constant returns to scale. Small efficient scale with entry dictates the result that firms have almost no effect on price when attention is restricted to the region in which they make positive profit. The addition of a demand sector determines the number of active firms in each industry. In each industry in which there is positive output, there are a large number of firms active, each of which is producing a small amount (at approximately efficient scale).

APPENDIX I

Partial Equilibrium (Novshek, [10])

Assumptions

For the cost function $C(y)$,

$$\text{(C)}\quad C(y) = \begin{cases} 0 & y = 0, \\ C_0 + V(y) & y > 0, \text{ where } C_0 > 0 \text{ and for all } y \geq 0, \end{cases}$$

$v \geq 0$, $v' > 0$, $v'' \geq 0$. Average cost is minimized uniquely at $y = 1$.

For the inverse demand function $F(Y)$,

(F) $F \in C^2([0, \infty))$ with $F' < 0$ whenever $F > 0$,

and there exists $Y^* > 0$ such that $F(Y^*) = C(1)$ (equals minimum average cost). Y^* is the competitive output.

Definitions

(II, 1) An *α size firm corresponding to C* is a firm with cost function $C_\alpha(y) = \alpha C(y/\alpha)$. For each α, C, and F, one considers a pool of available firms, each with cost function C_α, facing market inverse demand F.

(II, 2) Given C, F, and an $\alpha \in (0, \infty)$, an *(α, C, F) market equilibrium with free entry* is an integer n and a set $\{y_1, \ldots, y_n\}$ of positive outputs such that:

(a) $\{y_1, \ldots, y_n\}$ is an n firm Cournot equilibrium (without entry), i.e., for all $i = 1, \ldots, n$,

$$F\left(\sum_{j \neq i} y_j + y_i\right) y_i - C_\alpha(y_i) \geq F\left(\sum_{j \neq i} y_j + y\right) y - C_\alpha(y)$$

for all $y \geq 0$, and

(b) there is no profit incentive for additional entry, i.e.,

$$F\left(\sum_{j=1}^{n} y_j + y\right) y - C_\alpha(y) < 0 \text{ for all } y > 0.$$

The set of all (α, C, F) market equilibria with free entry is denoted $E(\alpha, c, F)$.

It is easily demonstrated that the Nash–Cournot equilibria exhibited converge to the competitive equilibrium Y^*; i.e., given C satisfying (C), F satisfying (F) and $\alpha \in (0, \infty)$, if n, $\{y_1, y_2, \ldots, y_n\} \in E(\alpha, C, F)$, then $\sum_{j=1}^{n} y_j \in [Y^* - \alpha, Y^*]$. This observation is made whole by the following theorem.

Theorem. *Given C satisfying* (C) *and F satisfying* (F), *there exists $\alpha^* > 0$ such that for all $\alpha \in (0, \alpha^*]$, $E(\alpha, C, F) = \emptyset$.*

Appendix II

In order to show the genericity of (j1), (j2), and (j3), we use as basic G_j, y_j^*, $E(y^*)$, B_j, and θ_{ij} subject to

(a) G_j is negative semidefinite and $x'G_jx < 0$ for $x \in P_j(R^l)\backslash\{0\}$;

(b) $y_j^{*\prime}F(y^*) = 0$ for all j;

(c) $F_1(y^*) = 1$;

(d) $y_j^{\dagger\prime} B_j y_j^{\dagger} < 0$ for all j;

(e) $(1, 0, 0, \ldots, 0)\, B_j = (0, 0, \ldots, 0)$ for all j; and

(f) $\sum_i \theta_{ij} = 1$ for all j.

The ownership shares, θ_{ij}, and the shape of production sets at competitive outputs, G_j, are clearly basic. The $F(y^*)$, y^*, and $\partial F/\partial \omega'$ are considered basic in the spirit of the result that given any prices, p, output, y, and Jacobian $[\partial y/\partial p]$, a set of consumers exist whose aggregate demand is y at prices p, and has Jacobian $[\partial y/\partial p]$ at (y, p) (Sonnenschein [15]). The B_j are used as basic rather than the $\partial F/\partial \omega$, since

$$[B_1 \cdots B_n]_{l\times ln} = \left[\frac{\partial F}{\partial \omega'}\right]_{l\times lm} \begin{bmatrix} \theta_{11} I_{l\times l} & \cdots & \theta_{1n} I_{l\times l} \\ \vdots & & \vdots \\ \theta_{m1} I_{l\times l} & \cdots & \theta_{mn} I_{l\times l} \end{bmatrix}_{lm\times ln}$$

so if $m \geq n$ and

$$\begin{bmatrix} \theta_{11} & \cdots & \theta_{1n} \\ \vdots & & \vdots \\ \theta_{m1} & \cdots & \theta_{mn} \end{bmatrix}_{m\times n}$$

has rank n (which is a generic property if $m \geq n$ since the θ_{ij} are basic) then if any $\partial F/\partial \omega'$ (with $\partial F_1/\partial \omega' = 0'$) is possible, so is any $[B_1 \cdots B_n]$ (with $(1\ 0 \cdots 0)_{1\times l}\, [B_1 \cdots B_n]_{l\times ln} = (0 \cdots 0)_{1\times ln}$).

When consumers are specified in terms of demand functions rather than preferences, the genericity arguments can be modified to remain valid under suitable conditions. At the end of this appendix, we briefly sketch the modifications.

We only consider economies which satisfy the required differentiability assumptions, and (a)–(f). Each economy $\mathscr{E}$ in this space of economies can be represented by a vector $((G_j)_j, (y_j^{\dagger})_j, (\theta_{ij})_{i,j}, (B_j)_j, F(y^*))$, where the matrices G_j are written as vectors (row 1, row 2,..., row l). The distance between any two economies is defined to be 1 if the number of consumers, commodities, or industries is different, or if for any industry j, the set of smoothly variable commodities is different. Otherwise, the distance is defined to be the usual distance in $R^{nl^2+nl+mn+nl^2+l}$. We show that the nondegeneracy conditions hold in an open dense subset of this metric space, and are therefore generic properties. Since eigenvalues and determinants are continuous functions of the entries of the corresponding matrices, and (j1)–(j3) are open conditions, they hold in a neighborhood of an economy at which they hold. It only remains to show that (j1)–(j3) hold in a dense subset of the space.

(j1) Using the form of $\begin{bmatrix} 0 & -\bar{p}' \\ 0 & I \end{bmatrix} G_j$ (the first row is a linear combination of the other rows, and row i is zero if i is not a smoothly variable component), (j1) can be transformed into an equivalent form (eliminating all components which are identically zero in $P_1(R^l) \times P_2(R^l) \times \cdots \times P_n(R^l)$), determinant $\{D + \tilde{G}^{-1}\} \neq 0$, where

$$\tilde{G} = \begin{bmatrix} \tilde{G}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{G}_n \end{bmatrix}$$

with $\tilde{G}_j$ the negative definitive submatrix of G_j corresponding to $P_j(R^l)$ (if $G_j = [0]$ then $\tilde{G}_j$ is skipped) and where D is independent of $\tilde{G}$. Since $\tilde{G}^{-1}$ has full rank, all entries of $\tilde{G}^{-1}$ can be varied, so if (j1) fails at $\mathscr{E}$, it is satisfied at a sequence of economies $\mathscr{E}_\nu \to \mathscr{E}$.

(j2) By (j1), the existence of $\{I + H_1\}^{-1}$ is generic. Let λ be the smallest strictly positive eigenvalue of $\{I + H_1\}^{-1}\{H_1 - H_2(1)\}$. If $H_2(m)x = -x$ then $\{I + H_1\}x = \{H_1 - H_2(m)\}x = m\{H_1 - H_2(1)\}x$ and $x = m\{I + H_1\}^{-1}\{H_1 - H_2(1)\}x$. When $m < (1/\lambda)$ ($m < 1$ if λ does not exist), then $x = m\{I + H_1\}^{-1}\{H_1 - H_2(1)\}x$ implies $x = 0$, so $H_2(m)$ does not have eigenvalue -1.

(j3) (a) It is generic that

$$N = \begin{bmatrix} y_1^{*\prime} & & 0 \\ & \ddots & \\ 0 & & y_n^{*\prime} \end{bmatrix}_{n \times ln} \begin{bmatrix} B_1 & \cdots & B_n \\ \vdots & & \vdots \\ B_1 & \cdots & B_n \end{bmatrix}_{ln \times ln} \begin{bmatrix} y_1^{*} & & 0 \\ & \ddots & \\ 0 & & y_n^{*} \end{bmatrix}_{ln \times n}$$

has rank n. To see this, let $B_j(x) = B_j + x\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}_{l \times l}$ for all j (I is $(l-1) \times (l-1)$), and let $N(x)$ be the matrix corresponding to N when B is replaced by $B(x)$. Then

$$N(x) = N + x\begin{bmatrix} z_1'z_1 & \cdots & z_1'z_n \\ \vdots & & \vdots \\ z_n'z_1 & \cdots & z_n'z_n \end{bmatrix}_{n \times n} .$$

where $z_j' = (y_{j2}^*, y_{j3}^* \cdots y_{jl}^*)$. The $z_1, z_2, \ldots, z_n \in R^{l-1}$ are generically independent since $n \leq l-1$ and the y_j^* are basic, so the matrix multiplied by x has rank n gerically. Expanding the determinant of $N(x)$, we find

$$\det N(x) = a_n + xa_{n-1} + x^2a_{n-2} + \cdots + x^{n-1}a_1 + x^na_0 .$$

Let s be the largest integer such that $a_s \neq 0$ (such an s exists since $a_0 = \det([z_i'z_j]_{ij}) \neq 0$). Then $1/x^{(n-s)} \det N(x) = a_s + xa_{s-1} + \cdots + x^sa_0 \to a_s \neq 0$ as $x \to 0$; For all x sufficiently small, $\det N(x) \neq 0$ so if N does not have rank n for economy $\mathscr{E}$, $N(x)$ does for a sequence of economies $\mathscr{E}_\nu \to \mathscr{E}$.

(b) It is generic that

$$M = \begin{bmatrix} y_1^{*\prime} \\ \vdots \\ y_n^{*\prime} \end{bmatrix} [B_1 \cdots B_n]\{I + H_1\}^{-1} \begin{bmatrix} y_1^* & & 0 \\ & \ddots & \\ 0 & & y_n^* \end{bmatrix}$$

$$= \begin{bmatrix} y_1^{*\prime} & & 0 \\ & \ddots & \\ 0 & & y_n^{*\prime} \end{bmatrix} \begin{bmatrix} B_1 & \cdots & B_n \\ \vdots & & \vdots \\ B_1 & \cdots & B_n \end{bmatrix} \{I + H_1\}^{-1} \begin{bmatrix} y_1^* & & 0 \\ & \ddots & \\ 0 & & y_n^* \end{bmatrix}$$

has rank n. To see this, let $B(x) = B + xB\{I - xH_1H_1\}^{-1} H_1\{I - H_1\}$, where

$$B = \begin{bmatrix} B_1 & \cdots & B_n \\ \vdots & & \vdots \\ B_1 & \cdots & B_n \end{bmatrix}_{ln \times ln}.$$

(the inverse exists for all small x), and let $M(x)$ be the matrix corresponding to M when B is replaced by $B(x)$ (note that B occurs in H_1, which becomes $H_1(x)$). Computation yields $B(x)\{I + H_1(x)\}^{-1} = B\{I + H_1\}^{-1}[(1 - x)I + x\{I + H_1\}] = (1 - x)B\{I + H_1\}^{-1} + xB$. Thus $M(x) = (1 - x)M + xN$ where N (the matrix from (a)) generically has rank n, and, proceeding as in (a), we see $M(x)$ has rank n for all small nonzero x. If (j3) fails for economy $\mathscr{E}$, it is satisfied for a sequence of economies $\mathscr{E}_x \to \mathscr{E}$.

We now briefly outline how, under suitable conditions, these genericity arguments can be carried out in terms of individual demand functions. Clearly only (j3) must be modified. Let the number of consumers, m, be greater than or equal to $n + l$, let $p' = (p_2, p_3, \ldots, p_l)$ $(p_1 = 1)$, and let $h_i(p, w_i)$ be the ith consumers demand, as a function of prices and wealth. Results similar to (j3) (a) and (b) can be obtained by altering the derivatives of the demand functions, evaluated at the equilibrium, prices and wealths. For any l-vector $r_i = (r_{i1}, r_{i2}, \ldots, r_{il})$, let $r^i = (r_{i2}, r_{i3}, \ldots, r_{il})'$.

First show that

$$B_j = \sum_{k=1}^{m} \theta_{kj} \begin{bmatrix} 0 \cdots 0 \\ \left[\dfrac{\partial p}{\partial \omega_k}\right] \end{bmatrix}$$

where

$$\left[\frac{\partial p}{\partial \omega_k}\right]_{(l-1)\times l} = \left\{ \sum_{i=1}^{m} \left\{\left[\frac{\partial h^i}{\partial p}\right] + \left(\frac{\partial h^i}{\partial w_i}\right)(\bar{\omega}^i)'\right\} \right\}^{-1} \left\{ \begin{bmatrix} 0 & \\ \vdots & I \\ 0 & \end{bmatrix} - \left(\frac{\partial h^k}{\partial w_k}\right)(1, p') \right\}$$

and $\bar{\omega}_i = \omega_i + \sum_{j=1}^{n} \theta_{ij} y_j^*$ (for convenience call the matrix which is inverted, M).

Second, show that with suitable assumptions about the possible values of initial endowments, it is generic that $[\theta\mathscr{W}]_{m\times(n+l-1)}$ has rank $n+l-1$, where $\theta = [\theta_{ij}]_{m\times n}$ and

$$\mathscr{W} = \begin{bmatrix} (\bar{\omega}^1 - h^1)' \\ (\bar{\omega}^2 - h^2)' \\ \vdots \\ (\bar{\omega}^m - h^m)' \end{bmatrix}_{m\times(l-1)}.$$

As in (j3), for each x there will be a different economy. Here we define new demand functions $h_i(x)$ for each x. In order to maintain the required properties of the derivatives of demand functions, we set

$$\left[\frac{\partial h^i}{\partial p}(x)\right] = \left[\frac{\partial h^i}{\partial p}\right] + \left\{\left(\frac{\partial h^i}{\partial w_i}\right) - \left(\frac{\partial h^i}{\partial w_i}(x)\right)\right\} h^{i\prime}$$

and vary

$$\left(\frac{\partial h^i}{\partial w_i}(x)\right)$$

(all derivates are evaluated at equilibrium prices and wealth, which do not change as x changes).

To obtain the same final result as (j3)(a), it is sufficient to find $((\partial h^i/\partial w_i)(x))$, $i = 1, 2,\ldots, m$, such that

$$\left[\left(\left(\frac{\partial h^1}{\partial w_1}(x)\right) - \left(\frac{\partial h^1}{\partial w_1}\right)\right) \cdots \left(\left(\frac{\partial h^m}{\partial w_m}(x)\right) - \left(\frac{\partial h^m}{\partial w_m}\right)\right)\right]\mathscr{W} = M(x) - M = O(x),$$

where $(M(x))^{-1} = M^{-1} + xI$. Since $\mathscr{W}$ generically has rank $l-1$ we can find $((\partial h^i/\partial w_i)(x)) \rightarrow (\partial h^i/\partial w_k)$ which satisfy the equation.

To obtain the same final result as (j3)(b), it is sufficient to find $((\partial h^i/\partial w_i)(x))$, $i = 1, 2,\ldots, m$, such that

$$\left[\left(\left(\frac{\partial h^1}{\partial w_1}(x)\right) - \left(\frac{\partial h^1}{\partial w_1}\right)\right) \cdots \left(\left(\frac{\partial h^m}{\partial w_m}(x)\right) - \left(\frac{\partial h^m}{\partial w_m}\right)\right)\right][\theta\mathscr{W}]$$
$$= [[0]_{(l-1)\times n}\, S(x)_{(l-1)\times(l-1)}],$$

where $S(x)$ is a $O(x)$ matrix determined by the choice of $B(x)$ in (j3) (b). Since $[\theta\mathscr{W}]$ generically has rank $n+l-1$, we can find $((\partial h^i/\partial w_i)(x)) \rightarrow (\partial h^i/\partial w_i)$ which satisfy the equation.

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REFERENCES

1. K. ARROW, Toward a theory of price adjustment *in* "The Allocation of Economic Resources: Essays in Honor of Bernard Francis Haley" (M. Abramovitz *et al.*), Stanford Univ. Press, Stanford, Calif. 1959.
2. Y. BALASKO, Some results on uniqueness and on stability of equilibrium in general equilibrium theory, *J. Math. Econ.* **2** (1975), 95–118.
3. G. DEBREU, "Theory of Value," Wiley, New York, 1959.
4. G. DEBREU, Four aspects of the mathematical theory of economic equilibrium, *Proceedings of the International Congress of Mathematicians*, Vancouver, 1974.
5. J. J. GABSZEWICZ AND J-P. VIAL, Oligopoly "à la Cournot" in general equilibrium analysis, *J. Econ. Theory* **4** (1972), 381–400.
6. O. HART, Monopolistic competition in a large economy with differentiated commodities, *Rev. Econ. Stud.* in press.
7. O. HART, On shareholder unanimity in large stock market economies, *Econometrica*, in press.
8. M. LOEVE, "Probability Theory," 2nd ed. Van Nostrand, Princeton, N.J., 1960.
9. A. MAS-COLELL, A model of equilibrium with differentiated commodities, *J. Math. Econ.* **2** (1975), 263–295.
10. W. NOVSHEK, Cournot equilibrium with free entry, *Rev. Econ. Stud.*, in press.
11. J. T. RADER, Edgeworth exchange and general economic equilibrium, *Yale Econ. Essays* **4** (1964), 133–180.
12. J. ROBERTS AND H. SONNENSCHEIN, On the foundations of the theory of monopolistic competition, *Econometrica* **45** (1977), 101–113.
13. R. J. RUFFIN, Cournot oligopoly and competitive behavior, *Rev. Econ. Stud.* **38** (1971), 493–502.
14. P. SAMUELSON, The monopolistic competition revolution, *in* "Monopolistic Competition Theory: Studies in Impact; Essays in Honor of Edward H. Chamberlin", (R. E. Kuenne, Ed.), Wiley, New York, 1968.
15. H. SONNENSCHEIN, The utility hypothesis and market demand theory, *West. Econ. J.* **11** (1973), 404–410.
16. M. SPENCE, Product selection, fixed costs, and monopolistic competition, *Rev. Econ. Stud.* **43** (1976), 217–235.
17. F. A. VALENTINE, "Convex Sets," McGraw-Hill, New York, 1964.

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Temporary Equilibrium in a Money Economy*

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1. INTRODUCTION

It had been customary, when concerned with the existence of equilibrium, to restrict efforts to economies without money. Hahn [6] in his paper "On some Problems of Proving the Existence of an Equilibrium in a Monetary Economy" made a first attempt to introduce monetary elements into an Arrow-Debreu model and since then many papers have been written on that theme (Hahn [7], Starrett [14], Kurz [10, 11], Hayashi [8], Heller [9], to mention only some of them).

While Hahn and Starrett point out that there exists an incentive to introduce money into an economy in order to prevent inefficient exchange by use of future markets which may use up more resources as transaction costs than spot markets, Kurz is mainly interested in a situation including several markets differing with respect again to efficiency as well as institutional arrangements such as barter and monetary exchange. Though Kurz mentions the role of money as medium of exchange, his framework is such that only the difference in value of purchases over sales has to be paid for in money. If, by coincidence, this difference is zero, no money has to be used at all as means of payment notwithstanding the fact that the volume of trade may be positive. In this sense, all the above contributions emphasize the *store of value* aspect of money, neglecting its role as a *medium of exchange*.

Clower [1] holds a different view: "The natural point of departure for a

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theory of monetary phenomena is a precise distinction between money and nonmoney commodities. In this connexion, it is important to observe that such a distinction is possible only if we assign a special role to certain commodities as means of payment. For any commodity may serve as a unit of account and standard of deferred payment; and every asset is, by its very nature, a potential store of value. If money is to be distinguished by the functions it performs, therefore, it is to the medium of exchange function that we must address our attention."

How can this be accomplished? In a money economy, a bundle of commodities might be exchanged for a certain amount of money and vice versa. But typically, as Clower points out, there is no way to exchange nonmoney commodities directly for each other: "Money buys goods and goods buy money; but goods do not buy goods." Exactly in that sense a money economy contrasts most sharply with a world of barter. Thus for a consumer the market m presents itself as divided into two sectors s and t. At sector s he sells a nonmoney commodity bundle S out of his initial endowments in exchange for the amount $m(s)$ of money and at sector t he buys a nonmoney commodity bundle D using $m(t)$ units of money as means of payment. If $p(s)$ and $q(t)$ denote buying prices at sector s and selling prices at sector t, respectively, then the consumer faces a monetary constraint at each sector which has the following form:

$$p(s)\,S = m(s) \text{ and } q(t)\,D = m(t). \tag{1}$$

If money income earned at sector s could be used immediately at sector t to buy commodities then the budget equation might be written simply as

$$M_0 + m(s) = m(t) + M \tag{2}$$

where M_0 and M denote initial money holdings and balances to be carried over to the next period, respectively. Furthermore, (1) and (2) could be reduced to the single equation

$$M_0 - M = q(t)\,D - p(s)\,S.$$

Therefore a state of equilibrium where $M_0 = M$ and our consumer needs no money at all to carry out his transactions cannot be ruled out, thus contrasting with what we observe in a real monetary world.

Clower objects strongly against such an outcome. As a remedy, he proposed to assume that money income $m(s)$ earned at sector s can be spent on purchases at sector t only in the next period. This time lag caused by the way transactions are handled at market m makes these "intraperiod" receipts of money $m(s)$ correspond to what is sometimes referred to as demand for transaction balances. Formally, Clower's requirement amounts to adding another monetary constraint expressed as

$$M_0 \geq m(t). \tag{3}$$

All purchases in the present period have to be covered by money balances carried over from earlier periods. To use the same terminology as in Clower [1], our $m(s)$ corresponds to Clower's *income demand for money*, our $M - m(s)$ to his *reservation demand for money*, our (2) to his *expenditure constraint*, our equation $p(s) S = m(s)$ to his *income constraint*, and our (3) to his requirement that reservation demand for money has to be nonnegative. If there is just one market m composed out of two sectors s and t in the above way and if money income earned at sector s cannot be spent during the present period, then Clower speaks of a *pure money economy*. Grandmont and Younès [5] proved the existence of temporary equilibrium in such a pure money economy without coping, however, with Clower's exact specification and not allowing for transaction costs.

Our description of a Clower market as consisting of two sectors with corresponding monetary constraints (1) may look rather clumsy in the case of a pure money economy but it allows for a convenient way of defining the set of choice alternatives in a general non-pure money economy.

Modern economic society can hardly be described as a pure money economy for at least two reasons.

(A) Typically, there are several items which serve as medium of exchange (domestic and foreign currency, checks, credit cards, etc.) and also there exist several markets. Not every medium of exchange is accepted as means of payment at every market and no *money commodity* exchanging directly for any other commodity might exist. Furthermore, a *barter subset* should be part of the system to be described as an additional market m_b where commodities and media of exchange belonging to a prespecified list are traded directly for each other. This barter subset could allow for an additional though possibly inefficient way of exchanging commodities through barter (cf. Kurz [11]). It also might describe an overall market for money commodities (banking system).

(B) As argued above, the notion of an overall budget equation has to be rejected in a money economy. Nevertheless, some transfer of value from one market to another might be allowed for, especially intraperiod receipts of money earned at some market could be used as security on credit to be obtained at the banking sector m_b. We call *marketing* any activity of transferring value by means of commodities between markets. Constraints on marketing activities are to enter essentially in the model of a money economy as a crucial element of specification.

It is the purpose of this paper to collect, taking (A) and (B) into account, the elements for a model of a *money economy* and to find sufficient conditions ensuring the existence of a temporary equilibrium. Surprisingly enough, the familiar methods of proving existence have to be modified considerably in order to handle a money economy thus suggesting that money and barter models differ analytically in a substantial way.

Finally, it might be worth noting that expectations on future prices influence the real value of money stocks, especially lacking confidence in one type of money may put downward pressure on its value and, as a consequence, costlier transactions using different media of exchange may be forced to take place thus wasting resources. This type of inefficiency might be observed rather frequently in money economies.

The present paper is a simplified and we hope, a more readable version of our earlier paper (cf. Müller and Schweizer [12]).

2. ELEMENTS FOR A MODEL OF A MONEY ECONOMY

Following Clower, we distinguish the set C of nonmoney commodities from the set E of money commodities (media of exchange).

For any money commodity there exists a corresponding intraperiod medium of exchange. The operator * associates to every set of money commodities $D \subseteq E$ the set of corresponding intraperiod media of exchange D^*. $N = C \cup E \cup E^*$ denotes the set of all commodities and at the same time the number of elements in it. No ambiguity should arise.

There is a finite set M of markets $m = (s, t)$ each of which contains two sectors s and t. At the first sector s of m any nonmoney commodity belonging to a prespecified list $K(s) \subseteq C$ can be exchanged for money commodities belonging to another prespecified list $L(s) \subseteq E^*$. At the second sector t money commodities of $K(t) \subseteq E$ are exchanged for nonmoney commodities belonging to $L(t) \subseteq C$. We assume that no production takes place and that only non-

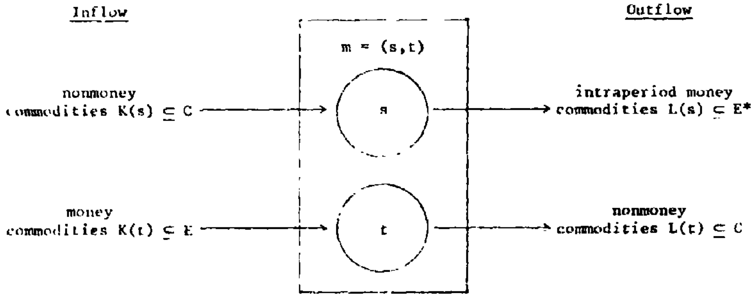
Inflow
Outflow
m = (s,t)
nonmoney commodities K(s) ⊆ C
s
intraperiod money commodities L(s) ⊆ E*
money commodities K(t) ⊆ E
t
nonmoney commodities L(t) ⊆ C

FIG. 1. Nonmoney and money commodities traded and accepted as means of payment at market $m = (s, t) \in M$.

money commodities are used up as transaction costs hence $K(t)^+ = L(s)$ and $L(t) \subsetneq K(s)$.

If only one medium of exchange is accepted as means of payment at market m then $K(t)$ consists of a single element. Figure 1 illustrates trade possibilities at market $m = (s, t)$.

The barter subset described as market $m_b = (s_b)$ contains a single sector s_b, $K(s_b)$ and $L(s_b) \subsetneq N$ denoting the list of inflow and outflow commodities, respectively, traded at m_b. Here no production takes place either hence $L(s_b) \subsetneq K(s_b)$. Figure 2 illustrates trade possibilities at the barter or banking sector s_b.

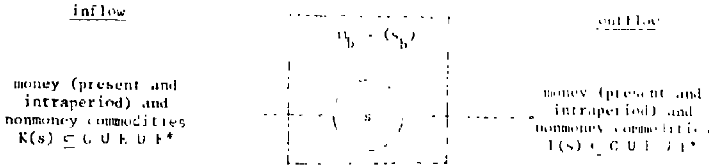
inflow
money (present and intraperiod) and nonmoney commodities K(s) ⊆ G ∪ F ∪ F*
m_b = (s_b)
s
outflow
money (present and intraperiod) and nonmoney commodities L(s) ⊆ G ∪ F ∪ F*

FIG. 2. Trade possibilities at market $m_b = (s_b)$.

For notational convenience we introduce two more sectors, an *endowment sector* s_e and a *consumption sector* s_c, the meaning of which will become clear in a moment. The following symbols are used throughout:

$$S := \{s_b\} \cup \bigcup_{(s,t) \in M} (\{s\} \cup \{t\})$$

and

$$S_0 := S \cup \{s_e\} \cup \{s_c\}.$$

Let I be the finite set of consumers. The action of consumer $i \in I$ can be described by his *consumption-marketing plan* $x_i = (\cdots x_i(s, t) \cdots)$, a notion borrowed from Schweizer, Varaiya and Hartwick [13]. Here $x_i(s, t) \in \mathbb{R}_+^N$, $s, t \in S_0$, denotes the commodity bundle transferred from sector s to sector t and

$$x_i(\cdot, s) := \sum_{t \in S_0} x_i(t, s) \text{ and } x_i(s, \cdot) := \sum_{t \in S_0} x_i(s, t)$$

stand for total inflow and outflow at sector s, respectively, due to i-th's marketing activity. $x_i(s_e, s_c)$ denotes consumption out of endowment and $x_i(\cdot, s_c)$ total consumption including stocks of present and intraperiod money commodities carried forward to the next period.

Furthermore, $\omega_i \in \mathbb{R}_+^N$ is the vector of the i-th consumer's initial endowment by which total outflow out of his endowment sector has to be bounded, i.e.

$$x_i(s_e, \cdot) \leq \omega_i. \tag{4}$$

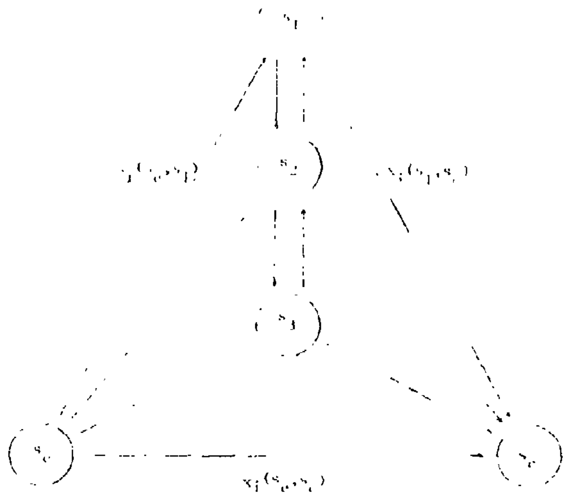

FIG. 3. Consumption-marketing plan of consumer *i*.

As mentioned above, particular attention must be given to the fact that the transfer of value between sectors is subject to certain prespecified constraints which may be expressed as

$$x_i(s, t) \in T(s, t),\ s, t \in S_0\,. \tag{5}$$

In general, the shape of $T(s, t)$ reflects two things. First, for the consumer, it makes sense to transfer a commodity *j* from *s* to *t* only if *j* is traded at both sectors, i.e. $x_i(s, t) \in T(s, t)$ and $x_{ij}(s, t) \neq 0$ only if $j \in L(s) \cap K(t)$. Second, $T(s, t)$ specifies which commodities can actually be transferred. Especially $T(s, t) = \{0\}$ might occur corresponding to a situation where the length of the period or any other limiting factor excludes such a transfer though $L(s) \cap K(t)$ could be nonempty.

Finally, consumers face a set of monetary constraints reflecting that with the exception of the barter subset exchange takes place using money commodities as media of exchange. Let $p(s)$ and $q(s) \in \mathbb{R}_+^N$ denote buying (inflow) and selling (outflow) prices prevailing at market sector *s*. These monetary constraints may be expressed as

$$d_i(s) + p(s)\, x_i(\cdot, s) - q(s)\, x_i(s, \cdot) \geq 0,\ s \in S \tag{6}$$

where $d_i(s)$ denotes the *dividend income* of consumer *i* at sector *s*, a notion to be explained below. We point out that there is a budget constraint at each market sector and that, typically, the system (6) of constraints cannot be replaced by an overall budget constraint in a money economy.

For consumer $i \in I$ $\tilde{\xi}_i(d_i, p, q)$ is the set of all consumption-marketing plans x_i which satisfy (4)-(6), given prices $(p, q) = (\cdots p(s) \cdots, \cdots q(s) \cdots)$ and a dividend configuration $d_i = (\cdots d_i(s) \cdots)$. He ranks alternatives in $\tilde{\xi}_i(d_i, p, q)$ by an *expected utility index* $U_i(x_i(\cdot, s_c), p, q)$ indicating the expected intertemporal utility to be derived from the sum $x_i(\cdot, s_c)$ of consumption of nonmoney commodities and of balances of money commodities carried over to the next period, given that present prices are (p, q). These prices will influence his expectations on future prices thus entering the expected utility index as an argument. Note that we distinguish stocks of present and intraperiod media of exchange. However, since we shall not assume strict quasi-concavity of U_i the case where such a distinction is irrelevant will be covered as well.

For a discussion of these concepts well-known in the theory of temporary equilibrium, we refer to Grandmont [4] and the references given there.

Consumer i, acting as a price taker in a competitive world, can be described by the correspondence $\xi_i(d_i, p, q)$ of all consumption-marketing plans which maximize his expected utility index over $\tilde{\xi}_i(d_i, p, q)$.

Market $m \subset M \cup \{m_b\}$ transforms inflow bundle $v_m \in \mathbb{R}_+^N$ into outflow bundle $w_m \in \mathbb{R}_+^N$. No production takes place but transaction costs may occur hence, for $j \in C$, $w_{mj} \leqq v_{mj}$ and $v_{mj} - w_{mj}$ corresponds to the quantity of commodity j used up as transaction costs whereas, for $j \in E$, $m \in M$, $w_{mj} = v_{mj}$. Markets are operated by *traders* choosing (v_m, w_m) from the set F_m of all feasible transactions. Traders too behave as price takers trying to maximize profit. For the barter subset $m = m_b = (s_b)$ no difficulty arises defining profit as

$$\pi_{m_b}(p, q) = \max[q(s_b)\, w_{m_b} - p(s_b)\, v_{m_b}] \text{ over } F_{m_b}. \tag{7}$$

For a market $m = (s, t) \in M$ of the nonbarter type, we have to take into account that exchange takes place by means of media of exchange. For notational convenience, we introduce the following decomposition, given $(v_m, w_m) \in F_m$:

$$v_m = v_m(s) + v_m(t), \quad w_m = w_m(s) + w_m(t) \tag{8}$$

where $v_m(s), v_m(t), w_m(s), w_m(t) \in \mathbb{R}_+^N$ and $v_{mj}(s) > 0$ only if $j \in K(s)$, $w_{mj}(s) > 0$ only if $j \in L(s)$ and similarly for t. $v_m(s)$ and $w_m(t)$ correspond to nonmoney inflow and outflow at market m whereas $v_m(t)$ and $w_m(s)$ denote money inflow and outflow. Since exchange is of monetary type, money inflow should not exceed nonmoney outflow in value, i.e.

$$q(t)\, w_m(t) \geqq p(t)\, v_m(t). \tag{9}$$

The value of money incomes to be paid out to consumers amounts to $p(s)\, v_m(s)$ whereas the value of gross money balances at the outflow side is

$q(s)\,w_m(s)$. Therefore the trader at market m clearly has an incentive to maximize the difference $q(s)\,w_m(s) - p(s)\,v_m(s)$. This leads to the following definition

$$\pi_m(p, q) = \max[q(s)\,w_m(s) - p(s)\,v_m(s)],\ m = (s, t) \in M \tag{10}$$

subject to (9) and $(v_m, w_m) \in F_m$. We assume that profits are paid out to consumers according to their relative shares in markets:

$$d_i(s; p, q) = \begin{cases} \alpha_i(m)\,\pi_m(p, q) & \text{if } m = (s_b) \\ & \text{or if } m = (s, t) \in M \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where relative shares $\alpha_i(m)$ are given such that $\sum_i \alpha_i(m) = 1$ for all markets m.

We define, for any market $m \in M \cup \{m_b\}$, the set $\xi_m(p, q)$ of all inflow-outflow combinations $(v_m, w_m) \in F_m$ which maximize profit as defined above, given prices (p, q).

A state of *temporary equilibrium* for the above money economy consists of a price vector $(p, q) > 0$, a consumption-marketing plan x_i and a dividend configuration $d_i = (\cdots d_i(s) \cdots)$ for each consumer $i \in I$, and an inflow-outflow combination (v_m, w_m) for each market $m \in M \cup \{m_b\}$ such that

$$x_i \in \xi_i(d_i, p, q),\ i \in I \tag{12}$$

$$(v_m, w_m) \in \xi_m(p, q),\ m \in M \cup \{m_b\} \tag{13}$$

$$\sum_i x_i(\cdot, s) = v(s),\ s \in S \tag{14}$$

$$\sum_i x_i(s, \cdot) = w(s),\ s \in S \tag{15}$$

$$d_i(s) = d_i(s; p, q),\ i \in I,\ s \in S \tag{16}$$

where $v(s_b) = v_{m_b}$, and $v(s) = v_m(s)$, $v(t) = v_m(t)$ for $m = (s, t) \in M$ and similarly for w. Note that the case, where all markets of the nonbarter type are shut down just because none of the accepted media of exchange is of positive value, has not been ruled out yet. Such a situation, however, would hardly correspond to an equilibrium of monetary type. Therefore, we will prove the existence of an equilibrium such that at any market $m \in M$ of the nonbarter type at least one medium of exchange yields positive value.

As should be clear from the introduction, Clower's pure money economy corresponds to the following specifications: no transaction costs, only one market $m = (s, t)$ consisting of two sectors, $K(s) = C$, $L(s) = E^*$, $K(t) = E$, $L(t) = C$, $\#E = 1$, no barter subset, and no possible transfer between market sectors.

There is, however, another way to fit a pure money economy into our framework, namely by specifying that there exists a market $m_c = (s_c, t_c)$ consisting of two sectors for each non-money commodity $c \in C$ (trading post), that $K(s_c) = \{c\}$, $L(s_c) = E^*$, $K(t_c) = E$ and $L(t_c) = \{c\}$ for $c \in C$, and, again, that no transfer between market sectors is possible. Note, that even in this version a "trading post" necessarily consists of two sectors. Sectors are at the level where monetary constraints can be defined most easily, whereas a trading post consisting of two sectors corresponds to the level at which feasibility of flows of commodities is achieved. The inflow of non-money commodity c at sector s_c results in outflow of that commodity at sector t_c whereas inflow of the money commodity at t_c corresponds to money outflow (E^*) at s_c.

We conclude this section with a remark on the inefficiency of temporary equilibrium in a money economy of the above type. Examples can be constructed easily where a consumption-marketing plan $x \in \Pi\Gamma(s, t)$ together with an inflow-outflow combination $(v_m, w_m) \in F_m$ for each market m exists such that $\bar{x}(s_0, \cdot) \le \Sigma\omega_i$, $\bar{x}(\cdot, s) = \bar{v}(s)$, $\bar{x}(s, \cdot) = \bar{w}(s)$ for all $s \in S$ and $\bar{x}(\cdot, s_c) > \Sigma x_i(\cdot, s_c)$. This means that more consumption could be assigned to each consumer using the same system of transaction technologies, in other words, resources are wasted in such a state of temporary equilibrium. The rules of monetary exchange prevent more efficient trade arrangements from being achieved.

This case arises if there are two money commodities, say, paper money and gold. If the market, where gold is the means of payment, uses up more resources as transaction costs than the market where nonmoney commodities exchange for paper money, but if people have pessimistic expectations on the value of paper money then its present purchasing power may decrease to the point where no longer all transactions can take place at the more efficient market. As a consequence, refuge has to be taken in the other market where gold serves as means of payment and, in this way, resources are wasted at the less efficient part of the market system.

3. THE EXISTENCE OF TEMPORARY EQUILIBRIUM

In this section, we discuss a set of assumptions which will turn out to be sufficient for proving existence of a temporary equilibrium in a money economy. We begin with the marketing technology $\Pi\Gamma(s, t)$. The following notation is used repeatedly:

$$\mathbb{R}[n] := \{x \in \mathbb{R}_+^N \mid x_j = 0\ \forall j \notin n\}$$

where n is any subset of N.

Assumption A1.

(a) $T(s, t) \subseteq \mathbb{R}[L(s) \cap K(t)]$ convex, closed cone for all $s, t \in S_0$

(b) $T(s, s_c) = \mathbb{R}[L(s)]$ if $s \in S \cup \{s_e\}$

$T(s_e, s) = \mathbb{R}[L(s_e) \cap K(s)]$ and

$L(s_e) = C \cup E$ if $s \in S \cup \{s_c\}$

$T(s, s) = (0)$ if $s \in S_0$.

(c) If $x \in T(s, t)$ and $0 \leq x' \leq x$ then $x' \in T(s, t)$.

(a) and (c) are standard and need no comment. As for (b) it is required, firstly, that each commodity occurring as outflow at any sector can be taken into the consumption sector of consumers, secondly, that any inflow commodity, with the exception of intraperiod media of exchange, occurring at a sector can be transferred to that sector out of endowments and, thirdly, that no commodity bought at a market sector can be sold to the same sector in the same period.

We admit, however, that $T(s, s_b) \neq (0)$ for some $m = (s, t) \in M$ to include money economies where borrowing on security at the barter subset (banking system) is allowed for. Security consists of intraperiod receipts of money offered for money commodities of the present period.

Next we come to an assumption regarding the expected utility index of consumers. First we have to specify the domain of definition of that index. A set Σ of prices is given as follows:

$(p, q) = (\cdots p(s) \cdots, \cdots q(s) \cdots) \in \Sigma$ if and only if $p(s) \in \mathbb{R}[K(s)]$, $q(s) \in \mathbb{R}[L(s)]$ and

$$\sum_{j \in N} (p_j(s) + q_j(s)) = 1 \text{ for all } s \in S \tag{17}$$

and $q_j(s) > 0$ whenever $j \in L(s)$.

Assumption A2.

(a) $U_i : \mathbb{R}_+^N \times \Sigma \to \mathbb{R}$ continuous and $U_i(\cdot, p, q)$ quasiconcave and strictly monotone on $\mathbb{R}_+^N$ for any $(p, q) \in \Sigma$.

(b) For any sequence $(p^\nu, q^\nu) \in \Sigma$ converging to a point in $\bar{\Sigma}\setminus\Sigma$ there exists a subsequence $(p^{\nu'}, q^{\nu'})$, for which $\lim_{\nu'} U_i(\cdot, p^{\nu'}, q^{\nu'})$ defines a continuous function on $\mathbb{R}_+^N$ which is strictly monotone.

Hereby $\bar{\Sigma}$ denotes the closure of Σ.

As for (a) we observe that normalization of prices as given by (17) is reasonable since the consumer's optimization problem exhibits the corresponding homogeneity in monetary constraints. For prices $(p, q) \in \bar{\Sigma}\setminus\Sigma$

the expected utility index U_i need not be defined but some boundary conditions as expressed by (b) should hold. Condition (b), sometimes derived from so-called "tightness" assumptions, is standard in temporary equilibrium theory (cf. Grandmont [3] and [4]).

Assumption A3.

(a) $\sum_i \omega_{ij} \gg 0$ for all $j \in C \cup E$.

(b) There exists a consumption-marketing plan $\bar{x} \in \Pi T(s, t)$ such that $\bar{x}_j(\cdot, s_b) \gg 0$ for all $j \in K(s_b)$.

A3 corresponds to the familiar assumption of strict positive total endowments. Since intraperiod media of exchange, by definition, are never part of initial endowments, (a) has to be supplemented by (b).

Assumption A4. ($m = m_b$)

(a) $F_m \subseteq \mathbb{R}[K(s_b)] \times \mathbb{R}[L(s_b)]$ closed, convex set, $0 \in F_m$, and F_m allows for free disposal.

(b) $(v_m, w_m) \in F_m$ only if $v_m \geq w_m$.

(c) There exists $(\bar{v}_m, \bar{w}_m) \in F_m$ such that $\bar{w}_j(m) \gg 0$ for all $j \in L(s_b)$.

(a), again is standard, (b) excludes production at the barter subset, and (c) means that a positive amount of any commodity $j \in L(s_b)$ can flow through the barter subset.

Assumption 5. ($m = (s, t) \in M$).

(a) $F_m = F_m^1 + F_m^2$ where
$F_m^1 \subseteq \mathbb{R}[K(s)] \times \mathbb{R}[L(t)]$ is a closed, convex set, allowing for free disposal, $0 \in F_m^1$ and $v_m \geq w_m$ for all $(v_m, w_m) \in F_m^1$

and

F_m^2 consists of all $(v_m, w_m) \in \mathbb{R}[K(t)] \times \mathbb{R}[L(s)]$ such that $v_{mj} \geqslant w_{mj}$, for all $j \in E$.

(b) There exists $(\bar{v}_m, \bar{w}_m) \in F_m^1$ such that $\bar{w}_{mj} \gg 0$ for all $j \in L(t)$.

F_m^1 is the set of inflow-outflow combinations consisting of nonmoney commodities only whereas F_m^2 denotes the set of combinations of present money inflow and corresponding intraperiod money outflow. (b) requires that a positive amount of each nonmoney commodity $j \in L(t)$ can flow through market m using as inflow nonmoney commodities only, though such a trade cannot be admitted by the rules of monetary exchange. (a) requires that the trading technology is of a simple form. Only nonmoney commodities are used up as transaction costs, money commodities cause no transaction costs and inflow of present and outflow of intraperiod money commodities coincide. Finally, no production takes place.

THEOREM. *Given that (A1)-(A5) hold, a temporary equilibrium exists such that $p(t) > 0$ for any market $m = (s, t) \in M$. This means that at any market of the nonbarter type at least one medium of exchange of positive value is accepted.*

A proof of this theorem is given in the following appendix.

APPENDIX

In order to prove existence we have to deal with the boundedness of the economy. The marketing activities of the consumers may lead to cycles. Therefore in contrast to the usual temporary equilibrium model, the excess demand correspondence on a market m may not be bounded from below.

To overcome this difficulty, we modify our original model by imposing an artificial constraint on all actions, then we increase the value of these constraints in a proper way, approximating an equilibrium of the original economy by a sequence of equilibria of the constrained type.

We take any continuous function h defined on Σ such that

$$\begin{aligned} &h(p, q) \geq 1 && \text{if } (p, q) \in \Sigma \text{ and} \\ &\lim_{\nu} h(p^{\nu}, q^{\nu}) = \infty && \text{if } \lim_{\nu}(p^{\nu}, q^{\nu}) \in \Sigma/\Sigma \end{aligned} \tag{18}$$

For any integer $\mu = 1, 2, \ldots$, and for markets $m = (s, t) \in M$ and $m = m_b = (s_b)$ we define

$$\begin{aligned} (v_m, w_m) \in \tilde{\eta}_m^{\mu}(p, q) \text{ iff } &(v_m, w_m) \in F_m \\ &\|(v_m, w_m)\| \leq \mu h(p, q) \end{aligned} \tag{19}$$

and, if $m = (s, t) \in M$, $q(t)\, w_m(t) - p(t)\, v_m(t) \geq 0$.

Furthermore, $(\hat{v}_m, \hat{w}_m) \in \eta_m^{\mu}(p, q)$ iff $(\hat{v}_m, \hat{w}_m) \in \tilde{\eta}_m^{\mu}(p, q)$ and

$$\rho_m^{\mu}(p, q) := \begin{cases} q(s_b)\, \hat{w}_m - p(s_b)\, \hat{v}_m & \text{if } m = m_b \\ q(s)\, \hat{w}_m(s) - p(s)\, \hat{v}_m(s) & \text{if } m = (s, t) \in M \end{cases}$$

where

$$\rho_m^{\mu}(p, q) := \max \begin{cases} q(s_b)\, w_m - p(s_b)\, v_m & \text{if } m = m_b \\ q(s)\, w_m(s) - p(s)\, v_m(s) & \text{if } m = (s, t) \in M \end{cases}$$

$$\text{subject to } (v_m, w_m) \in \tilde{\eta}_m^{\mu}(p, q).$$

LEMMA 1. *For any* $m \in M \cup \{m_b\}$, *the correspondence* $\eta_m^\mu(p, q)$ *is* u.h.c., *convex and nonempty for all* $(p, q) \in \Sigma$. *Furthermore,* $\rho_m^\mu(p, q)$ *is a continuous function on* Σ.

Proof. Obviously, $\bar{\eta}_m^\mu$ is u.h.c. on Σ. Next, we claim that $\bar{\eta}_m^\mu$ is l.h.c. For $m = m_b$, this is clear whereas for $m = (s, t) \in M$ we find, using *A*5, $(\bar{v}_m, \bar{w}_m) \in \bar{\eta}_m^\mu(p, q)$ such that $q(t)\,\bar{w}_m(t) - p(t)\,\bar{v}_m(t) > 0$ and $\|(\bar{v}_m, \bar{w}_m)\| < \mu h(p, q)$. Lower hemi-continuity can now be proved by a standard argument. It follows that $\bar{\eta}_m^\mu$ is continuous. The proof of Lemma 1 can now be completed easily. Q.E.D.

For consumer $i \in I$ the μ-th modification of his correspondences are defined as follows:

$$x \in \bar{\eta}_i^\mu(p, q) \text{ iff } x = (\cdots x(s, t) \cdots) \in \Pi\Gamma(s, t)$$

$$x(s_r, \cdot) \leq \omega_i$$

$$\beta_i(s; x; p, q) \geq 0 \text{ for all } s \in S \tag{20}$$

$$\|x\| \leq \mu \cdot h(p, q) \tag{21}$$

where $\beta_i^\mu(s, x, p, q) := f_i^\mu(s; p, q) + p(s)\,x(\cdot, s) - q(s)\,x(s, \cdot)$ and

$$f_i^\mu(s; p, q) = \begin{cases} \nu_i(m)\,\rho_m^\mu(p, q) & \text{if } s = s_b \text{ or if } m = (s, t) \in M \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

$$\hat{x} \in \eta_i^\mu(p, q) \text{ iff } U_i(\hat{x}(\cdot, s_r), p, q) = \max U_i(x(\cdot, s_r), p, q)$$

$$\text{subject to } x \in \bar{\eta}_i^\mu(p, q).$$

Finally, we define consumption correspondences $\bar{C}_i^\mu$ and C_i^μ in the following way:

$$C \in \bar{C}_i^\mu(p, q) \text{ iff } C = x(\cdot, s_r) \text{ for some } x \in \bar{\eta}_i^\mu(p, q)$$

$$C \in C_i^\mu(p, q) \text{ iff } C = x(\cdot, s_r) \text{ for some } x \in \eta_i^\mu(p, q).$$

We claim:

LEMMA 2. *For any* $i \in I$ *the correspondence* $\eta_i^\mu(p, q)$ *is* u.h.c., *convex and nonempty for all* $(p, q) \in \Sigma$.

Proof. During this proof, we omit writing indices μ and i. The only difficulty arises in proving upper hemicontinuity. First, it is easily seen that $\bar{\eta}(p, q)$ and $\bar{C}(p, q)$ are u.h.c. Suppose that $\bar{C}(p, q)$ is l.h.c. Then it follows from a standard argument that $C(p, q)$ has to be u.h.c. Once $C(p, q)$ is shown to be u.h.c., it follows immediately that $\eta(p, q)$ is u.h.c. Thus it remains to be shown that $\bar{C}(p, q)$ is l.h.c. on Σ.

Suppose we are given a sequence $(p^\nu, q^\nu) \in \Sigma$ of prices such that $\lim(p^\nu, q^\nu) = (p, q) \in \Sigma$ and $C = x(\cdot, s_c)$ for some $x \in \tilde{\eta}(p, q)$. We define S^+ as the set of all sectors $s \in S$ for which there exists $\tilde{x} \in \tilde{\eta}(p, q)$ such that $\beta(s; \tilde{x}, p, q) > 0$ and $S^- = S \backslash S^+$. It follows from convexity that there exists $\tilde{x} \in \tilde{\eta}(p, q)$ such that $\beta(s, \tilde{x}, p, q) > 0$ for all $s \in S^+$.

Since $(p, q) \in \Sigma$ and since x and $\tilde{x} \in \tilde{\eta}(p, q)$ it follows that $x(s, s_c) = \tilde{x}(s, s_c) = 0$ for $s \in S^-$. Furthermore, $x(s, t) = \tilde{x}(s, t) = 0$ for $s \in S^-$ and $t \in S^+$. Finally, if x' is such that

$$x'(s, t) = \begin{cases} 0 & \text{if } s \in S^- \text{ or } t \in S^- \\ x(s, t) & \text{otherwise} \end{cases} \tag{23}$$

then $x' \in \tilde{\eta}(p, q)$ and $x'(\cdot, s_c) = x(\cdot, s_c)$ and similarly for $\tilde{x}$. Therefore, we may assume w.l.o.g. that

$$x(s, t) = \tilde{x}(s, t) = 0 \text{ for } s \in S^- \text{ or } t \in S^-.$$

Put $\lambda^\nu := \min\{\lambda \geq 0 \mid \lambda \tilde{x} + (1 - \lambda)\, x \in \tilde{\eta}(p^\nu, q^\nu)\}$. Then $\beta(s, \tilde{x}, p^\nu, q^\nu) = \beta(s, x, p^\nu, q^\nu) \geq 0$ if $s \in S^-$ and

$$\left.\begin{array}{l} \lim\limits_\nu \beta(s, x, p^\nu, q^\nu) \geq 0 \\ \lim\limits_\nu \beta(s, \tilde{x}, p^\nu, q^\nu) > 0 \end{array}\right\} \text{ if } s \in S^+.$$

It follows that $\lim_\nu \lambda^\nu = 0$ hence $\lim_\nu [\lambda^\nu \tilde{x} + (1 - \lambda^\nu)\, x] = x$ and $\lambda^\nu \tilde{x} + (1 - \lambda^\nu)\, x \in \tilde{\eta}(p^\nu, q^\nu)$ thus proving that $\tilde{C}$ is l.h.c. at (p, q). Q.E.D.

Note that we had to replace x by x' according to (23) in the above proof. This reflects the fact that $\tilde{\eta}$ itself need not be l.h.c.

Lemma 3. *Suppose, for fixed μ and $i \in I$, we are given a sequence $(p^\nu, q^\nu) \in \Sigma$ converging towards (p, q) and $x^\nu \in \eta_i^\mu(p^\nu, q^\nu)$. If either*

(i) $\lim_\nu f_i^\mu(\bar{s}; p^\nu, q^\nu) = \infty$

or

(ii) $q_j(\bar{s}) = 0$ *for some* $j \in L(\bar{s})$ *and* $\underline{\lim}_\nu f_i^\mu(\bar{s}; p^\nu, q^\nu) > 0$

or

(iii) $q_j(\bar{s}) = 0$ *for some* $j \in L(\bar{s})$ *and there exists*

$$\tilde{x}^\nu \in \tilde{\eta}_i^\mu(p, q) \text{ such that } \underline{\lim}_\nu\, \beta(\bar{s}, \tilde{x}^\nu, p^\nu, q^\nu) > 0$$

for some market sector $\bar{s} \in S$ *then* $\lim_\nu \| C^\nu \| = \infty$ *where* $C^\nu = x^\nu(\cdot, s_c)$.

Proof. It follows from (18) that $(p, q) \in \bar{\Sigma} \backslash \Sigma$ in either case. Choose $\bar{C} > 0$, $\bar{C} \in \mathbb{R}[L(\bar{s})]$ and $\bar{x} \in \Pi T(s, t)$ such that

$$\bar{x}(s, t) = \begin{cases} \bar{C} \text{ if } (s, t) = (\bar{s}, s_c) \\ 0 \text{ otherwise} \end{cases}$$

and, in cases (ii) and (iii), $q(\bar{s})\bar{C} > 0$. This is possible because $A1$(b) holds. Now assume the lemma is false. Then we may assume w.l.o.g. that $\lim C^\nu = C$. Using $A2$(b) we find a subsequence $\nu' = \nu_1 < \nu_2 < \cdots$ and $\epsilon > 0$ such that

$$\lim_{\nu'} U_i((1 - \epsilon) C^{\nu'} + \bar{C}, p^{\nu'}, q^{\nu'}) > \lim_{\nu'} U_i(C^{\nu'}, p^{\nu'}, q^{\nu'})$$

hence

$$U_i((1 - \epsilon) C^{\nu'} + \bar{C}, p^{\nu'}, q^{\nu'}) > U_i(C^{\nu'}, p^{\nu'}, q^{\nu'}) \tag{24}$$

for ν' large enough. But, it cases (i) and (ii), $(1 - \epsilon) x^{\nu'} + \bar{x} \in \tilde{\eta}^\mu(p^{\nu'}, q^{\nu'})$ for ν' large enough, whereas in case (iii) $(1 - \epsilon) x^{\nu'} + \epsilon/2\, \bar{x}^{\nu'} + \bar{x} \in \tilde{\eta}_i^\mu(p^{\nu'}, q^{\nu'})$ for ν' large enough.(24) contradicts optimality of $C^{\nu'}$ and the lemma is established. Q.E.D.

Next we define an excess demand correspondence ϵ^μ for the μ-th modification as follows:

$(\alpha, \beta) = (\cdots \alpha(s) \cdots, \cdots \beta(s) \cdots) \in \epsilon^\mu(p, q)$ iff there exist $x_i \in \eta_i^\mu(p, q)$, $i \in I$, and $(v_m, w_m) \in \eta_m^\mu(p, q)$, $m \in M \cup \{m_b\}$, such that

$$\alpha(s) = v_m(s) - \sum_i x_i(\cdot, s) \text{ and } \beta(s) = \sum_i x_i(s, \cdot) - w_m(s)$$

for all $s \in S$. It follows from Lemmas 1 and 2 that ϵ^μ is a nonempty, convex and u.h.c. correspondence on Σ. The following form of Walras' law holds:

$$p(s)\alpha(s) + q(s)\beta(s) = 0, s \in S. \tag{25}$$

for $(\alpha, \beta) \in \epsilon^\mu(p, q)$ and $(p, q) \in \Sigma$.

Next, we introduce the following sequence Σ^ν of price sets:

$$(p, q) \in \Sigma^\nu \text{ iff } (p, q) \in \Sigma \text{ and } q_j(s) \geq \frac{1}{\nu} \tag{26}$$

for all $j \in L(s)$ and $s \in S$. Σ^ν is compact and convex. Furthermore, $\Sigma = \bigcup_\nu \Sigma^\nu$.

LEMMA 4. *Suppose, for μ fixed, we are given a sequence $(p^\nu, q^\nu) \in \Sigma^\nu$ such that $\lim(p^\nu, q^\nu) = (p, q) \in \bar{\Sigma} \backslash \Sigma$ and $(\alpha^\nu, \beta^\nu) \in \epsilon^\mu(p^\nu, q^\nu)$. Then $\lim \| e^\nu \| = \infty$ and $\sum_{j \in J} e_j^\nu$ is bounded from below where*

$$e^\nu = \sum_{s \in S} (\alpha^\nu(s) + \beta^\nu(s)).$$

Proof. By definition, α^ν and β^ν might be written as

$$\alpha^\nu(s) = v_m^\nu(s) - \sum_i x_i^\nu(\cdot, s) \text{ and } \beta^\nu(s) = \sum_i x_i^\nu(s, \cdot) - w_m^\nu(s) \tag{27}$$

where $x_i^\nu \in \eta_i^\mu(p^\nu, q^\nu)$ and $(v_m^\nu, w_m^\nu) \in \eta_m^\mu(p^\nu, q^\nu)$. Let $C_i^\nu = x_i(\cdot, s_c)$ and $C^\nu = \sum_i C_i^\nu$.

Furthermore, we may assume w.l.o.g. that all coordinates of the above sequences are converging, possibly towards $\pm\infty$. We show first that

$$\lim_\nu C^\nu = \infty. \tag{28}$$

Several cases have to be distinguished:

(a) If there is a market $m = (s, t) \in M$ such that

$$q_j(s) = 0 \text{ for some } j \in L(s) \text{ or } q_j(t) = 0 \text{ for some } j \in L(t) \tag{29}$$

then the following five cases exhaust all possibilities:

(a1) $p(t) = 0$ and $q(s) > 0$;

(a2) $p(t) = 0$, $q(s) = 0$ and $p(s) > 0$;

(a3) $p(t) > 0$ but $q_j(t) = 0$ for some $j \in L(t)$;

(a4) $p(t) > 0$, $q_j(t) > 0$ for all $j \in L(t)$, $p(s) > 0$ and $q_j(s) = 0$ for some $j \in L(s)$;

(a5) $p(t) > 0$, $q_j(t) > 0$ for all $j \in L(t)$, $p(s) = 0$ and $q_j(s) = 0$ for some $j \in L(s)$.

In each of cases (a1)-(a5), a consumer $i \in I$ can be found such that

$$\lim C_i^{\nu'} = \infty. \tag{30}$$

This follows in case (a1) from A5(a) and Lemma 3(i), in cases (a2)-(a4) from A1(b), A3 and Lemma 3(iii), and in case (a5) from A5(b) and Lemma 3(ii).

(b) If there is no market $m \in M$ satisfying (29) then, by assumption, it follows that

$$q_j(s_b) = 0 \text{ for some } j \in L(s_b).$$

If $p(s_b) = 0$ then (30) follows from A4(c) and Lemma 3(ii) whereas if $p_j(s_b) > 0$ for some $j \in K(s_b)$, then we claim that there exist a consumer $i \in I$ and a consumption-marketing plan x_0 such that

$$x_0 \in \tilde{\eta}_i^\mu(p^\nu, q^\nu) \text{ for } \nu \text{ large enough and} \tag{31}$$

$$\varliminf_\nu \beta_i^\mu(s_b, x_0, p^\nu, q^\nu) > 0. \tag{32}$$

To prove this, we distinguish two cases:

(b1) $p_j(s_b) > 0$ for some $j \in C \cup E$;

(b2) $p_j(s_b) \neq 0$ for some $j \in E^*$.

In case (b1), using A1(b) and A3(a), x_0 can be constructed easily such that (31) and (32) hold. As for (b2) suppose $p_j(s_b) > 0$, $j \in E^*$. By A3(b), there exist $m = (r, t) \in M$ and $\bar{x}(r, s_b) \in T(r, s_b)$ such that $\bar{x}_j(r, s_b) > 0$.

Now either $p(r) = 0$. Then, by A5(b), $\underline{\lim}_\nu\, \rho_m{}^\nu(p^\nu, q^\nu) = 0$ and there exists $i \in I$ such that

$$f_i{}^\nu(r, p^\nu, q^\nu) > 0 \text{ for } \nu \text{ large enough.}$$

Or $p(r) > 0$. Then, by A3(a), there exist $i \in I$ and $\bar{x}(s_r, r) \in T(s_r, r)$ such that $\bar{x}(s_r, r) \leq \omega_i$ and $p(r)\,\bar{x}(s_r, r) > 0$. In either case, a consumer $i \in I$ and a consumption-marketing plan x_0 can be found such that (31) holds and such that

$$0 < x_0(r, s_b) \sim \bar{x}(r, s_b) \text{ and } x_0(s_b, \cdot) = 0.$$

(32) is established and, by Lemma 3(iii), so is (30). Since (a) and (b) cover all possibilities, (28) is proved.

Using (27), A4(b) and A5(a), it can be shown that

$$\begin{aligned}\sum_{j \in N} e_j{}^\nu &= \sum_{j \in N} \sum_{s \in S} (y^\nu(s) + \beta^\nu(s))_j \\ &\quad + \sum_{j \in N} \sum_{m \in M \cup \{m_0\}} (v_m{}^\nu - w_m{}^\nu)_j + \sum_{j \in N} \sum_{s \in S} \sum_{i \in I} (x_i{}^\nu(s, \cdot) - x_i{}^\nu(\cdot, s))_j \\ &\quad + \sum_{j \in N} \sum_{i \in I} (C_i{}^\nu - \omega_i)_j \geq - \sum_{j \in N} \sum_{i \in I} \omega_{ij}.\end{aligned}$$

Hence $\sum_{j \in N} e_j{}^\nu$ has to be bounded from below and the lemma is established.

Q.E.D.

LEMMA 5. *Suppose, for each $s \in S$, we are given a sequence $z^\nu(s) \in \mathbb{R}^N$ whose components all converge (possibly towards $\pm\infty$). If $\lim |\sum_{s \in S} z^\nu(s)| = \infty$ and if $\sum_{j \in N} \sum_{s \in S} z_j{}^\nu(s)$ is bounded from below, then there exists $\tilde{s} \in S$ and a subsequence ν' such that*

$$\lim_{\nu'} z_j^{\nu'}(\tilde{s}) = \infty \textit{ for some } j \in N$$

and

$$\sum_{j \in N} z_j^{\nu'}(\tilde{s}) \qquad \textit{bounded from below.}$$

Proof. Firstly, it is easily seen that $\bar{S}$ is nonempty where $\bar{S} := \{s \in S \mid \lim_\nu z_j(s) = \infty$ for some $j \in N\}$. Secondly, if the lemma was wrong then

$$\overline{\lim} \sum_{j \in N} z_j^\nu(s) = -\infty \text{ for all } s \in \bar{S}$$

and

$$\overline{\lim} \sum_{j \in N} z_j^\nu(s) < \infty \text{ for all } s \in S \setminus \bar{S}$$

hence

$$\overline{\lim} \sum_{j \in N} \sum_{s \in S} z_j^\nu(s) = -\infty$$

thus contradicting our assumptions and the lemma is established. Q.E.D.

Through the next lemma, a fixed-point argument enters the proof of our theorem (see Debreu [2] and Grandmont [4]).

LEMMA 6. *For any parameter μ, there exists a price vector $(\hat{p}^\mu, \hat{q}^\mu) \in \Sigma$ together with a vector $(\hat{\alpha}^\mu, \hat{\beta}^\mu) \in \epsilon^\mu(\hat{p}^\mu, \hat{q}^\mu)$ of excess demand such that $(\hat{\alpha}^\mu, \hat{\beta}^\mu) = 0$.*

Proof. We omit writing μ as an index during this proof. For any ν, Σ^ν as defined in (26) is convex and compact. Using a familiar fixed-point argument, sequences $(p^\nu, q^\nu) \in \Sigma^\nu$ and $(\alpha^\nu, \beta^\nu) \in \epsilon(p^\nu, q^\nu)$ can be constructed such that

$$(p(s), q(s))(\alpha^\nu(s), \beta^\nu(s)) \leq (p^\nu(s), q^\nu(s))(\alpha^\nu(s), \beta^\nu(s))$$

for all $s \in S$ and $(p, q) \in \Sigma^\nu$. Hence, by Walras' law (25),

$$(p(s), q(s))(\alpha^\nu(s), \beta^\nu(s)) \leq 0 \text{ for all } (p, q) \in \Sigma^\nu, s \in S. \tag{33}$$

Taking subsequences, if necessary, we may assume w.l.o.g. that $\lim(p^\nu, q^\nu) =: (\hat{p}, \hat{q})$. Now suppose $(\hat{p}, \hat{q}) \in \bar{\Sigma} \setminus \Sigma$. Define $z^\nu(s) := \alpha^\nu(s) + \beta^\nu(s)$. Then, by Lemma 4,

$$\lim_\nu \Big\| \sum_{s \in S} z^\nu(s) \Big\| = \infty \text{ and } \sum_{j \in N} \sum_{s \in S} z_j^\nu(s) \text{ is bounded from below.}$$

Therefore, by Lemma 5, there exists $\bar{s} \in S$ such that

$$\lim z_j^\nu(\bar{s}) = \infty \text{ for some } j \in N \text{ and}$$

$$\overline{\lim} \sum_{j \in N} z_j^\nu(\bar{s}) > -\infty.$$

This contradicts (33) hence $(\hat{p}, \hat{q}) \in \Sigma$ and, as a consequence, the sequence (α^ν, β^ν) has to be bounded. Again, w.l.o.g., we may assume that $\lim(\alpha^\nu, \beta^\nu) =:$

$(\hat\alpha, \hat\beta)$ and, since ϵ is u.h.c., that $(\hat\alpha, \hat\beta) \in \epsilon(\hat p, \hat q)$. Furthermore, by (33), $(p, q)(\hat\alpha, \hat\beta) \leq 0$ for all $(p, q) \in \Sigma$ hence $(\hat\alpha, \hat\beta) \leq 0$. Defining $(\hat p^\mu, \hat q^\mu) := (\hat p, \hat q)$ and $(\hat\alpha^\mu, \hat\beta^\mu) := (\hat\alpha, \hat\beta)$ a pair of vectors is shown to exist as claimed. Q.E.D.

Modifying Lemmas 3 and 4 slightly, it can be shown in a similar way as above that the sequence $(\hat p^\mu, \hat q^\mu)$ as constructed in Lemma 6 is bounded away from $\bar\Sigma \setminus \Sigma$ and, again w.l.o.g., we may assume that

$$\lim_\mu (\hat p^\mu, \hat q^\mu) = (p, q) \in \Sigma. \tag{34}$$

Furthermore, by definition,

$$0 = \hat\alpha^\mu(s) = \hat v_m^\mu(s) - \sum_i \hat x_i^\mu(\cdot, s) \text{ and } 0 = \hat\beta^\mu(s) = \sum_i \hat x_i^\mu(s, \cdot) - \hat w_m^\mu(s) \tag{35}$$

for some $\hat x_i^\mu \in \eta_i^\mu(\hat p^\mu, \hat q^\mu)$, $i \in I$ and $(\hat v_m^\mu, \hat w_m^\mu) \in \eta_m^\mu(\hat p^\mu, \hat q^\mu)$, $m \in M \cup \{m_b\}$.

LEMMA 7. *For $m = (s, t) \in M$, the sequences $\hat x_i^\mu(s, r)$ and $\hat x_i^\mu(r, t)$ are bounded for all $i \in I$ and $r \in S_0$.*

Using A1(b), A4(b) and A5(a), this lemma can be proved easily. We omit details. At this point, the assumption is used that there exists only one market of the barter type. For a discussion of difficulties arising without such an assumption, the reader is referred to Müller and Schweizer [12],

LEMMA 8. *The sequences $\hat x_i^\mu$, $i \in I$, are bounded.*

Proof. Using (20), (34) and Lemma 7, we find that $\hat x_i^\mu(t, r)$ is bounded for all $i \in I$ and $r \in S_0$ provided that $m = (s, t) \in M$. Hence $\hat x_i^\mu(\cdot, s_b)$ is bounded and, by A4(b) and (35), so is $\hat x_i^\mu(s_b, \cdot)$ for all $i \in I$. Finally, it follows that $x_i^\mu(\cdot, s)$ is bounded for all $i \in I$ and $m = (s, t) \in M$, thus proving the lemma. Q.E.D.

We finish the proof of our theorem as follows. Condition (35) on market clearing implies that $\hat v_m^\mu$ has to be bounded for all $m \in M \cup \{m_b\}$ and, as a consequence of A4(b) and A5(a), so has to be $\hat w_m^\mu$ and, w.l.o.g., we may assume that

$$\lim_\mu \hat x_i^\mu = x_i, \; i \in I \text{ and } \lim_\mu (\hat v_m^\mu, \hat w_m^\mu) = (v_m, w_m), \; m \in M \cup \{m_b\}.$$

Using Lemmas 1, 2 and 8, it can be verified readily that

$$x_i \in \eta_i^\mu(p, q), \; i \in I \text{ and } (v_m, w_m) \in \eta_m^\mu(p, q), \; m \in M \cup \{m_b\}$$

for μ large enough since in this case, constraints (19) and (21) cannot be binding.

It follows that $\rho_m^{\mu}(p, q) = \pi_m(p, q)$ and $(v_m, w_m) \in \xi_m(p, q)$ for all $m \in M \cup \{m_0\}$ and μ large enough hence $x_i \in \xi_i(d_i, p, q)$ and $d_i = d_i(s, p, q)$ for μ large enough and $i \in I$.

It remains to be shown that

$$p(t) > 0 \text{ for all } m = (s, t) \in M. \tag{36}$$

Assume the contrary, i.e. $p(t) = 0$ for some $m = (s, t) \in M$. Since $(p, q) \in \Sigma$ it follows that $q(s) = 0$ hence, by A5(a), $\pi_m(p, q) = \infty$, a contradiction. (36) is proved.

This brings the proof of our theorem on the existence of temporary equilibrium in a money economy to an end.

REFERENCES

1. R. W. CLOWER, A reconsideration of the microfoundations of monetary theory, *Western Econ. J.* 6 (1967), 1–9.
2. G. DEBREU, "Theory of Value," Wiley, New York, 1959.
3. J.-M. GRANDMONT, On the short-run equilibrium in a monetary economy, *in* "Allocation under Uncertainty, Equilibrium, and Optimality," (J. H. Drèze, Ed.), Macmillan, New York, 1974.
4. J.-M. GRANDMONT, Temporary general equilibrium theory, *Econometrica* **45** (1977), 535–572.
5. J.-M. GRANDMONT AND Y. YOUNÈS, On the role of money and the existence of a monetary equilibrium, *Rev. Econ. Stud.* **39** (1972), 355–372.
6. F. HAHN, On some problems of proving the existence of an equilibrium in a monetary economy, *in* "The Theory of Interest Rates" (F. Hahn and F. P. Brechling, Eds.), Macmillan, New York, 1965.
7. F. HAHN, On transaction costs, inessential sequence economies, and money, *Rev. Econ. Stud.* **40** (1973), 449–462.
8. T. HAYASHI, The non-Pareto efficiency of initial allocation of commodities and monetary equilibrium: An inside money economy, *J. Econ. Theory* **7** (1974), 173–187.
9. W. P. HELLER, The holding of money balances in general equilibrium, *J. Econ. Theory* **7** (1974), 93–108.
10. M. KURZ, Equilibrium in a finite sequence of markets with transaction cost, *Econometrica* **42** (1974), 1–20.
11. M. KURZ, Equilibrium with transaction cost and money in a single market exchange economy, *J. Econ. Theory* **7** (1974), 418–452.
12. H. MÜLLER AND U. SCHWEIZER, "Temporary Equilibrium with Several Media of Exchange and Transaction Costs," Working Paper in Economic Theory and Econometrics IP-242, University of California, Berkeley, 1976.
13. U. SCHWEIZER, P. VARAIYA, AND J. HARTWICK, General equilibrium and location theory, *J. Urban Econ.* **3** (1976), 285–303.
14. D. STARRETT, Inefficiency and the demand for "money" in a sequence economy, *Rev. Econ. Stud.* **40** (1973), 437–448.

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A Simple General Equilibrium Model of Production: Comparative Statics with Price Uncertainty

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This paper examines the possibility of extending the basic theorems of the risk-free, two-sector, two-factor, constant returns to scale model of production to cover situations with price uncertainty. It is shown that the Rybczynski and Stolper-Samuelson theorems may fail to hold for certain cases while the factor price equalization theorem cannot carry over to the stochastic world, provided firms in the uncertainty sector exhibit decreasing absolute risk aversion. The implications of uniform (relative) changes in both factor endowments and in both (expected) commodity prices are also explored.

1. INTRODUCTION

In the past a simple general equilibrium model of production has served as an important workhorse for the evolution of many areas of economics, notably international economics, public finance and economic growth. The most famous model of this sort is the deterministic, two-sector, two-factor, constant returns to scale model of production.[1] In this paper we attempt to blend the recent developments in the theory of uncertainty with some important problems which have been traditionally examined within the framework of the two-by-two, full certainty model.

We are concerned with the competitive economy in which one of the two sectors is confronted with uncertainty as the result of price fluctuations in its product; therefore, the economy comprises the uncertainty sector and the certainty sector. Extending the usual, small-country assumption to the price uncertainty milieu, we assume that firms in the uncertainty sector have no control over the distribution function of the price of their product, whose shape depends upon weather, taste pattern and many other natural and social factors influencing the demand-supply condition in the world market.[2] It is further assumed that those uncertianty firms exhibit decreasing absolute risk aversion and seek to maximize expected utility from profits.

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[1] For the standard treatment of the nonstochastic, two-by-two model, see [5, 8, 11, 18].

[2] Empirical analysis has shown that many countries suffer from high price fluctuations in their exporting commodities. See Coppock [6]. This motivates us to study the competitive economy facing price uncertainty as a good approximation to reality.

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In deriving comparative statics results from this simple model with price uncertainty, we highlight the dual relationship between factor endowments and commodity outputs on the one hand and factor prices and commodity price parameters on the other. It proves convenient for this purpose to make full use of the variable, input-output coefficients of both sectors instead of the production functions *per se*. Clearly, seeing to what extent the presence of price risk affects this dual feature of the model and thus forces the well-known, nonstochastic results to be modified should be the main task assigned to the present paper.[3]

We shall examine the possibility of extending the basic theorems of the traditional, risk-free model to cover situations with price uncertainty. As will be shown below, even at the long-run equilibrium, the uncertainty sector's expected profits are positive in the case of risk aversion. Accordingly, the part of the expected price of the uncertainty output exceeding the wage rate and rental should constitute the per unit payment for the "risk factor." Existence of such a risk bearing fee and its responsiveness to changes in factor endowments and in commodity price parameters require special care in exploring the comparative statics properties of our stochastic model. The following results among other things are worth mentioning. (i) The factor price equalization theorem is not valid—at least in its strict version -in the stochastic context. (ii) The Rybczynski and Stolper-Samuelson theorems may fail to hold for certain cases to be specified below. This partially invalidates the Heckscher-Ohlin theorem as well. (iii) A uniform (relative) change in both factor endowments affects commodity outputs unevenly, while a uniform (relative) change in both (expected) commodity prices affects factor prices unevenly. The so-called magnification effect of Jones cannot carry over to the stochastic world.

In Section 2 we shall make a set of assumptions to establish our two-sector, two-factor model with price uncertainty. In Section 3 we shall place the key equations describing the model on a common footing by rewriting them in terms of relative changes in variables and parameters. Section 4 will discuss the relationship between factor prices and commodity price parameters. The subsequent sections will be devoted to the comparative statics properties of the uncertainty model. Namely, we shall explore the implications of changes in factor endowments in Section 5, of changes in (expected) commodity prices in Section 6, and of changes in price uncertainty in Section 7.

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[3] It was Batra [2, 3, 4] who first investigated the basic characteristics of the two-by-tw equilibrium model under uncertainty. Apart from the fact that his interest was main centered around *production* uncertainty rather than *price* uncertainty, his formulation of th model is, unfortunately, fairly tedius and tends to obscure the duality feature of the mode My present work may be regarded as a complement to Batra's pioneering work. Recentl a similar model was developed by Mayer [21], which came to my attention after the com pletion of this paper.

2. Assumptions and the Model

Throughout this paper, we maintain the following assumptions under which our uncertainty model is to be established. Some of those assumptions are usually made for the certainty model, while several others are newly installed to take account of price uncertainty.

Specifically, the following assumptions and notations are maintained:

(i) Two primary factors, labor (L) and capital (K), are used in producing two distinct commodities, U and C. Technology in each of the two sectors exhibits constant returns to scale.

(ii) The money wage rate and the money rent for capital service are respectively denoted by w and r, whereas the market prices of the two commodities are denoted by p_U and p_C.

(iii) Firms in sector U face price uncertainty, but firms in sector C is free from uncertainty. Free competition prevails in all markets. The probability distribution of the price of commodity U as well as the price of commodity C is exogeneously given to the country; in other words, we keep the "small country" assumption in the probabilistic sense.

(iv) Factors are fully employed and inelastically supplied. Although factors are perfectly mobile between the two sectors, factor intensity rankings never reverse when factor prices change.

(v) Firms in the uncertainty sector, which have to make their input-output decision prior to the knowledge of the price of their output, seek to maximize expected utility from profits. It is assumed that they are not only risk averters but also exhibit decreasing absolute risk aversion. On the other hand, firms in the certainty sector seek to maximize utility from profits.

(vi) All quantities should be valued in terms of money. Money is implicitly present in the economy and serves merely as the unit of measurement.

(vii) The demand side and other socio-politico-economic sides that must be introduced for a fuller description of the economy are all but neglected here.

With those assumptions in mind, let us build a two-sector, two-factor, general equilibrium model with price uncertainty. The linearly homogeneous production function and the profit level in the uncertainty sector are given by

$$U = f_U(L_U, K_U)$$

and

$$\pi_U = p_U U - wL_U - rK_U,$$

respectively. The random variable p_U is rewritten as

$$p_U = \mu_U + \gamma\epsilon,$$

where $E[\epsilon] = 0$ and γ is a shift parameter which we may take as being equal to one initially. An increase in γ leads to an increased spread of the probability distribution $\phi(\epsilon)$ around the constant mean μ_U, and this will be regarded as our definition of an increase in the riskiness of p_U.[4]

Let v_U be the utility attainable from π_U. Assuming risk aversion on the part of U-producers, we have,

$$v'_U(\pi_U) > 0 \qquad \text{and} \qquad v''_U(\pi_U) < 0.$$

Those producers are interested in maximizing $E[v_U(\pi_U)]$ with respect to their two decision variables, L_U and K_U. This gives the following relationships:

$$E[v'_U(\pi_U)\{(\mu_U + \gamma\epsilon) f_{LU} - w\}] = 0, \tag{1}$$

$$E[v'_U(\pi_U)\{(\mu_U + \gamma\epsilon) f_{KU} - r\}] = 0, \tag{2}$$

where $f_{LU} = \partial f_U/\partial L_U$ and $f_{KU} = \partial f_U/\partial K_U$.[5] From Eqs. (1) and (2), we immediately have

$$w = (\mu_U - \gamma\rho) f_{LU}, \tag{3}$$

$$r = (\mu_U - \gamma\rho) f_{KU}, \tag{4}$$

where

$$\rho = -E[v'_U(\pi_U)\epsilon]/E[v'_U(\pi_U)]. \tag{5}$$

The value of ρ represents the per unit (psychological) cost of risk bearing associated with price uncertainty, which we may call the risk bearing fee. As will be shown in Lemma 1 below, ρ is always positive whenever U-producers are risk averse. Therefore, a nice interpretation may be given to Eqs. (3) and (4): At U-producers' uncertainty equilibrium, the price of each factor is equal to the "net" or "effective" expected price of its marginal product. Since factor prices are positive, μ_U must be greater than $\gamma\rho$.

LEMMA 1. *If* $v''_U(\pi_U) < 0$ *then* $\rho > 0$.

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[4] It is assumed here that p_U is positive for any value of ϵ and that $\phi(\epsilon)$ takes on positive values for at least two values of ϵ. Therefore, $E[p_U]$ and $\text{var}(p_U)$ are both positive. This mean-preserving spread type of shift in the distribution function was first introduced into the theory of the firm by Sandmo [17]. Also see Sakai [14].

[5] The second-order condition for a regular maximum requires that the Hessian matrix of $E[v_U(\pi_U)]$ with respect to L_U and K_U be negative definite. This condition is weaker than the condition that the Hessian matrix of $f_U(L_U, K_U)$ is negative definite, provided U-firms are risk averse. See Sakai [14, p. 33].

Proof. From the definition of π_U, we find $\pi_U - E[\pi_U] = \gamma\epsilon U$. Therefore, if $v_U''(\pi_U) < 0$,

$$v_U'(\pi_U) \lesseqgtr v_U'(E[\pi_U]) \quad \text{according to whether} \quad \epsilon \gtreqless 0.$$

It follows immediately that

$$v_U'(\pi_U)\epsilon \leqq v_U'(E[\pi_U])\epsilon \qquad \text{for } any\ \epsilon, \tag{6}$$

with the equality only when $\epsilon = 0$. Recall the assumption that $\phi(\epsilon)$ are positive for at least two values of ϵ. Hence, taking expectations on both sides of Eq. (6), we obtain

$$E[v_U'(\pi_U)\epsilon] < v_U'(E[\pi_U])\,E[\epsilon] = 0,$$

so that ρ must be positive. Q.E.D.

On the other hand, C-producers are interested in maximizing utility from profits:

$$v_C(\pi_C) = v_C(p_C f_C(L_C, K_C) - wL_C - rK_C),$$

where v_C is an increasing function and f_C the linearly homogeneous production function. The optimum conditions are furnished by

$$w = p_C f_{LC}, \tag{7}$$

$$r = p_C f_{KC}, \tag{8}$$

where $f_{LC} = \partial f_C/\partial L_C$ and $f_{KC} = \partial f_C/\partial K_C$. In the light of Eqs. (3), (4), (7) and (8), we have

$$\frac{f_{LU}}{f_{KU}} = \frac{w}{r} = \frac{f_{LC}}{f_{KC}}. \tag{9}$$

Therefore, as with the certainty case, the factor price ratio must be equal to the marginal substitution between factors for each sector.

The technology of the economy is also described by the columns of the following matrix:

$$a = \begin{bmatrix} a_{LU} & a_{LC} \\ a_{KU} & a_{KC} \end{bmatrix}$$

where a_{ij} stands for the quantity of factor i required to produce a unit of commodity j; i.e., $a_{LU} = L_U/U$, $a_{KU} = K_U/U$, etc. Let us try to reformulate our two-by-two model in terms of the technology matrix (a_{ij}) instead of the production functions f_U and f_C. First of all, the requirement that both factors be fully employed is obviously given by

$$L = a_{LU}U + a_{LC}C, \tag{10}$$

$$K = a_{KU}U + a_{KC}C. \tag{11}$$

Secondly, noting that the production function f_C of the uncertainty sector is linearly homogeneous, application of the Euler theorem yields

$$U = f_{LU}L_U + f_{KU}K_U,$$

whence

$$\mu_U - \gamma\rho = (\mu_U - \gamma\rho) f_{LU}a_{LU} + (\mu_U - \gamma\rho) f_{KU}a_{KU}.$$

In view of Eqs. (3) and (4), this results in

$$\mu_U = wa_{LU} + ra_{KU} + \gamma\rho. \tag{12}$$

In similar fashion, we can obtain for the certainty sector:

$$p_C = wa_{LC} + ra_{KC}. \tag{13}$$

As is clear from comparison between Eqs. (10) and (11) and Eqs. (12) and (13), the formulation based on the input-output matrix (a_{ij}) is very instructive in showing the similarity of the relationship between factor endowments and commodity outputs on the one hand to that between commodity price parameters and factor prices on the other. Such a duality feature will be proved more deeply in carrying out comparative statics analysis in the subsequent sections.

In the general case of variable coefficients, each a_{ij} depends on the rate of factor prices:[6]

$$a_{LU} = a_{LU}(w/r), \quad a_{KU} = a_{KU}(w/r), \quad a_{LC} = a_{LC}(w/r), \quad a_{KC} = a_{KC}(w/r). \tag{14}$$

It is now a simple matter to show that expected profits are positive if U-producers are risk averse. Indeed, in view of Eq. (12),

$$\begin{aligned} \pi_U &= \{(\mu_U + \gamma\epsilon) - (wa_{LU} + ra_{KU})\}U \\ &= \{(\mu_U + \gamma\epsilon) - (\mu_U - \gamma\rho)\}U, \end{aligned}$$

so that

$$\pi_U = (\rho + \epsilon)\gamma U. \tag{15}$$

We thereby have $E[\pi_U] = \gamma\rho U$, implying that expected profits are positive if $v_U''(\pi_U)$ is negative (see Lemma 1). This is in marked contrast to the certainty sector in which profits are of coures zero.[7]

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[6] Linear homogeneity of the production function f_U implies $f_U(a_{LU}, a_{KU}) = 1$. Besides, since the first derivatives f_{LU} and f_{KU} are homogeneous of degree zero, it follows from Eq. (9) that $f_{LU}(a_{LU}, a_{KU})/f_{KU}(a_{LU}, a_{KU}) = w/r$. Therefore, privided f_U is well behaved, these two relations yield $a_{LU} = a_{LU}(w/r)$ and $a_{KU} = a_{KU}(w/r)$. Similarly, we can obtain $a_{LC} = a_{LC}(w/r)$ and $a_{KC} = a_{KC}(w/r)$.

[7] For this point, see Sandmo [17].

The production structure in the economy is thus determined by ten independent equations including (5) and (10)–(15). (Note that the system (14) contains four equations.) Our uncertainty model serves to determine ten variables: U, C, w, r, a_{LU}, a_{KU}, a_{LC}, a_{KC}, ρ and π_U. Hence, we have a determinate system with L, K, μ_U, p_C and γ being treated as parameters.

3. THE EQUATIONS OF CHANGE

We are now in a position to explore the comparative statics properties of the model introduced in the previous section. In so doing, it will prove useful to place the key equations describing our uncertainty model on a common footing by rewriting them in terms of relative changes in variables and parameters.

Let a hat "^" indicate the relative or percentage rate of change in a variable or parameter; for example, $\hat{L} = dL/L$, $\hat{w} = dw/w$, $\hat{a}_{KU} = da_{KU}/a_{KU}$, etc. Then differentiating the equilibrium full-employment equations (10) and (11) and rearranging slightly, we obtain

$$\lambda_{LU}\hat{U} + \lambda_{LC}\hat{C} = \hat{L} - (\lambda_{LU}\hat{a}_{LU} + \lambda_{LC}\hat{a}_{LC}), \tag{16}$$

$$\lambda_{KU}\hat{U} + \lambda_{KC}\hat{C} = \hat{K} - (\lambda_{KU}\hat{a}_{KU} + \lambda_{KC}\hat{a}_{KC}), \tag{17}$$

where $\lambda_{LU} = a_{LU}U/L$, $\lambda_{LC} = a_{LC}C/L$, $\lambda_{KU} = a_{KU}U/K$ and $\lambda_{KC} = a_{KC}C/K$. Clearly, the λ's refer to factor endowment's relative allocations in each sector. A fraction of labor force is allocated to the uncertainty sector (λ_{LU}), and this plus the fraction of the labor force allocated to the certainty sector (λ_{LC}) must add to unity:

$$\lambda_{LU} + \lambda_{LC} = 1. \tag{18}$$

Similar argument can proceed for the capital endowment to have

$$\lambda_{KU} + \lambda_{KC} = 1. \tag{19}$$

The equilibrium price-cost equations (12) and (13) are also rewritten as the following equations of relative rates of change:

$$\theta_{LU}\hat{w} + \theta_{KU}\hat{r} + \theta_R\hat{\rho} = \hat{\mu}_U - \theta_R\hat{\gamma} - (\theta_{LU}\hat{a}_{LU} + \theta_{KU}\hat{a}_{KU}). \tag{20}$$

$$\theta_{LC}\hat{w} + \theta_{KC}\hat{r} = \hat{p}_C - (\theta_{LC}\hat{a}_{LC} + \theta_{KC}\hat{a}_{KC}), \tag{21}$$

where $\theta_{LU} = a_{LU}w/\mu_U$, $\theta_{KU} = a_{KU}r/\mu_U$, $\theta_R = \gamma\rho/\mu_U$, $\theta_{LC} = a_{LC}w/p_C$ and $\theta_{KC} = a_{KC}r/p_C$. The θ's represent factor's relative shares in each sector. θ_{LU} and θ_{KU} are labor's share and capital's share in the uncertainty sector, respectively; while θ_R is "risk factor"'s share in the same sector. These three shares must add to unity:

$$\theta_{LU} + \theta_{KU} + \theta_R = 1. \tag{22}$$

In similar fashion we can find

$$\theta_{LC} + \theta_{KC} = 1, \tag{23}$$

which shows the zero profit condition for the certainty sector, with no risk factor being present. As far as physical factors are both employed and firms are risk averse, each θ should be positive and less than unity.

Let A be the determinant of the technology matrix, i.e., $A = |a|$. Obviously, A is positive if the uncertainty sector is relatively labor intensive (i.e., $a_{LU}/a_{KU} > a_{LC}/a_{KC}$) and negative if it is relatively capital intensive (i.e., $a_{LU}/a_{KU} < a_{LC}/a_{KC}$). Further, let us define the allocation matrix λ and the share matrix θ as follows:

$$\lambda = \begin{bmatrix} \lambda_{LU} & \lambda_{LC} \\ \lambda_{KU} & \lambda_{KC} \end{bmatrix}, \qquad \theta = \begin{bmatrix} \theta_{LU} & \theta_{KU} \\ \theta_{LC} & \theta_{KC} \end{bmatrix}.$$

It follows from Eqs. (18), (19), (22) and (23) that

$$\Lambda = |\lambda| = \lambda_{LU} - \lambda_{KU} = \lambda_{KC} - \lambda_{LC} \tag{24}$$

and

$$\Theta = |\theta| = \theta_{LU} - (1 - \theta_R)\,\theta_{LC} = (1 - \theta_R)\,\theta_{KC} - \theta_{KU}\,. \tag{25}$$

It is not difficult to show that Λ and Θ are both positive if the uncertainty sector is relatively labor intensive and negative if it is relatively capital intensive.[8]

Our next step is to simplify Eqs. (16), (17), (20) and (21) by eliminating every $\hat{a}_{ij}$ and, therefore, the λ and θ weighted sums of the $\hat{a}_{ij}$'s. First, minimize unit cost ($wa_{LU} + ra_{KU}$); since w and r are given to U-firms, this implies $w\,da_{LU} + r\,da_{KU} = 0$, hence the following relationship holds:

$$\theta_{LU}\hat{a}_{LU} + \theta_{KU}\hat{a}_{KU} = 0. \tag{26}$$

Similarly,

$$\theta_{LC}\hat{a}_{LC} + \theta_{KC}\hat{a}_{KC} = 0. \tag{27}$$

Secondly, we may establish relationships between factor price changes and factor proportions as follows:

$$\hat{a}_{KU} - \hat{a}_{LU} = \sigma_U(\hat{w} - \hat{r}), \tag{28}$$

$$\hat{a}_{KC} - \hat{a}_{LC} = \sigma_C(\hat{w} - \hat{r}), \tag{29}$$

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[8] This relation among the signs of A, Λ and Θ is now well established in the international trade literature. For example, see Jones [8] or Takayama [19, Chapter 2].

where σ_U and σ_C denote the elasticities of substitution between factors in each sector. In conjunction with Eqs. (26) and (27) and Eqs. (22) and (23), these two equations give solutions for the $\hat{a}$'s of both sectors:

$$\hat{a}_{LU} = -\frac{\theta_{KU}}{1-\theta_R}\sigma_U(\hat{w} - \hat{r}), \tag{30}$$

$$\hat{a}_{KU} = \frac{\theta_{LU}}{1-\theta_R}\sigma_U(\hat{w} - \hat{r}), \tag{31}$$

$$\hat{a}_{LC} = -\theta_{KC}\sigma_C(\hat{w} - \hat{r}), \tag{32}$$

$$\hat{a}_{KC} = \theta_{LC}\sigma_C(\hat{w} - \hat{r}). \tag{33}$$

We then substitute these solutions for the $\hat{a}$'s into Eqs. (16), (17), (20) and (21) to obtain

$$\lambda_{LU}\hat{U} + \lambda_{LC}\hat{C} - \delta_L(\hat{w} - \hat{r}) = \hat{L}, \tag{34}$$

$$\lambda_{KU}\hat{U} + \lambda_{KC}\hat{C} + \delta_K(\hat{w} - \hat{r}) = \hat{K}, \tag{35}$$

$$\theta_{LU}\hat{w} + \theta_{KU}\hat{r} = \hat{\mu}_U - \theta_R(\hat{\rho} + \hat{\gamma}), \tag{36}$$

$$\theta_{LC}\hat{w} + \theta_{KC}\hat{r} = \hat{p}_C, \tag{37}$$

where

$$\delta_L = \lambda_{LU}\frac{\theta_{KU}}{1-\theta_R}\sigma_U + \lambda_{LC}\theta_{KC}\sigma_C,$$

$$\delta_K = \lambda_{KU}\frac{\theta_{LU}}{1-\theta_R}\sigma_U + \lambda_{KC}\theta_{LC}\sigma_C.$$

It is noted that δ_L and δ_K are both positive.

Now, from Eq. (5), we find

$$E[v'_U(\pi_U)]\rho + E[v'_U(\pi_U)\epsilon] = 0, \tag{38}$$

whose total differentiation yields

$$E[v''_U(\pi_U)(\rho + \epsilon)\,d\pi_U] + E[v'_U(\pi_U)]\,d\rho = 0. \tag{39}$$

Since $d\pi_U = (\rho + \epsilon)(\gamma\,dU + U\,d\gamma) + \gamma U\,d\rho$ from (15), Eq. (39) is rewritten as

$$\alpha\hat{U} - \beta\hat{\rho} = -\alpha\hat{\gamma}, \tag{40}$$

where

$$\alpha = -E[v''_U(\pi_U)(\rho + \epsilon)^2]\,\gamma U, \tag{41}$$

$$\beta = \{E[v''_U(\pi_U)(\rho + \epsilon)]\,\gamma U + E[v'_U(\pi_U)]\}\rho. \tag{42}$$

Obviously, when U-firms are risk averse, α must be positive. Besides, the assumption that they exhibit decreasing absolute risk aversion is strong enough to determine the sign of β. For if we define the absolute risk aversion function by

$$R_a(\pi_U) = -v''_U(\pi_U)/v'_U(\pi_U),^{9}$$

we can obtain the following result:

LEMMA 2. *If* $R'_a(\pi_U) < 0$ *then* $\beta > 0$.

Proof. From Eq. (5), we find

$$E[v'_U(\pi_U)(p + \epsilon)] = 0. \tag{43}$$

Let ϵ^* be such that $v'_U(\pi_U^*)(p + \epsilon^*) = 0$ where $\pi_U^* = (\pi_U + \gamma\epsilon^*)U - wL_U - rK_U$. Since $\pi_U = \pi_U^* + \gamma(\epsilon - \epsilon^*)U$, it follows that

$$\pi_U \gtrless \pi_U^* \quad \text{according to whether} \quad \epsilon \gtrless \epsilon^* = -p.$$

If $R_a(\pi_U)$ is decreasing, we must have

$$-\frac{v''_U(\pi_U)}{v'_U(\pi_U)} = R_a(\pi_U) \lessgtr R_a(\pi_U^*) \quad \text{according to whether} \quad \pi_U \gtrless \pi_U^*.$$

Accordingly,

$$v''_U(\pi_U)(p + \epsilon) > -R_a(\pi_U^*)\, v'_U(\pi_U)(p + \epsilon) \qquad \text{for } any\ \epsilon.$$

Taking expected values, we obtain

$$E[v''_U(\pi_U)(p + \epsilon)] > -R_a(\pi_U^*)\, E[v'_U(\pi_U)(p + \epsilon)] = 0$$

because of (43). This ensures that β is positive. Q.E.D.

The system of equations of relative change for our uncertainty model is exhibited in Eqs. (34)–(37) and (40). In matrix form, it can be rewritten as

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & 0 & -\delta_L & \delta_L \\ \lambda_{KU} & \lambda_{KC} & 0 & \delta_K & -\delta_K \\ -\alpha & 0 & \beta & 0 & 0 \\ 0 & 0 & \theta_R & \theta_{LU} & \theta_{KU} \\ 0 & 0 & 0 & \theta_{LC} & \theta_{KC} \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{C} \\ \hat{p} \\ \hat{w} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} \hat{L} \\ \hat{K} \\ \alpha\hat{\gamma} \\ \hat{\mu}_U - \theta_R\hat{\gamma} \\ \hat{p}_C \end{bmatrix}. \tag{44}$$

The subsystem consisting of the first two equations in the system (44) represents the commodity output-factor endowment relationship and the subsystem consisting of the last two equations the factor price-commodity

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[9] The absolute risk aversion function $R_a(\pi_U)$ was first introduced by Arrow [1] and Pratt [12].

price parameter relationship. The third equation, which links $\hat{\rho}$ to $\hat{\gamma}$, tells us how the presence of price uncertainty contributes to make those two subsystems connected and thus the dual feature shared by any two-by-two model more complicated than in the traditional, risk-free case.

In the special case in which U-firms are just risk neutral, the terms θ_R, α and β all vanish; so that (44) is reduced to

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & -\delta_L & \delta_L \\ \lambda_{KU} & \lambda_{KC} & \delta_K & -\delta_K \\ 0 & 0 & \theta_{LU} & \theta_{KU} \\ 0 & 0 & \theta_{LC} & \theta_{KC} \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{C} \\ \hat{w} \\ \hat{r} \end{bmatrix} \begin{bmatrix} \hat{L} \\ \hat{K} \\ \hat{\mu}_U \\ \hat{p}_C \end{bmatrix}. \tag{45}$$

The risk-neutral system (45) has essentially the same structure as the risk-free system with respect to variables, parameters and coefficients; the only difference is that the relative change in *expected* price of the uncertainty sector is now present as a variable.[10] In this special system, the "financial" subsystem containing the last two equations may be entirely separated from the "physical" subsystem containing the first two equations in the following sense: The former subsystem itself is complete enough to determine the values of $\hat{w}$ and $\hat{r}$ given $\hat{\mu}_U$ and $\hat{p}_C$. As seen above, however, such a simple separation of the system ceases to be feasible in the more plausible case of risk aversion.

4. THE RELATIONSHIP BETWEEN FACTOR PRICES AND COMMODITY PRICE PARAMETERS

One of the salient features of the nonstochastic two-sector, two-factor model is that under certain conditions, factor prices are dependent only upon commodity prices, which is called the factor-price equalization theorem. We shall show in this section that such a nice relationship between factor prices and commodity prices cannot carry over to the world with price uncertainty, except for the special case that firms in the uncertainty sector are just risk neutral.

In order to make the system (44) more manageable, let us attempt to reduce the number of variables from five to three. From the last two equations in (44), we can solve for $\hat{w}$ and $\hat{r}$ to obtain

$$\hat{w} = \frac{1}{\Theta}(-\theta_{KC}\theta_R\hat{\rho} + \theta_{KC}\hat{\mu}_U - \theta_{KC}\theta_R\hat{\gamma} - \theta_{KU}\hat{p}_C), \tag{46}$$

$$\hat{r} = \frac{1}{\Theta}(\theta_{LC}\theta_R\hat{\rho} - \theta_{LC}\hat{\mu}_U + \theta_{LC}\theta_R\hat{\gamma} + \theta_{LU}\hat{p}_C). \tag{47}$$

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[10] For the risk-free system of equations of relative change, see Jones [8], for example.

In view of Eqs. (22) and (23), we thereby obtain

$$\hat{w} - \hat{r} = \frac{1}{\Theta}\{-\theta_R\hat{\rho} + \hat{\mu}_U - \theta_R\hat{\gamma} - (1 - \theta_R)\hat{p}_C\}. \tag{48}$$

As is clear from the above equations, the way in which factor prices are affected by changes in commodity price parameters is dependent upon in which direction and how much the risk bearing fee (ρ) responds to the changes. But $\hat{\rho}$ is linked to $\hat{L}$ and $\hat{K}$ through $\hat{U}$ and $\hat{C}$ (see the physical subsystem in (44)). This demonstrates the general dependence of factor price changes upon changes in the factor endowment. Therefore, the factor price equalization theorem no longer holds in our stochastic model, except for the special risk-neutral case in which θ_R, the risk factor's share, disappears.[11]

Substituting for $\hat{w}$ and $\hat{r}$ in (46) and (47) into the first three equations in the system (44) and rearranging slightly, we obtain

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & \frac{\delta_L\theta_R}{\Theta} \\ \lambda_{KU} & \lambda_{KC} & -\frac{\delta_K\theta_R}{\Theta} \\ -\alpha & 0 & \beta \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{C} \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \hat{L} + \frac{\delta_L}{\Theta}\hat{\mu}_U - \frac{\delta_L\theta_R}{\Theta}\hat{\gamma} - \frac{\delta_L(1-\theta_R)}{\Theta}\hat{p}_C \\ \hat{K} - \frac{\delta_K}{\Theta}\hat{\mu}_U + \frac{\delta_K\theta_R}{\Theta}\hat{\gamma} + \frac{\delta_K(1-\theta_R)}{\Theta}\hat{p}_C \\ \alpha\hat{\gamma} \end{bmatrix}. \tag{49}$$

Given changes in parameters, the values of $\hat{U}$, $\hat{C}$ and $\hat{\rho}$ are simultaneously determined in the reduced system (49), and then the values of $\hat{w}$ and $\hat{r}$ are separately determined by substituting for this $\hat{\rho}$ into Eqs. (46) and (47). Letting D be the determinant of the coefficient matrix in (49), namely,

$$D = \begin{vmatrix} \lambda_{LU} & \lambda_{LC} & \frac{\delta_L\theta_R}{\Theta} \\ \lambda_{KU} & \lambda_{KC} & -\frac{\delta_K\theta_R}{\Theta} \\ -\alpha & 0 & \beta \end{vmatrix},$$

we obtain

$$D = \beta A + \alpha\frac{\theta_R}{\Theta}(\delta_L\lambda_{KC} + \delta_K\lambda_{LC}). \tag{50}$$

It is noted that D is positive if the uncertainty sector is relatively labor intensive and negative if it is relatively capital intensive.

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[11] Even in the risk-free case, factor prices may fail to be equalized when there are more factors than commodities. In the light of this well-known result, the non-equalization of factor prices in the case of risk aversion could be quite understandable if the risk bearing fee (ρ) is interpreted as the payment for a *third* factor whose presence must be counted. See Samuelson [16].

5. Changes in Factor Endowments

In this and following sections, we shall investigate the comparative statics properties of the uncertainty model described above more deeply by focusing on a change in one parameter only. First of all, we are interested in seeing what happens to the equilibrium variables of the system when the labor or capital endowment changes. Specifically, we wish to know whether the following theorem of Rybczynski [13] remains unscrathed by the recognition of price uncertainty: An increase in the endowment of any factor leads to an expansion in the output of whichever sector is relatively intensive in its use of that factor, and to a contraction in the output of the other sector.

We shall state and prove the following result:

THEOREM 1. (i) *Suppose* $\hat{L} > 0$. *Then*

$$\hat{U} > 0, \qquad \hat{C} \lessgtr 0, \qquad \hat{\rho} > 0, \qquad \hat{w} < 0, \qquad \hat{r} > 0$$

if the uncertainty sector is relatively labor intensive;

$$\hat{U} < 0, \qquad \hat{C} > 0, \qquad \hat{\rho} < 0, \qquad \hat{w} \lessgtr 0, \qquad \hat{r} > 0$$

if it is relatively capital intensive.

(ii) *Next suppose* $\hat{K} > 0$. *Then*

$$\hat{U} < 0, \qquad \hat{C} > 0, \qquad \hat{\rho} < 0, \qquad \hat{w} > 0, \qquad \hat{r} \lessgtr 0$$

if the uncertainty sector is relatively labor intensive;

$$\hat{U} > 0, \qquad \hat{C} \lessgtr 0, \qquad \hat{\rho} > 0, \qquad \hat{w} > 0, \qquad \hat{r} < 0$$

if it is relatively capital intensive.

Proof. Letting all the parameters but $\hat{L}$ be zero in Eq. (49), we have

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & \dfrac{\delta_L \theta_R}{\Theta} \\ \lambda_{KU} & \lambda_{KC} & -\dfrac{\delta_K \theta_R}{\Theta} \\ -\alpha & 0 & \beta \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{C} \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \hat{L} \\ 0 \\ 0 \end{bmatrix}. \tag{51}$$

If we solve for the variables, it is not hard to see that

$$\hat{U} = \frac{\beta \lambda_{KC}}{D} \hat{L}, \tag{52}$$

$$\hat{C} = -\frac{\beta \lambda_{KU} \Theta - \alpha \delta_K \theta_R}{D\Theta} \hat{L}, \tag{53}$$

$$\hat{\rho} = \frac{\alpha \lambda_{KC}}{D} \hat{L}. \tag{54}$$

On the other hand, from Eqs. (46) and (47), we obtain

$$\hat{w} = -\frac{\theta_{KC}\theta_R}{\Theta}\hat{\rho}, \tag{55}$$

$$\hat{r} = \frac{\theta_{LC}\theta_R}{\Theta}\hat{\rho}. \tag{56}$$

Keeping in mind that Θ and D are positive (or negative) if the uncertainty sector is relative labor (or capital) intensive, Statement (i) immediately follows from Eqs. (52)-(56). Note that the sign of $\hat{C}$ is indeterminate when $\hat{L} > 0$ and $\Theta > 0$.

Now focusing on $\hat{K}$, the proof of Statement (ii) proceeds in a similar way and is omitted. For a later discussion, we only write the following formula:

$$\hat{\rho} = -\frac{\alpha\lambda_{LC}}{D}\hat{K}. \tag{57}$$

Q.E.D.

Theorem 1 has several interesting implications in comparison with the nonstochastic results. First, it is noted that the Rybczynski theorem may fail to hold for the certainty sector although it does hold for the uncertainty sector, provided that firms in the latter sector exhibit decreasing absolute risk aversion. More precisely, when the uncertainty sector is relatively labor intensive (or capital intensive), an increase in the labor endowment (or the capital endowment) *does not necessaroly result in a decrease in the output* of the certainty sector. Since there is an intimate link between the Heckscher-Ohlin theorem and the Rybczynski theorem, this also implies the partial invalidity of the former theorem.[12] Secondly, the risk bearing fee varies in either direction in response to an increase in the factor endowment, depending upon the factor intensity between the two sectors. This explains why the Rybczynski theorem may become partially invalid in the case of price uncertainty. In fact, from the first two equations in (51), we obtain

$$\lambda_{LU}\hat{U} + \lambda_{LC}\hat{C} = \hat{L} - \frac{\delta_L\theta_R}{\Theta}\hat{\rho},$$

$$\lambda_{KU}\hat{U} + \lambda_{KC}\hat{C} = -\frac{\delta_K\theta_R}{\Theta}\hat{\rho}.$$

---

[12] After the major part of my present work was finished, Das' paper [7] drew my attention. Following Batra's model closely, Das came to the same conclusion as above when the production technology is subject to uncertainty. Put differently, this means that the assumption of decreasing absolute risk aversion is not strong enough to guarantee the full validity of the Rybczynski theorem. Similar qualifications should be kept in mind for later discussions on the validity of other theorems.

Therefore, in case the uncertainty sector is relatively labor intensive (i.e., Θ is positive), the increase in the risk bearing fee has an effect that is similar to a decrease in the labor supply *and* an increase in capital, showing the possibility that both sectors expand simultaneously as the result of the initial increase in the labor supply.[13] An analogous argument can proceed for the implications of an increase in the capital supply (see Eq. (57)). Thirdly, an increase in the endowment of any factor must cause the price of that factor to decline and the price of the other factor to rise, whatever the factor intensity between the two factors. This result is well contrasted with the nonstochstic case in which factor prices are not affected at all by changes in factor endowments.

THEOREM 2. *Suppose* $\hat{L} = \hat{K} > 0$. *Then* (i) $\hat{C} > \hat{U} > 0$; (ii) $\hat{\rho} > 0$; (iii) $\hat{w} < 0$ *and* $\hat{r} > 0$ *if the uncertainty sector is relatively labor intensive*; $\hat{w} > 0$ *and* $\hat{r} < 0$ *if it is relatively capital intensive.*

Proof. Making use of Eqs. (49), (46) and (47), it is straightforward to show that for $\hat{L} = \hat{K} = \hat{E}$,

$$\hat{U} = \frac{\beta A}{D}\hat{E}; \quad \hat{C} = \frac{\beta A\Theta + \alpha\theta_R(\delta_L + \delta_K)}{D\Theta}\hat{E}; \quad \hat{U} - \hat{C} = -\frac{\alpha\theta_R(\delta_L + \delta_K)}{D\Theta}\hat{E};$$

$$\hat{\rho} = \frac{\alpha A}{D}\hat{E}; \qquad \hat{w} = -\frac{\alpha\theta_{KC}\theta_R A}{D\Theta}\hat{E}; \qquad \hat{r} = \frac{\alpha\theta_{LC}\theta_R A}{D\Theta}\hat{E}.$$

Using these formulas, the desired results follow immediately. Q.E.D.

We say that a uniform (relative) change in the two factor endowments has a "neutral" effect if it does not alter the composition of outputs and the ratio of factor prices. Theorem 2 shows that as contrasted with the nonstochastic, constant-returns-to-scale economy, this neutral effect of factor endowments is no longer valid in our stochastic framework. Suppose that both endowments expand at the same rate. Then the uncertainty output grows at a lower rate than the certainty output. Besides, the price of whichever factor is used relatively intensively in the uncertainty sector decreases and the price of the other factor increases. The reason why the uniform growth of both endowments produces such uneven effects on commodity outputs and factor prices is that it forces the risk bearing fee to increase, which in turn affects the uncertainty sector relatively unfavorably.[14]

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[13] It can be shown that $\hat{L} - (\delta_L\theta_R/\Theta)\hat{\rho} > 0$ if $\hat{L} > 0$ (use Eqs. (54) and (50)). Therefore, the response of $\hat{\rho}$ to $\hat{L}$ makes a positive $\hat{L}$ smaller than in the risk-free case, but not negative.

[14] As a corollary of Theorem 2, we see that there exists a sufficiently small, positive value ξ such that $\hat{L} = \hat{K} + \xi$, $\hat{U} < \hat{C}$ and the uncertainty sector is relatively labor intensive. Therefore, the magnification effect of factor supplies on commodity outputs cannot carry over to the case of risk aversion. For the magnification effect, see Jones [8].

6. CHANGES IN (EXPECTED) COMMODITY PRICES

In this section we shall consider the implications of a change in the expected price of the uncertainty commodity and of a change in the price of the certainty commodity. The theorem of Stolper and Samuelson [18, 15], the one that is dual to the Rybczynski theorem, says that an increase in the price of any commodity results in an increase in the price of whichever factor is used relatively intensively in the production of that commodity, and in a decline in the price of the other factor. We shall be especially interested in the extent to which the Stolper-Samuelson theorem carries over to the price uncertainty world.

THEOREM 3. (i) *Suppose* $\hat{\mu}_U > 0$. *Then*

$$\hat{U} > 0, \qquad \hat{C} < 0, \qquad \hat{\rho} > 0, \qquad \hat{w} > 0, \qquad \hat{r} < 0$$

if the uncertainty sector is relatively labor intensive;

$$\hat{U} > 0, \qquad \hat{C} < 0, \qquad \hat{\rho} > 0, \qquad \hat{w} < 0, \qquad \hat{r} > 0$$

if it is relatively capital intensive.

(ii) *Next suppose* $\hat{p}_C > 0$. *Then*

$$\hat{U} < 0, \qquad \hat{C} > 0, \qquad \hat{\rho} < 0, \qquad \hat{w} \lesseqgtr 0, \qquad \hat{r} > 0$$

if the uncertainty sector is relatively labor intensive;

$$\hat{U} < 0, \qquad \hat{C} > 0, \qquad \hat{\rho} < 0, \qquad \hat{w} > 0, \qquad \hat{r} \lesseqgtr 0$$

if it is relatively capital intensive.

Proof. Taking advantage of Eq. (50), the proof of this theorem is almost parallel to that of Theorem 1 and is omitted here. We only record the following formulas: For a change in μ_U,

$$\hat{\rho} = \frac{\alpha(\delta_L \lambda_{KC} + \delta_K \lambda_{LC})}{D\Theta} \hat{\mu}_U, \tag{58}$$

$$\hat{\mu}_U - \theta_R \hat{\rho} = \frac{\beta \Lambda}{D} \hat{\mu}_U; \tag{59}$$

and for a change in p_C,

$$\hat{\rho} = -\frac{\alpha(1 - \theta_R)(\delta_L \lambda_{KC} + \delta_K \lambda_{LC})}{D\Theta} \hat{p}_C, \tag{60}$$

$$\hat{r} - \hat{p}_C = \frac{\beta(1 - \theta_R)\, \theta_{LC} \Lambda}{D\Theta} \hat{p}_C, \tag{61}$$

$$\hat{p}_C - \hat{w} = \frac{\beta(1 - \theta_R)\, \theta_{KC} \Lambda}{D\Theta} \hat{p}_C. \tag{62}$$

Q.E.D.

We can give some interesting interpretations to Theorem 3. First of all, we see that although the Stolper-Samuelson theorem is valid for the case with changes in the expected price of the uncertainty commodity, it may fail to be valid for the case with changes in the price of the certainty commodity; in fact, an increase in the price of the certainty commodity does not always lead to a decrease in the price of the factor that is less intensively used in its production. The key to such a nonsymmetric result lies in the fact that the risk bearing fee is present in the stochastic world and that it rises when the expected price of the uncertainty output rises but falls when the price of the certainty output rises. To see this point more sharply, let all the parameters but $\hat{\mu}_U$ be zero in the last equations in (44) and rearrange them slightly to obtain

$$\begin{aligned} \theta_{LU}\hat{w} + \theta_{KU}\hat{r} &= \hat{\mu}_U - \theta_R\hat{\rho}, \\ \theta_{LC}\hat{w} + \theta_{KC}\hat{r} &= 0. \end{aligned}$$

In the light of Eq. (59), the term $(\hat{\mu}_U - \theta_R\hat{\rho})$ is positive if $\hat{\mu}_U$ is so, whatever the intersectoral factor intensity condition. Consequently, the effect of the positive response of $\hat{\rho}$ to $\hat{\mu}_U$ is to make a positive $\hat{\mu}_U$ merely smaller, but not negative; so that the Stolper-Samuelson result continues to hold in this case. On the other hand, letting all the parameters but $\hat{p}_C$ be zero in the last two equations in (44) yields

$$\begin{aligned} \theta_{LU}\hat{w} + \theta_{KU}\hat{r} &= -\theta_R\hat{\rho}, \\ \theta_{LC}\hat{w} + \theta_{KC}\hat{r} &= \hat{p}_C. \end{aligned}$$

Therefore, the decrease in ρ accompanied with an increase in p_C has the same effect on factor prices as would an increase in μ_U. This tell us that there is a possibility that contrary to the Stolper-Samuelson theorem, both w and r go in the same direction when p_C varies.[15] It should also be noticed that an increase in the (expected) price of any sector—the uncertainty or certainty sector—causes that sector to expand and the other sector to contract. This is exactly the same result as in the risk-free case.

THEOREM 4. *Suppose* $\hat{\mu}_U - \hat{p}_C > 0$. *Then* (i) $\hat{U} > 0$, $\hat{C} < 0$; (ii) $\hat{\rho} > 0$; (iii) $\hat{w} > 0$, $\hat{r} \gtreqless 0$, $\hat{w} > \hat{r}$ *if the uncertainty sector is relatively labor intensive*; $\hat{w} \gtreqless 0$, $\hat{r} > 0$, $\hat{r} > \hat{w}$ *if it is relatively capital intensive.*

Proof. The proof of this theorem is quite analogous to that of Theorem 2 and is omitted. Q.E.D.

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[15] Even so, w and r increase at different rates, depending upon the intersectoral factor intensity. As can be readily seen by Eqs. (61) and (62), w rises at a faster rate than r (i.e., the wage-rental ratio rises) if the certainty sector is relatively labor intensive, and conversely, if it is relatively capital intensive.

We say that a uniform (relative) change in the two (expected) commodity prices has a "neutral" effect if it does not influence the output ratio and the factor price ratio. As a dual of Theorem 2, Theorem 4 demonstrates that this neutral effect of (expected) commodity prices does not hold in the presence of price uncertainty, which is in marked contrast with the risk-free case. Suppose that both (expected) commodity prices rise at the same rate. Then first of all, the uncertainty output must increase and the certainty output must decrease. Secondly, the price of whichever factor is intensive in the use of the uncertainty output must rise at a faster rate than the price of the other factor, with the possibility that the latter price might fall. Note that the risk bearing fee is present in the risk aversion case and is positively responsive to the uniform commodity price change. This is the reason why the uniform change affects both physical and financial variables unevenly as stated above.[16]

7. CHANGES IN PRICE UNCERTAINTY

We turn to the implications of a change in price uncertainty for the equilibrium values of the model. Although this was already discussed by Batra [4, Chapter 6], our approach taken here gives an alternative, and hopefully better, method of proof by shedding a new light on the duality featute of the economy.

THEOREM 5. *Suppose* $\hat{\gamma} > 0$. *Then*

(i) $\hat{C} < 0, \quad \hat{C} > 0, \quad \hat{\rho} + \hat{\gamma} > 0, \quad \hat{w} < 0, \quad \hat{r} > 0$

if the uncertainty sector is relatively labor intensive;

(ii) $\hat{C} < 0, \quad \hat{C} > 0, \quad \hat{\rho} + \hat{\gamma} < 0, \quad \hat{w} > 0, \quad \hat{r} < 0$

if it is relatively capital intensive.

Proof. Limiting our attention to $\hat{\gamma}$, the proof is direct. We only write the following formulas:

$$\hat{\rho} = -\frac{\nu\{\theta_R(\delta_L\lambda_{KC} + \delta_K\lambda_{LC}) - \Lambda\Theta\}}{D\Theta}\hat{\gamma}. \tag{63}$$

$$\hat{\rho} + \hat{\gamma} = \frac{(\nu + \beta)\Lambda}{D}\hat{\gamma}. \tag{64}$$

$$\hat{w} = -\frac{\theta_{KC}\theta_R}{\Theta}(\hat{\rho} + \hat{\gamma}). \tag{65}$$

$$\hat{r} = \frac{\theta_{LC}\theta_R}{\Theta}(\hat{\rho} + \hat{\gamma}). \tag{66}$$

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[16] Theorem 4 clearly implies that the magnification effect of commodity prices on factor prices cannot be extended to the world with price uncertainty.

Theorem 5 is interesting in several respects. An increase in price uncertainty contracts the uncertainty sector and expands the certainty sector. Although the risk bearing fee may go either way, the sign of the "combined risk effect" represented by the sum $(\hat{\rho} + \hat{\gamma})$ must be positive (compare Eqs. (63) and (64)). This plays a crucial role in determining the signs of $\hat{w}$ and $\hat{r}$ as is seen by Eqs. (65) and (66). In fact, the price uncertainty increase results in a decline in the price of whichever factor is used relatively intensively and in a rise in the price of the other factor. Comparison of Theorems 3 and 5 shows that as may be naturally expected, the implications of a change in the expected price of the uncertainty output and those of a change in the price variation are antithetical in the case of risk aversion.

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REFERENCES

1. K. J. Arrow, "Essays in the Theory of Risk-Bearing," North-Holland, Amsterdam, 1970.
2. R. N. Batra, Resource allocation in a general equilibrium model of production under uncertainty, *J. Econ. Theory* **8** (1974), 50–63.
3. R. N. Batra, Production uncertainty and the Heckscher–Ohlin theorem, *Rev. Econ. Stud.* **42** (1975), 259–268.
4. R. N. Batra, "The Pure Theory of International Trade under Uncertainty," Macmillan & Co., London/Basingstoke, 1975.
5. R. E. Caves and R. W. Jones, "World Trade and Payments," Little, Brown, Boston, 1973.
6. J. D. Coppock, "International Economic Instability: The Experience after World War II," McGraw-Hill, New York, 1962.
7. S. K. Das, Uncertainty and the Heckscher–Ohlin theorem: a comment, *Rev. Econ. Stud.* **44** (1977), 189–190.
8. R. W. Jones, The structure of simple general equilibrium models, *J. Polit. Econ.* **53** (1965), 557–572.
9. M. C. Kemp, "The Pure Theory of International Trade and Investment," Prentice-Hall, Englewood Cliffs, N.J., 1969.
10. M. C. Kemp, "Three Topics in the Theory of International Trade," North-Holland, Amsterdam, 1976.
11. R. Komiya and A. Amano, "International Economics," Iwanami, Tokyo, 1972.
12. J. W. Pratt, Risk aversion in the small and in the large, *Econometrica* **32** (1964), 122–136.
13. T. N. Rybczynski, Factor endowments and relative commodity prices, *Economica* **33** (1955), 336–341.
14. Y Sakai, The theory of the firm under price uncertainty. *Econ. Stud. Quart.* **28** (1977), 31–39.

15. P. A. SAMUELSON, International factor price equalization once again, *Econ J.* **59** (1949), 181–197.
16. P. A. SAMUELSON, Prices of factors and goods in general equilibrium, *Rev. Econ. Stud.* **21** (1953–54), 1–20.
17. A. SANDMO, On the theory of the competitive firm under price uncertainty, *Amer. Econ. Rev.* **61** (1971), 65–73.
18. W. F. STOLPER AND P. A. SAMUELSON, Protection and real wages, *Rev. Econ. Stud.* **9** (1941), 58–73.
19. A. TAKAYAMA, "International Economics," Holt, Rinehart & Winston, New York, 1972.
20. J. S. FLEMMING, S. J. TURNOVSKY, AND M. C. KEMP, On the choice of numeraire and certainty price in general equilibrium models of price uncertainty, *Rev. Econ. Stud.* **44** (1977), 573–583.
21. W. MAYER, The Rybczynski, Stolper-Samuelson, and factor-price equalization theorems under price uncertainty, *Amer. Econ. Rev.* **66** (1976), 797–808.

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A Spatial Version of the Nonsubstitution Theorem*

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The purpose of nonsubstitution theory is to find conditions on the production sector of an economy such that equilibrium prices and the choice of techniques do not depend on the structure of final demand. Typically the following assumptions are needed to obtain a nonsubstitution theorem:

(I) There is only one non-produced good
(II) Constant returns to scale hold for all productive activities
(III) There is no "genuine" joint production.

For a thorough discussion we refer to Bliss [1, chapter 11].

The only non-produced good might be identified as labor provided that labor is of a homogeneous type and freely movable between sectors. Land differs from labor as a factor of production because land can hardly be considered mobile even if all land is of the same quality. Therefore, land at different locations has to be treated like different factors and, as a consequence, the structure of final demand, given by its distribution over space, influences local f.o.b. prices and land rents. A nonsubstitution theorem in the narrow sense does not hold and the question arises whether any other set of invariants related to the price-land rent system exists which does not depend on final demand.

Such invariants have been found by Goldstein and Moses [2] in the case of two commodities and by Schweizer and Varaiya [4, 5] in the general case. The present paper extends results due to Schweizer and Varaiya [4] in the direction that we allow here in addition to the set L of commodities produced within the monocentric city for a set F of factors of production to be imported at the central business district (CBD). These factors can be thought of as different types of raw material or as intermediate goods produced outside the

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city. It is shown that, given a Leontief technology with fixed coefficients, production takes place in rings around the CBD and that the production pattern given by the ordering of these rings does not depend on final demand at the CBD. The notion of production pattern, forming a system of invariants in the above sense, allows to calculate equilibrium prices, given final demand, and to study how the structure of spatial equilibrium changes with the interest rate.

Note that the assumption of fixed input coefficients is crucial for this ordering to be independent on the structure of final demand at the CBD. This follows from results due to Schweizer and Varaiya [5]. Moreover, the assumption that no production process with output belonging to the set F of imported factors makes any land rent offers cannot be relaxed either. Nevertheless the results of the present paper seem to be noteworthy for simplicity as well as generality. The way of proving them turns out rather intricate, especially as compared to corresponding proofs for the case $F =$ empty in Schweizer and Varaiya [4].

The model can be described as follows:

At distance u, $u \geq u^0$, from the CBD the quantity $\theta(u)$ of land is available for production. L denotes the set of produced commodities, F the set of non-produced factors of production to be imported at the CBD. Transportation occurs radially and to ship one unit of commodity i, $i \in L \cup F$, over one unit of distance costs t_i, $t_i > 0$ for $i \in L \cup F$, units of "money". $t = (\cdots t_i \cdots)$ is the (row) vector of these cost coefficients. Production possibilities are given by a Leontief technology with fixed coefficients and without joint products. To produce one unit of commodity j, $j \in L$, a_{ij} units of commodity i, $i \in L \cup F$, and a_j units of land are needed. $A = (a_{ij})$ denotes the input matrix with as many rows and as many columns as there are goods in $L \cup F$ and in L respectively and $I = (\delta_{ij})$ stands for the matrix of the same format such that $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise. $a = (\cdots a_j \cdots)$ is the (row) vector of land use coefficients. Final demand occurs only at the CBD and only factors of production are imported from outside, all other commodities needed to cover final as well as intermediate demand being produced within the monocentric city. The arrangement of production over space can be described by vectors of production levels $x(u)$ and by vectors $s(u)$ and $\sigma(u)$ denoting commodity flow towards the center and towards the periphery, respectively, of the city at distance u from the CBD.

DEFINITION 1. A triplet $(x(u), s(u), \sigma(u))$, $x(u) \in R_+^L$, $s(u)$ and $\sigma(u) \in R_+^{L+F}$, $u^0 \leq u \leq \bar{u}$, of profiles is called an *activity profile* if, for $u^0 \leq u \leq \bar{u}$,

$$(I - A)\, x(u) = -\dot{s}(u) + \dot{\sigma}(u) \tag{1}$$

$$ax(u) = \theta(u) \tag{2}$$

and

$$\sigma_L(u^0) = 0, \ \sigma(\bar{u}) = s(\bar{u}) = 0. \tag{3}$$

$s(u^0)$ is called the *final output* at the CBD of the activity profile.

L and F denote the sets as well as their corresponding cardinality. The following notation is used throughout: R_+^L is the nonnegative orthant of L-dimensional real vector space R^L and similarly for R_+^{L+F}. Given $z \in R^{L+F}$, z_L denotes the vector in R^L of the first L coordinates of z. "Dot" means derivative with respect to distance u.

By definition, activity profiles represent stationary states of production in space, i.e. steady states of zero growth in the sense of capital theory. There is a unit period of production at the beginning of which inputs are needed to obtain output at the end of that period. Since activity levels and flows do not depend on time parameters in stationary states, reference to time can be omitted.

We restrict our attention to activity profiles which can be sustained by a system of f.o.b. prices and of land rents as spatial equilibrium.

DEFINITION 2. Given a vector of factor prices P_F at the CBD and an interest factor R, a continuous profile $(p(u), r(u))$ of f.o.b. prices $p(u) \in R_+^{L+F}$ and land rents $r(u) \in R_+$ sustains the activity profile $(x(u), s(u), \sigma(u))$, $u^0 \leqslant u \leqslant \bar{u}$, as a *spatial equilibrium* (at interest factor R) if, for any distance u, $u^0 \leqslant u \leqslant \bar{u}$,

$$\begin{aligned} p(u)(I - RA) &\leqslant r(u)\,a \\ p(u)(I - RA)\,x(u) &= r(u)\,a x(u) \end{aligned} \tag{4}$$

$$\dot{p}_i(u) = \begin{cases} -t_i & \text{if } s_i(u) > 0 \\ t_i & \text{if } \sigma_i(u) > 0 \\ \in [-t_i, t_i] & \text{otherwise} \end{cases} \tag{5}$$

and

$$r(\bar{u}) = 0, \ p_F(u^0) = P_F \tag{6}$$

If (1)-(6) hold then $(x(u), s(u), \sigma(u), p(u), r(u))$, $u^0 \leqslant u \leqslant \bar{u}$ is called an *equilibrium* profile.

(4) means that profits are maximized at all distances, (5) prevents arbitrage in the transport sector but if transportation takes place then costs are covered by the difference in f.o.b. prices and (6) requires that land at the edge of the city yields zero rent and that f.o.b. prices of factors coincide with given factor prices at the CBD.

Before proceeding a remark on optimality might be due. Suppose the activity profile $(x(u), s(u), \sigma(u))$, $u^0 \leqslant u \leqslant \bar{u}$, can be sustained as a spatial equilibrium at interest factor $R = 1$ and suppose we are given another activity profile $(\tilde{x}(u), \tilde{s}(u), \tilde{\sigma}(u))$, $u^0 \leqslant u \leqslant \bar{u}$, with the same final output, i.e. $\tilde{s}_F(u^0) = s_F(u^0)$, then

$$P_F\sigma_F(u^0) + \int_{u^0}^{\bar{u}} t[s(u) + \sigma(u)]\,du \leqslant P_F\tilde{\sigma}_F(u^0) + \int_{u^0}^{\bar{u}} t[\tilde{s}(u) + \tilde{\sigma}(u)]\,du \qquad (7)$$

which means that the sum of costs of factors and of transportation is minimized at an equilibrium profile provided that the interest rate equals zero, i.e. $R = 1$. A proof of (7) can be adapted easily from Schweizer and Varaiya [4, p. 235]. In this sense $p(u)$ and $r(u)$ play the role of shadow prices for an optimization problem. If $R \neq 1$ no such interpretation seems to be possible.

THEOREM 1. *Given an interest factor* R, $0 \leqslant R < \bar{R}$, *there exists a sequence* (I^k, S^k, Σ^k) of sets, $I^k \subseteq L$, S^k and $\Sigma^k \subseteq L \cup F$, $k = 1,\ldots, K$, with the following property:

Suppose we are given an activity profile $(x(u), s(u), \sigma(u))$, $u^0 \leqslant u \leqslant \bar{u}$, such that, for $u^{k-1} < u < u^k$,

$$x_j(u) > 0 \text{ only if } j \in I^k$$
$$s_j(u) > 0 \text{ only if } j \in S^k$$
$$\sigma_j(u) > 0 \text{ only if } j \in \Sigma^k,$$

$k = 1,\ldots, K$, where $u^0 < u^1 < \cdots < u^K = \bar{u}$. Then such an activity profile can be sustrained by a system of f.o.b. prices and land rents as a spatial equilibrium.

Hereby $\bar{R}$ is defined as follows: The set of interest factors $R \geqslant 0$ for which $[(I - RA)\,y]_L \gg 0$ admits a nonnegative solution y can be shown to be an interval of the form $[0, \rho)$. Define $\bar{R} = \rho$. Note that, for $0 \leqslant R < \bar{R}$, the $L \times L$ matrix $(I - RA)_L$ formed by all rows of $I - RA$ whose indices belong to L is nonnegative invertible and so is any principal minor of it. For a proof we refer to Nikaido [3, pp. 100-105].

The sequence (I^k, S^k, Σ^k), $k = 1,\ldots, K$ of theorem 1 is called the *canonical production pattern*. It depends on the coefficients as given by A, a and t as well as on the interest factor R but not on final demand Q nor on the shape of the city described by $\theta(\cdot)$ nor on factor prices P_F at the CBD. The canonical production pattern describes the ordering of production rings around the CBD, ring $k = 1$ being closest to the CBD.

THEOREM 2. *The canonical production pattern given in theorem* 1 *has the following properties*: $(k = 1,\ldots, K)$

$$I^k \cup S^k \cup \Sigma^k = L \cup F \qquad (8)$$

$$S^k = \bigcup_{l>k} (I^l \cap S^l) \tag{9}$$

$$\Sigma^k = F \bigcup_{l<k} (I^l \cap \Sigma^l) \tag{10}$$

Moreover, $S^1 = L$, $\Sigma^1 = F$ and

$$I^k \cap S^{k+1} \text{ empty} \tag{9a}$$

$$I^{k+1} \cap \Sigma^k \text{ empty}, \ k = 1,\ldots, K-1 \tag{10a}$$

Furthermore it is shown that each production process j operates over a connected set of rings $k = k_j, k_j + 1, \ldots, K_j$ covering intermediate demand for commodity j locally over these rings. Intermediate demand for j in rings closer to the CBD as well as final demand is covered from production in ring k_j whereas intermediate needs arising farther out are produced in ring K_j.

THEOREM 3. *Suppose* $\bar{R} > 1$. *Then, given any vector of final demand* $Q > 0$ *at the CBD, a sequence* $x^k \geq 0$, $k = 1,\ldots, K$, *of activity vectors can be found such that*

$$x_j^k = 0 \text{ if } j \in L \setminus I^k$$

and

$$[(I-A)\, x^k]_j = \begin{array}{ll} Q_j + [A(x^1 + \cdots + x^{k-1})]_j & \text{if } j \in I^k \cap S^k \\ [A(x^{k+1} + \cdots + x^K)]_j & \text{if } j \in I^k \cap \Sigma^k \\ 0 & \text{if } j \in I^k \setminus (S^k \cup \Sigma^k) \end{array}$$

(*Note that, by convention,* $[A(x^{k+1} + \cdots + x^K)]_j = 0$ *for* $k = K$).

We point out that theorem 3 might be used to construct an activity profile with final output Q_L of the canonical production pattern in the following way: First, distances $u^0 < u^1 < \cdots < u^K$ are chosen such that

$$\int_{u^{k-1}}^{u^k} \theta(u)\, du = ax^k, \ k = 1,\ldots, K.$$

This can be done provided that enough land is available to produce Q_L, i.e. if

$$a(x^1 + \cdots + x^K) < \int_{u>u^0} \theta(u)\, du.$$

Second, the sequence of activity vectors x^k of theorem 3 are allocated as follows:

$$x(u) = [ax^k]^{-1}\, \theta(u)\, x^k, \ u^{k-1} < u < u^k, \ k = 1,\ldots, K.$$

Third, a continuous flow of commodities inwards and outwards is given as

$$s(u^K) = 0, \quad \sigma_L(u^0) = 0, \quad \sigma_F(u^0) = [A(x^1 + \cdots + x^K)]_F$$

and

$$\dot{s}_j(u) = \begin{cases} -[(I - A)\,x(u)]_j & \text{if } j \in S^k \\ 0 & \text{if } j \notin S^k \end{cases}$$

$$\dot{\sigma}_j(u) = \begin{cases} [(I - A)\,x(u)]_j & \text{if } j \in \Sigma^k \\ 0 & \text{if } j \notin \Sigma^k \end{cases}$$

$u^{k-1} \leq u \leq u^k$, $k = 1, \ldots, K$.

THEOREM 4. *The profile as constructed above is an activity profile with final output Q_L of the canonical production pattern.*

We conclude that, no matter what final demand amounts to nor what shape the monocentric city is of, there always exists an equilibrium profile of the canonical pattern. Such a result on nonsubstitution of production patterns might be called a spatial version of the nonsubstitution theorem.

In addition it can be shown that any spatial equilibrium has to be of the canonical production pattern. For the case $R = 1$ the proof is adapted easily from Schweizer and Varaiya [4, Theorem 3] whereas for $R \neq 1$ an auxiliary construction allows to reduce that case to the former. Details of that construction will not be given here.

The proofs for theorems 1-4 are worked out in several steps. For a given interest factor R, $0 \leq R < \bar{R}$, the canonical production pattern can be constructed inductively as follows:

$$t_j^1 = \begin{cases} t_j & \text{if } j \in L \\ -t_j & \text{if } j \in F \end{cases} \quad \text{and } \gamma^1 = \max_{j \in L} t^1 b_j$$

where $b_j = (b_{1j}, \ldots, b_{L+F,j})'$ and

$$b_{ij} = \begin{cases} a_j^{-1}(1 - Ra_{ij}) & \text{if } i = j \\ -a_j^{-1} Ra_{ij} & \text{if } i \neq j \end{cases}$$

If $t^k \in R^{L+F}$ and $\gamma^k \in R$ are constructed then

$$I^k = \{j \in L \mid t^k b_j = \gamma^k\}$$
$$S^k = \{j \in L \cup F \mid t_j^k = t_j\}$$
$$\Sigma^k = \{j \in L \cup F \mid t_j^k = -t_j\}$$

Note that t^k and γ^k are going to be the gradients of f.o.b. prices and land rent at ring k, respectively.

If $S^k \subseteq I^k$ define $K = k$ and the construction is finished. Otherwise we proceed distinguishing two cases:

Case (*a*) $I^k \setminus \Sigma^k$ nonempty.
A nonnegative (row) vector Δ^k of dimension $L + F$ is defined by

$$\Delta^k b_j = 1 \text{ if } j \in I^k \setminus \Sigma^k$$
$$\Delta_j^k = 0 \text{ if } j \notin I^k \setminus \Sigma^k.$$

Note that the principal minor $\tilde{B}$ of the matrix $(b_1 \cdots b_L)_L$ composed out of all columns and rows whose indices belong to $I^k \setminus \Sigma^k$ can be obtained by multiplying the j-th column of the corresponding principal minor of $(I - RA)_L$ with a_j^{-1}. It follows from familiar results on nonnegative matrices (see Nikaido (3, pp. 100-105]) that $\tilde{B}$ has to be nonnegative invertible. Hence Δ^k is well-defined and nonnegative.

Moreover, we define

$$\lambda_+^k = \sup\{\lambda \mid (t^k - \lambda\Delta^k)\, b_j \leq \gamma^k - \lambda \text{ for all } j \in L \setminus \Sigma^k\}$$
$$\lambda_-^k = \sup\{\lambda \mid t^k - \lambda\Delta^k \geq -t\}$$
$$\lambda^k = \inf\{\lambda_+^k, \lambda_-^k\}.$$

Set $t^{k+1} = t^k - \lambda^k\Delta^k$ and $\gamma^{k+1} = \gamma^k - \lambda^k$.

Case (b) $I^k \setminus \Sigma^k$ empty.
Set $t^{k+1} = t^k$ and $\gamma^{k+1} = \mathrm{Max}_{j \in S^k \setminus I^k}\, t^{k+1} b_j$

PROPOSITION. The sequence $(t^k, \gamma^k, I^k, S^k, \Sigma^k)$ is well-defined by the above procedure and finite, i.e. there exists a number K such that $S^k \not\subseteq I^k$, $k < K$, and $S^K \subseteq I^K$. Moreover

$$\Sigma^1 = F \text{ and } I^1 \cap \Sigma^1 \text{ empty} \tag{11}$$

$$\gamma^{k-1} > \gamma^k,\ k \geq 2 \tag{12}$$

$$t \geq t^{k-1} \geq t^k \geq -t,\ k \geq 2 \tag{13}$$

$$t_j^{k-1} > t_j^k \text{ iff } j \in I^{k-1} \setminus \Sigma^{k-1},\ k \geq 2 \tag{14}$$

$$t^k b_j \begin{array}{c} > \\ = \\ < \end{array} \gamma^k \text{ according as } j \in \begin{array}{c} (\Sigma^k \cap L) \setminus I^k \\ I^k, \\ S^k \setminus I^k \end{array} \quad k \geq 1 \tag{15}$$

$$I^k \cap (S^k \cup \Sigma^k) \text{ nonempty},\ k \geq 1 \text{ and} \tag{16}$$

$$\gamma^K > 0 \tag{17}$$

COROLLARY. (a) $S^1 = L$

(b) $\Sigma^k \supset F$, $k = 1, \ldots, K$

(c) $I^k \cup S^k \cup \Sigma^k = L \cup F$, $k = 1, \ldots, K$

(d) $S^k \subset S^{k-1} \subset S^k \cup (I^{k-1} \setminus \Sigma^{k-1})$, $k = 2, \ldots, K$.

(Proof of the corollary: (a) holds by definition. It follows from (13) that $\Sigma^k \subseteq \Sigma^{k+1}$ and, since $\Sigma^1 = F$ by definition, (b) is established. As for (c) suppose $j \in L$. Then, by (15), $j \in (\Sigma^k \cap L) \cup I^k \cup$ [illegible] $\cup S^k$ [illegible] hence $j \in I^k \cup S^k \cup \Sigma^k$. Using (b), (c) can now be proved easily. As for (d), [illegible] $S^k \supseteq S^{k+1}$ follows from (13). Suppose $j \in S^{k+1}$ [illegible] S^k. Then, by definition, [illegible] hence, by (14), $j \in I^{k+1} \setminus \Sigma^{k+1}$ and (d) is proved.)

In the following the proposition will be proved by induction on k. Note that, if (12)-(16) is established for $k \leq \bar{k}$, corollary (b)-(d) holds for $k \leq \bar{k}$ too. This will be used repeatedly in the proof of the proposition.

Proof. (11) holds by definition. We proceed by induction on k. For $k = 1$, (15) and (16) can be verified easily. If $S^1 \subseteq I^1$ then $K = 1$ and, since $0 < R < \bar{R}$, $\gamma^1 > 0$ and the proposition is established in this case. Otherwise, suppose (12)-(16) hold for $k \leq \bar{k}$. If $S^{\bar{k}} \subseteq I^{\bar{k}}$ then $K = \bar{k}$ and, by corollary (c), $I^{\bar{k}} \cup \Sigma^{\bar{k}} = L \cup F$.

It follows from (15) that

$$t^k b_j = \gamma^k \quad \text{if } j \in I^k$$

$$t_j^k = -t_j \text{ if } j \notin I^k,\ k = \bar{k}.$$

Suppose $\gamma^k \leq 0$. Then, by nonnegative invertibility, $t^k \leq 0$ hence S^k empty, $k = \bar{k}$. It follows from corollary (d) that $S^{k-1} \subseteq I^{k-1}$ contradicting the assumption on when to stop the construction of the canonical production pattern. Therefore $\gamma^k > 0$ thus proving the proposition in the case where $S^k \subseteq I^k$, $k = \bar{k}$.

If $S^k \not\subseteq I^k$, $k = \bar{k}$, we consider the case first where $I^k \setminus \Sigma^k$ nonempty, $k = \bar{k}$. We claim that

$$L \setminus \Sigma^k = (I^k \setminus \Sigma^k) \cup (S^k \setminus I^k),\ k = \bar{k} \tag{18}$$

To show this, take any $j \in L \setminus \Sigma^k$. Then, by corollary (b), $j \in L \setminus \Sigma^1$ hence, by corollary (c), $j \in (I^k \setminus \Sigma^k) \cup (S^k \setminus I^k)$ thus proving that the lefthand side of (18) is contained in the righthand side. As for the converse we note that, by definition, $I^k \subseteq L$ and, by (13) and corollary (a), $L = S^1 \supseteq S^k$. (18) is established.

If $j \in I^k \setminus \Sigma^k$ then by construction, $(t^k - \lambda \Delta^k)\, b_j = \gamma^k - \lambda$ for all λ whereas if $j \in S^k \setminus I^k$ (which is nonempty) then by (15), $t^k b_j < \gamma^k$, $k = \bar{k}$. Since $\Delta^k b_j \leq 0$ it follows from (18) that $0 < \lambda_+{}^k < \infty$, $k = \bar{k}$.

Since $I^k \setminus \Sigma^k$ is nonempty and since, for $j \in I^k \setminus \Sigma^k$, $t_j^k > -t_j$, it follows that $0 < \lambda < \infty$, $k = \bar{k}$. Therefore (12)-(14) hold for $k = \bar{k} + 1$.

As for (15) choose $j \in (\Sigma^{k+1} \cap L) \setminus I^{k+1}$, $k = \bar{k}$. Then, since $\Sigma^{k+1} \supseteq \Sigma^k$, $j \in \Sigma^{k+1} \setminus \Sigma^k$ or $j \in \Sigma^k$, $k = \bar{k}$. In the first case it follows from (14) that $j \in I^k \setminus \Sigma^k$ hence, by construction, $t^{k+1} b_j = (t^k - \lambda^k \Delta^k)\, b_j = \gamma^k - \lambda^k = \gamma^{k+1}$ and $j \in I^{k+1}$, a contradiction. Therefore, $j \in \Sigma^k$ and, as a consequence of (12) and (13), $t^{k+1}b_j \geq t^k b_j \geq \gamma^k > \gamma^{k+1}$, $k = \bar{k}$, as was to be shown. If $j \in L$ but $\notin (\Sigma^{k+1} \cap L)\backslash I^{k+1}$, $k = \bar{k}$, then either $j \in \Sigma^{k+1} \cap I^{k+1}$ or $j \in L\backslash\Sigma^{k+1}$. In the first case $t^{k+1}b_j = \gamma^{k+1}$ and in the second case, since $\Sigma^k \subseteq \Sigma^{k+1}$, $j \in L\backslash\Sigma^k$ hence, by construction, $t^{k+1}b_j < \gamma^{k+1}$ and (15) is established for $k = \bar{k} - 1$.

As for (16) observe that if $\lambda^k = \lambda_\cdot^k$, $k = \bar{k}$, then, by (18), there exists $j \in S^k\backslash I^k$ such that $j \in I^{k+1}$. It follows from (14) that $j \in S^{k+1}$ hence $j \in I^{k+1} \cap S^{k+1}$, whereas if $\lambda^k = \lambda^{\prime k}$, $k = \bar{k}$, then there exists $j \in I^k\backslash\Sigma^k$ such that $j \in \Sigma^{k+1}$. Since, by construction, $I^k\backslash\Sigma^k \subset I^{k+1}$ it follows that $j \in I^{k+1} \cap \Sigma^{k+1}$ thus proving (16) for $k = \bar{k} + 1$.

The case where $I^k\backslash\Sigma^l$ empty, $k = \bar{k}$, can be handled similarly. We omit details.

Finally it remains to be shown that the sequence as constructed above is finite. Since

$$t^k b_j = \gamma^k \quad \text{if } j \in I^k$$

$$t_j^k = \begin{cases} t_j & \text{if } j \in S^k \\ -t_j & \text{if } j \in \Sigma^k \end{cases}$$

and since $I^k \cup S^k \cup \Sigma^k = L \cup F$ and since $I^k \cap (S^k \cup \Sigma^k)$ nonempty it can be verified easily that, given I^k, S^k and Σ^k, t^k and γ^k are determined uniquely. Since the choice of I^k, S^k and Σ^k is restricted to a finite class and since (12) holds it follows that the sequence has to be finite. Q.E.D.

Proof of theorem 2 (8) follows from (c). (9) can be proved by induction on k backwards: Since $S^K \subseteq I^K$, (9) holds for $k = K$. If (9) holds for $k = \bar{k} + 1$ then it holds for $k = \bar{k}$. To prove this, choose $j \in S^k$, $k = \bar{k}$. Then, by (14), $j \in S^{k+1}$ or $j \in I^k\backslash\Sigma^k$ hence $S^k \subseteq \bigcup_{l \geq k} (I^l \cap S^l)$. Since $S^k \supseteq S^{k+1}$ and since $S^{k+1} = \bigcup_{l \geq k+1} (I^l \cap S^l)$, $k = \bar{k}$, it follows that $S^k \supset \bigcup_{l \geq k} (I^l \cap S^l)$ thus proving (9) for $k = \bar{k}$.

As for (10) we observe that, for $k = 1$, (10) follows from (11). If (10) holds for $k = \bar{k}$ then it holds for $k = \bar{k} + 1$. To show this take $j \in \Sigma^{k+1}$ then either $j \in \Sigma^k$ or $j \in I^k\backslash\Sigma^k$ and, by definition, $j \in I^{k+1}$ thus proving that $\Sigma^{k+1} \subset F$ $\bigcup_{l \leq k+1} \cup (I^l \cap \Sigma^l)$, $k = \bar{k}$. Since $\Sigma^{k+1} \supseteq \Sigma^k$, it follows that $\Sigma^{k+1} \supset \bigcup_{l \leq k+1}$ $(I^l \cap \Sigma^l)$ thus proving that (10) holds for $k = \bar{k} + 1$.

Next, suppose that $j \in S^{k+1}$. Then, by (9), $j \in S^k$ hence $t_j^k = t_j^{k+1}$ and by (14), $j \notin I^k$ thus proving (9a).

As for (10a) suppose $j \in \Sigma^k \cap L$. Then, by (10), $j \in \Sigma^{k+1}$ and, by (13), $t^k b_j \leqslant t^{k+1}b_j$. Furthermore, either $j \in I^k \cap \Sigma^k$ hence, by (15), $t^k b_j = \gamma^k$ or $j \in (\Sigma^k \cap L)\backslash I^k$ hence, again by (15), $t^k b_j > \gamma^k$. In any case, $t^{k+1}b_j \geqslant t^k b_j \geqslant \gamma^k > \gamma^{k+1}$ thus proving that $j \notin I^{k+1}$ and (10a) is established. Q.E.D.

Proof of theorem **3.** Since $\bar{R} > 1$, the matrix $(I - A)_L$ has to be non-negative invertible and so are all principal minors of $(I - A)_L$.

Two sequences y^k and x^k, $k = 1,\dots, K$, of activity vectors are defined inductively as follows:

$$y_j^k = 0 \text{ if } j \in (\Sigma^k \cap L)\backslash I^k$$

$$[(I - A)\,y^k]_j = \begin{cases} Q_j + [A(x^1 + \cdots + x^{k-1})]_j & \text{if } j \in S^k \\ 0 & \text{if } j \in I^k\backslash S^k \end{cases} \tag{19}$$

and

$$x_j^k = \begin{cases} 0 & \text{if } j \in L\backslash I^k \\ [Ay^k]_j & \text{if } j \in I^k \cap \Sigma^k \end{cases}$$

$$[(I - A)\,x^k]_j = \begin{cases} Q_j + [A(x^1 + \cdots + x^{k-1})]_j & \text{if } j \in I^k \cap S^k \\ 0 & \text{if } j \in I^k\backslash(S^k \cup \Sigma^k) \end{cases} \tag{20}$$

It follows from (15) that

$$[(\Sigma^k \cap L)\backslash I^k] \cup S^k \cup [I^k\backslash S^k] = L \tag{21}$$

Furthermore

$$[L\backslash I^k] \cup [I^k \cap \Sigma^k] \cap [I^k \cap S^k] \cup [I^k\backslash(S^k \cup \Sigma^k)] = L \tag{22}$$

The decompositions of L in (21) and (22) are pairwise disjoint. In the case of (22) this is obvious whereas for (21) it follows from (15). Therefore, systems of equations (19) and (20) both are of full rank and the sequences y^k and x^k are well-defined and nonnegative.

Next we claim that

$$x^K = y^K \tag{23}$$

and

$$y^{k+1} = y^k - x^k,\ k = 1,\dots, K - 1. \tag{24}$$

To show this we note that, as a consequence of the above proposition, the left-hand side and the right-hand side of (23) and (24) respectively satisfy the same system of linear equations of full rank and therefore have to be equal. We omit the proof which is simple but tedious. (23) and (24) together imply that $y^k = x^k + x^{k+1} + \cdots + x^K$, $k = 1,\dots, K$, and therefore x^k, $k = 1,\dots, K$, is the sequence to be found in order to prove theorem 3. Q.E.D.

Proof of theorem 1. First we claim that, for given $r \in R_+$, the following system of equations admits a unique solution $(p^k, r^k) \in R^{L+F+1}$, $k = 0,\dots, K$:

$$r^K = r \text{ and } p_F{}^0 = P_F \tag{25}$$

$$\left.\begin{aligned} r^{k-1} &= (u^k - u^{k-1})\,\gamma^k + r^k \\ p^{k-1} &= (u^k - u^{k-1})\,t^k + p^k \end{aligned}\right\} k = 1,\dots, K \tag{26}$$

$$\begin{aligned} p^K b_j &= r^K \text{ if } j \in I^K \\ p^k b_j &= r^k \text{ if } j \in I^k \cap \Sigma^k,\ k = 1,\dots, K - 1 \end{aligned} \tag{27}$$

(25)-(27) can be shown to have a solution if and only if the following system admits a solution p^K:

$$p_F{}^K = P_F + (u^K - u^0)\, t_F \tag{28}$$

$$p^K b_j = \begin{cases} r & \text{if } j \in I^K \\ r + \sum_{l=k+1}^{K} (u^l - u^{l-1})(\gamma^l - t^l b_j) & \text{if } j \in I^k \cap \Sigma^k\ k = 1,\ldots, K-1 \end{cases} \tag{29}$$

Since $S^K \subseteq I^K$, it follows from (10) that

$$I^K \cup \bigcup_{k=1}^{K-1} (I^k \cup \Sigma^k) = L \tag{30}$$

We claim that the decomposition of L in (30) is pairwise disjoint. To see this choose $j \in I^k \cap \Sigma^k$ for some $k = \bar{k} < K$. It follows from (12) and (13) that $t^k b_j \geq t^{\bar{k}} b_j = \gamma^{\bar{k}} > \gamma^k$ for $k > \bar{k}$ hence $j \notin I^k$ for $k > \bar{k}$ as was to be shown.

Since the decomposition of L in (30) is pairwise disjoint it follows that (28)-(29) is of full rank thus admitting a unique solution and, as a consequence, so does (25)-(27).

Given a solution (p^k, r^k) $k = 0,\ldots, K$ of (25)-(27), we claim that

$$p^k b_j = r^k \text{ if } j \in I^k. \tag{31}$$

Note first that

$$I^k \backslash I^{k+1} \subseteq I^k \cap \Sigma^k \tag{32}$$

To prove (32), assume the contrary. Then there exists $j = j^* \in I^k \backslash I^{k+1}$ such that $j^* \notin \Sigma^k$ hence $j^* \in I^k \backslash \Sigma^k \subseteq I^{k+1}$, a contradiction, thus proving (32). It follows from (32) and the fact that the decomposition of L in (30) is pairwise disjoint that, for $j \in I^k$, either $j \in I^k \cap I^{k+1} \cap \cdots \cap I^K$ or there is l, $k \leq l < K$, such that $j \in I^k \cap I^{k+1} \cap \cdots \cap I^l$ but $j \notin I^{l+1}$. In the first case $p^K b_j = r^K$ whereas in the second case it follows from (32) and (27) that $p^l b_j = r^l$. Using (15), (31) can now be established easily.

We claim next that, for $r = 0$, the solution of (25)-(27) is nonnegative. Assume the contrary. Then there exists $\bar{r} > 0$ such that, for $r = \bar{r}$, the solution (p^k, r^k), $k = 0,\ldots, K$, of (25)-(27) is nonnegative and $p_j{}^k = 0$ for some $j = j^* \in L$ and $k = k^* \in \{1,\ldots, K\}$. It follows that $j^* \notin S^{k+1}, \notin \Sigma^k$ hence, by (8), $j^* \in I^k \cup (S^k \backslash S^{k+1})$ and, by (9), $j^* \in I^k$, $k = k^*$. Hence $p^k b_j \leq 0$ thus contradicting the fact that, by virtue of (31) $p^k b_j = r^k > 0$, for $k = k^*$ and $j = j^*$.

Finally, we construct a continuous profile $(p(u), r(u))$, $u^0 \leqslant u \leqslant \bar{u}$, of f.o.b. prices and land rents, which sustains the activity profile of theorem 1 as a spatial equilibrium, in the following way:

For $r > 0$, take a solution (p^k, r^k), $k = 0, \dots, K$, of (25)-(27). Define $p(u^K) = p^K$ and $r(u^K) = r^K = 0$ and

$$\dot{p}(u) = -t^k, \ \dot{r}(u) = -\gamma^k, \ u^{k-1} < u < u^k, \ k = 1, \dots, K.$$

It may be verified readily that $p(u^k) = p^k$ and $r(u^k) = r^k$ for all k and that $(p(u), r(u))$, $u^0 \leq u \leq \bar{u} = u^K$, forms a system of prices as required. Theorem 1 is established. Q.E.D.

Since $\Sigma^k \subseteq \Sigma^l$, $l \geq k$, since the decomposition in (30) is pairwise disjoint, and since (32) holds it can be seen immediately that each production process is operated only over a connected set of adjoining rings. More precisely, for any $j \in L$ there exist two numbers k_j and K_j such that $j \in I^k$ iff $k_j \leq k \leq K_j$. This result provides some additional information on the shape of the canonical production pattern.

Proof of theorem 4. By construction,

$$\int_{u^{k-1}}^{u^k} x(u)\, du = x^k \tag{33}$$

and

$$s_j(u^{k-1}) = s_j(u^k) + \left| \begin{array}{ll} [(I - A)\, x^k]_j & \text{if } j \in S^k \\ 0 & \text{otherwise} \end{array} \right. \tag{34}$$

$k = 1, \dots, K$. We claim that

$$s_j(u^{k-1}) = \begin{array}{ll} Q_j + [A(x^1 + \cdots + x^{k-1})]_j & \text{if } j \in S^k \\ 0 & \text{otherwise} \end{array} \tag{35}$$

(35) is proved by induction on k backwards. As for $k = K$ it follows from (9) that $S^K \subseteq I^K$. Suppose $j \in S^K$. Then, by (34), $s_j(u^{K-1}) = [(I - A)\, x^K]_j$ hence, by theorem 3,

$$s_j(u^{K-1}) = Q_j + [A(x^1 + \cdots + x^{K-1})]_j \, .$$

If $j \notin S^K$ then, by (34) $s_j(u^{K-1}) = 0$ and (35) holds for $k = K$.

Suppose (35) holds for $k = \bar{k} + 1$. If $j \in S^k$ then, by (9), $j \in S^{k+1}$ or $j \in I^k \cap S^k$, $k = \bar{k}$. In the first case $s_j(u^k) = Q_j + [A(x^1 + \cdots + x^k)]_j$, $k = \bar{k}$, hence, by (34), $s_j(u^{k-1}) = Q_j + [A(x^1 + \cdots + x^k)]_j + [(I - A)\, x^k]_j$. Since, by (9a), $j \notin I^k$ and, by theorem 3, $x_j{}^k = 0$, it follows that $[(I - A)\, x^k]_j = -[Ax^k]_j$ hence $s_j(u^{k-1}) = Q_j + [A(x^1 + \cdots + x^{k-1})]_j$.

In the second case it follows from (9a) that $j \notin S^{k+1}$ hence $s_j(u^k) = 0$ and by (34), $s_j(u^{k-1}) = [(I - A)\, x^k]_j$. Therefore, by theorem 3,

$$s_j(u^{k-1}) = Q_j + [A(x^1 + \cdots + x^{k-1})]_j \, .$$

If $j \notin S^k$ then, by (9), $j \notin S^{k-1}$ hence, by (34), $s_j(u^{k-1}) = s_j(u^k) = 0$. (35) is established.

It follows that $s(u) \geq 0$ for all u and that $s_j(u^0) = Q_j$ if $j \in S_1 = L$, i.e. $s_L(u^0) = Q_L$.

In a similar way it can be shown that

$$\sigma_j(u^k) = \begin{cases} [A(x^{k+1} + \cdots + x^K)]_j & \text{if } j \in \Sigma^k \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

We omit details. It follows from (36) that $\sigma(u) \geq 0$ for all u and $\sigma(u^K) = 0$.

Q.E.D.

The following numerical example may serve as illustration for the notion of a canonical production pattern. Coefficients are chosen as follows:

$$(I - A) = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \\ -4 & -\frac{1}{2} \end{pmatrix}, \; a = (1, 1) \text{ and } t = (1, 1, 1)$$

There exists one factor of production to be imported at the CBD. Hence, by definition,

$$t^1 = (1, 1, -1)$$

and, since $t^1 b_1 = 9/2 > t^1 b_2 = 1$,

$$\gamma^1 = 9/2,\ I^1 = \{1\},\ S^1 = \{1, 2\},\ \Sigma^1 = \{3\}.$$

It can be verified easily that

$$t^2 = (-1, 1, -1).$$

Note that $t^2 b_1 = 5/2 > t^2 b_2 = 2$ hence

$$\gamma^2 = 5/2,\ I^2 = \{1\},\ S^2 = \{2\},\ \Sigma^2 = \{1, 3\}.$$

Since $I^2 \backslash \Sigma^2$ empty it follows that

$$t^3 = (-1, 1, -1) \text{ and } \gamma^3 = 2.$$

Furthermore,

$$I^3 = \{2\},\ S^3 = \{2\},\ \Sigma^3 = \{1, 3\}.$$

We point out that, in the above example production in the second ring takes place for shipment outwards, i.e. towards the periphery of the city. If the set F of factors to be imported at the CBD is empty no shipment outwards occurs (see Schweizer and Varaiya [4]). For that reason, the possibility of

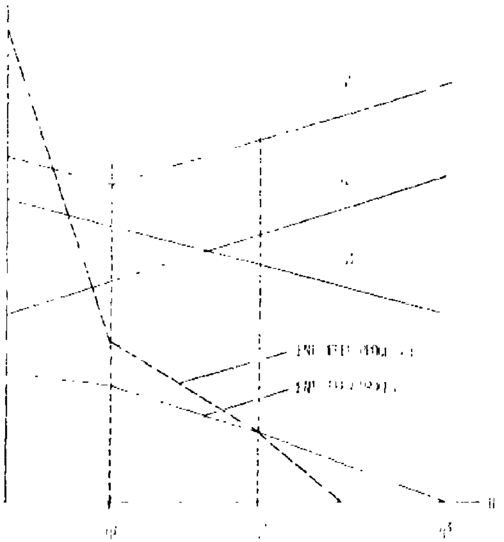

Fig. 1. Land rent and f.o.b. prices as functions of distance from *CBD*

shipping outwards of commodities produced within the city may come as a surprise. In figure 1 land rent and f.o.b. prices are shown as functions of distance u from the CBD for the above numerical example.

REFERENCES

1. C. J. Bliss, "Capital Theory and the Distribution of Income," North-Holland, Amsterdam, 1975.
2. G. S. Goldstein and L. N. Moses, Interdependence and the location of economic activity, *J. Urban Econ.* **2** (1975), 63–84.
3. H. Nikaido, "Convex Structures and Economic Theory," Academic Press, New York, 1968.
4. U. Schweizer and P. Varaiya, The spatial structure of production with a Leontief technology, *Regional Sci. and Urban Econ.* **6** (1976), 231–251.
5. U. Schweizer and P. Varaiya, The spatial structure of production with a Leontief technology. II. Substitute techniques, *Regional Sci. and Urban Econ.* **7** (1977), 293–320.

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On the Indeterminacy of Interest Rates and Wages with Perfect Foresight*

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The main purpose of this paper is to present two examples of infinite-horizon economies where the assumption of perfect foresight (or self-fulfilling expectations) is not sufficient to rule out the existence of a continuum of equilibrium paths, each of which converges to the *same* steady state. Earlier examples of this type were shown by Shell, Sidrauski and Stiglitz [22], Shell [21], Burmeister *et al.* [5], Black [3], Mirrlees [15], Taylor [25] and myself [7] in models with ad-hoc behavioral functions and/or with fiat money. In contrast, in the present paper we deal with nonmonetary economies of the Samuelson-Diamond type [10] [17] where all the behavioral functions can be derived from utility or profit maximization. Another interesting characteristic of the examples is that there is, in each case, a unique (homogeneous) production asset (land in Section 1 and reproducible capital in Section 2), thus showing that this type of nonuniqueness (or indeterminacy) is not necessarily linked to the presence of heterogeneous capital goods.

In my view, these examples are important for at least two areas of economic theory; the first one is income distribution theory because the examples imply the existence of cases where the Walras-Fischer general equilibrium approach is not sufficient for uniquely determining the path of income distribution, thus providing more, and different ammunition to the England side of the Cambridge-Cambridge controversy.[1]

The second area to which our results apply is that of macroeconomic models where agents are assumed to have "rational" expectations. Current renditions of the rationality hypothesis assume that individuals know the relevant economic model and figure out their expectations on the basis of *the*

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[1] It should be noted, however that since in our examples all equilibrium paths converge to the same steady state, they do not necessarily contradict the view that the Walras-Fisher approach is sufficient for uniquely determining income distribution in the "long run."

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equilibrium path that converges to a stationary distribution (see Sargent and Wallace [19, 20], Lucas [13, 14], Phelps and Taylor [16], Fischer [12]). In the absence of random shocks the hypothesis is equivalent to saying that individuals pick the self-fulfilling expectations path that converges to a steady state. However, in our examples there is an infinite number of such paths, thus showing that complete knowledge of the model plus the convergence assumption may not be sufficient information to figure out the future course of events. In other words, our results show the existence of cases where individuals are unable to figure out their expectations on the basis of the model and, therefore, where it is, in principle, unwarranted to assume that the economy will travel along a self-fulfilling (or rational) expectations path.[2]

The paper is organized in the following manner. Sections 1 and 2 present the main examples. Section 3 shows that indeterminacy does not disappear if prices are rationally set one period in advance; finally, Section 4 closes the paper with a discussion on the relevance of the results.

1. THE LABOR-AND-LAND ECONOMY

1.1. Let us assume that each individual lives for two periods and that his utility, u, satisfies

$$u = U(c, x) \tag{1}$$

where c and x are first- and second-period consumption, respectively. His budget constraint is

$$w = c + \frac{x}{1 + i} \tag{2}$$

where w is his wage in terms of (homogeneous) consumption in the first period, and i is the consumption own-rate of interest from the first to the second period of the individual's life. We assume he chooses c and x in order to maximize (1) subject to (2) taking w and i as given. In order to simplify the exposition we also assume function U is such that demands are uniquely determined for every $w > 0$ and $i > -1$ and that they are continuously differentiable everywhere.

Denoting the demand function for c as $C(1 + i, w)$, we define

$$S(1 + i, w) \equiv w - C(1 + i, w) = \text{savings in the first period of life in terms of consumption in that period} \tag{3}$$

Let w_t and i_t be the values of w and i which are arguments in the demand functions of the members of *generation t, i.e.* the set of individuals who live in periods t and $t+1$. Therefore, assuming a constant population and choosing units so that its total equals 1 we have that $S(1+i_t, w_t)$ is also the *aggregate savings function in period t* (in terms of consumption in period t).

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[2] For a further discussion of this issue see Section 3 and 4.4.

1.2. In the labor-and-land economy we assume that (homogeneous nondepreciating and nonreproducible) land is the only store of value, and we choose units such that its total is 1. Consumption is produced by means of labor and land with constant returns to scale and positive marginal productivities. Since the quantities of land and labor are constant over time and we assume stationary production conditions, at full employment the marginal productivities of the two factors of production are constant over time. Thus in a competitive equilibrium:

$$\left.\begin{aligned} w_t &= \bar{w} \text{ (a positive constant)} \qquad (4a)\\ R_t &= \bar{R} \text{ (a positive constant)} \qquad (4b)\end{aligned}\right\} \text{for all } t$$

where R_t is land rent in terms of consumption at t.

We assume land is traded in the following manner. At the beginning of period t the old are the owners of the land; they collect the rent and sell the land to the young. Therefore, if we denote q_t as the price of land in period t (in terms of consumption at t) the above commercialization procedure implies that

$$1+i_t = \frac{q_{t+1}+\bar{R}}{q_t} \text{ (for all } t) \tag{5}$$

for positive q_t and q_{t+1}.

We define a *perfect foresight* (or *self-fulfilling expectations*) *path* as a sequence $\{q_0, q_1, ...\}$, $q_t > 0$ for all t, such that

$$S\left(\frac{q_{t+1}+\bar{R}}{q_t}, \bar{w}\right) = q_t, \text{ for all } t \tag{6}$$

Recalling the discussion in Section 1.1 and Eqs. 4 and 5, (6) says that aggregate savings is equal to the total value of land for all t when i_t and w_t take up their competitive values. It is easy to verify that in the present context this condition is necessary and sufficient to ensure that all markets are cleared at all times if individuals are assumed to know all present and future prices.

Examples of nonuniqueness can now be generated in the following simple manner: choose $S(\ )$, $\bar{w}$ and $\bar{R}$ such that (6) has a locally stable steady-state solution for a positive q. Under those conditions it follows that there is a continuum of positive q_0's such that for each of them there is a sequence $\{q_1, q_2, ...\}$, $q_t > 0$ for all t, for which (6) is satisfied, and such that, in addition, every equilibrium q-sequence converges to the same steady-state q. Although the real wage will always equal $\bar{w}$, it is easily seen from (5) and (6) that for any two perfect foresight paths with different q_0's the economy generates different values of i_0; in other words, the rate of interest is undetermined.

We will use a hat (^) on the variables and functions to denote their steady-state levels. Thus linearization of (6) around $\hat q$ allows us to express it as

$$\tilde q_{t+1} = \left(\frac{\hat S}{\hat S_1} - 1 + \hat i\right) \tilde q_t \tag{7}$$

where $\tilde q_t = (q_t - \hat q)$ and S_1 is the derivative of S with respect to $(1 + i)$. Therefore, a *necessary* condition for the local stability of (6) at $q = \hat q$ is

$$-1 \leq \frac{\hat S}{\hat S_1} - 1 + \hat i \leq 1 \tag{8}$$

On the other hand, stability at $q = \hat q$ is ensured if the above holds with strict inequalities. Since $\hat q > 0$ and $\bar R > 0$ then, by (5), $\hat i > 0$ which implies, by (8), $\hat S_1 < 0$. In the present context the set of examples with a continuum of perfect-foresight paths is, therefore, characterized by the fact that *aggregate savings is a decreasing function of the rate of interest.*[3]

While assuming the propensity to save moves in the opposite direction to the rate of interest is perfectly consistent with utility analysis, economists familiar with the literature on uniqueness in static models (see, *e.g.*, Arrow and Hahn [1, Ch. 9]) will probably be inclined to think that the above phenomenon may have something to do with the implied lack of gross substitutability between young- and old-age consumption. That this is not so is demonstrated by the example of the following Section.

II. THE TWO-SECTOR ECONOMY

In this Section we will study a two-sector economy producing consumption and investment goods by means of capital and labor, and with fixed technical coefficients. The production side is, therefore, similar to that in the models of Shinkai [24] and Corden [9]. On the other hand, consumers are modelled like in Section 1.1.

Let us denote

a^j = capital-output ratio in sector j

b^j = labor-output ratio in sector j

k_t = stock of capital at the start of period t

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[3] The reader who, like myself, is not quite comfortable with local analysis is invited to examine the case where indifference curves are L-shaped: (6) boils down to a linear difference equation with at most one (stable) stationary solution.

λ_t^j output of sector j during period t

sector 1, 2 = consumption and investment goods sectors, respectively.

a^j, b^j, λ_t^j and k_t are constrained to be nonnegative.

As in the previous Section, we assume labor supply (inelastically provided by the young in each period) is constant and equal to one. Assuming full employment of capital and labor we then get

$$\begin{aligned} k_t &= a^1\lambda_t^1 + a^2\lambda_t^2 \\ 1 &= b^1\lambda_t^1 + b^2\lambda_t^2 \end{aligned} \tag{9}$$

which implies

$$\lambda_t^2 = \frac{a^1 - b^1 k_t}{\Delta} \tag{10}$$

$$\Delta = a^1 b^2 - b^1 a^2 \tag{11}$$

To simplify the exposition we will further assume that consumption goods are more capital intensive than investment goods, or, equivalently,

$$\Delta > 0 \tag{12}$$

Capital accumulates according to

$$k_{t+1} = k_t(1 - \delta) + \lambda_t^2 \tag{13}$$

where δ is the one-period rate of depreciation. Hence, recalling (10),

$$k_{t+1} = \frac{a^1}{\Delta} + \left[1 - \delta - \frac{b^1}{\Delta}\right] k_t . \tag{14}$$

Except in a borderline case this equation has a unique stationary value, $\hat{k}$, and satisfies:

$$\hat{k} = \frac{a^1}{\delta\Delta + b^1} \tag{15}$$

Furthermore, the necessary and sufficient condition for the stability of $\hat{k}$ is[4]

$$-1 < 1 - \delta - \frac{b^1}{\Delta} < 1 \tag{16}$$

In what follows we assume (16) to hold and we will examine the economy when capital is equal to $\hat{k}$. This we are allowed to do because the other variables of the model do not enter in the determination of k (recall (14)). The stability condition is imposed because, otherwise, only by fluke will the economy be found at or near $\hat{k}$.

---

[4] To simplify the exposition we will always assume the expression in (16) is different from zero. This will be crucial for (27) below to be defined.

We turn now to the price equations. Let us define

p_t = price of capital in period t (*after* appropriation of its rental of period t) = price of investment goods in period t

w_t = wage rate in period t

r_t = capital rental in period t;

all prices are in terms of the consumption good.

The zero-profit condition in both sectors is equivalent to

$$\begin{aligned} p_t &= r_t a^2 + w_t b^2 \\ 1 &= r_t a^1 + w_t b^1 \end{aligned} \tag{17}$$

Solving (17) for r_t and w_t in terms of p_t we get

$$\begin{aligned} r_t &= \frac{b^2 - p_t b^1}{\Delta} \\ w_t &= \frac{a^1 p_t - a^2}{\Delta} \end{aligned} \tag{18}$$

Again, denoting i_t the one-period consumption own-rate of interest from t to $t+1$, we now have

$$1 + i_t = \frac{p_{t+1}(1 - \delta) + r_{t+1}}{p_t} \tag{19}$$

Thus, by (18),

$$1 + i_t = \frac{p_{t+1}\left[1 - \delta - \dfrac{b^1}{\Delta}\right] + \dfrac{b^2}{\Delta}}{p_t} \tag{20}$$

The savings = assets equation—*i.e.*, the analog of (6) for the present case—now reads

$$S(1 + i_t, w_t) = p_t \hat{k} \tag{21}$$

We define a *perfect foresight* (or *self-fulfilling expections*) *path* as a sequence $\{p_0, p_1, ...\}$, $p_t > 0$ for all t, such that (18)-(21) hold.[5] In view of (18) and (20), (21) defines a first-order difference equation in p. As the reader who has patiently followed the argument up to this point must already know, our next step will now be to find necessary and sufficient conditions under which a stationary solution of (21) is stable.

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[5] For the present purposes it is enough to restrict our attention to paths where $k_t = \hat{k}$. Notice that, as in the previous Section, along a perfect foresight path all markets are cleared if individuals know present and future prices.

In order to simplify the exposition we will further assume that the map of indifference curves associated with function U in Section 1.1 is homothetic. Thus

$$S(1+i, w) = s(i)\,w \tag{22}$$

for some function $s(\cdot)$, and (21) becomes

$$s(i_t)\,w_t = p_t\dot{k} \tag{21'}$$

There are several details that must be cleared up before we formally present our example. However, we are already prepared to understand the reasons that may account for multiple equilibria. Suppose the system is at a stationary equilibrium (again here, all functional values and variables associated with a stationary solution of (21) will be denoted by a hat (^)). Thus, by (21'),

$$s(\hat{i})\,\hat{w} = \hat{p}\dot{k}.$$

Consider now a rise of p_0 over $\hat{p}$. Momentarily leaving $i_0 = \hat{i}$ we see from (18) that w_0 will also rise[6] and, as a result, both sides of (21') at $t = 0$ will increase. But using (18) in (21') we also see that

$$s(i_t)\cdot\frac{a^1 p_t - a^2}{\Delta} = p_t\dot{k},$$

therefore showing that if $a^2 > 0$ the L.H.S. of (21') will rise even more than the R.H.S.[7]. At the same time, however, if p_1 (and hence r_1) is kept constant, i_0 will fall when p_0 is increased; thus if $\hat{s}' > 0$ the fall in i_0 will tend to restore equilibrium. In fact, by picking the "right" slope for s the rise in p_0 would not call for changing p_1 from the original $\hat{p}$ level, and the economy would return to the stationary path in period one. Furthermore, if $\hat{s}'$ is close to the "right" one the accommodating change in p_1 would have to be small relative to that in p_0 which suggests that the system would have a tendency to eventually return to $\hat{p}$. We now turn to a more exact demonstration of this fact. By (20)

$$\hat{p} = \frac{b^2}{(\hat{i} + \delta)\Delta + b^1} \tag{23}$$

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[6] This constitutes an important difference with the land model because in the latter, w is unaffected by changes in the price of land.

[7] Anyone who recalls Stolper and Samuelson [25] can see this result right away because w/p is the wage in terms of investment goods and must, therefore, rise together with the price of the relatively labor intensive good, *i.e.*, with p.

Therefore, by (18) and (23),

$$\begin{aligned} \hat r &= \hat i + \delta \\ \hat w &= \frac{1 - a^2(\hat i + \delta)}{(\hat i + \delta)\Delta + b^1} \end{aligned} \tag{24}$$

Clearly, then, a necessary and sufficient condition for $\hat w$ to be positive (the only stationary values of w that we will consider in this paper) is

$$1 - a^2(\hat i + \delta) > 0 \tag{25}$$

Linearizing (21′) around a stationary solution and taking into account (18) and (20) yields

$$\tilde p_{t+1} = v\tilde p_t \tag{26}$$

where $\tilde p_t = (p_t - \hat p)$, and

$$v = \frac{\dfrac{\hat s}{\hat s'}\left[1 - \dfrac{\hat p}{\hat w}\dfrac{a^1}{\Delta}\right] + 1 + i}{1 - \delta + \dfrac{b^1}{\Delta}} \tag{27}$$

Therefore, a necessary condition for stability of the steady state is

$$-1 \leq v \leq 1 \tag{28}$$

while a sufficient condition is (28) with strict inequalities. Therefore, recalling (16), if $\hat i > 0$ (the only case we will consider here) then (28) requires that

$$\frac{\hat s}{\hat s'}\left[1 - \frac{\hat p}{\hat w}\frac{a}{\Delta}\right] \tag{29}$$

be sufficiently negative. This will be easy to ensure because, as we will next show, the expression in square brackets is *negative* if $a^2 > 0$, so (28) would hold if we set $\hat s'$ *positive* (not negative as in the example of Section 1) and sufficiently close (but not *too* close) to zero.

By (23) and (24)

$$\hat p a^1 - a^2 = \frac{\Delta}{(\hat i + \delta)\Delta + b^1}[1 - a^2(\hat i + \delta)] = \Delta\hat w > 0 \tag{30}$$

Hence, in view of (23), (24) and (30), if $a^2 > 0$ we have

$$\frac{\hat p}{\hat w}\frac{a^1}{\Delta} = \frac{1}{1 - \dfrac{a^2}{\hat p a^1}} > 1 \tag{31}$$

from which it follows that the square-bracketed expression in (29) is negative as asserted.

We will now show the existence of cases where (12), (16), (25) and (28) (with strict inequalities) are satisfied at a stationary solution of (21′) with $\hat i > 0$ and

$$s(\hat i) < 1 \tag{32}$$

It is easy to see that all such cases will be characterized by the type of lack of uniqueness we are examining in this paper.

Consider first the case where $a^2 = 0$ (investment goods are produced by means of labor alone) and $\delta = 1$. For (12) to hold in this instance we just have to set a^1 and b^2 positive; (25) is, obviously, satisfied, and since, by (15), (21′), (23) and (24),

$$s(\hat i) = \frac{\hat p}{\hat w}\hat k = \frac{b^2}{1 - a^2(\hat i + \delta)}\,\frac{a^1}{\delta\Delta + b^1} \tag{33}$$

then (32) also automatically holds when $a^2 = 0$ and $\delta = 1$. Furthermore, for (16) —the stability condition for k—we just have to constrain b^1, b^2 and a^1 such that

$$\frac{b^1}{a^1 b^2} < 1 \tag{34}$$

which we are clearly free to do. Thus, we have shown that except for (28) all the other conditions for a plausible example can be simultaneously satisfied when $a^2 = 0$ and $\delta = 1$. Unfortunately, however, (28) will never hold in this case because by (16), (27) and (31), $|r| > 1$ whenever $\hat i > 0$. But a slight modification will do.

Take an arbitrary $\hat i > 0$. By continuity and the above considerations if $\delta = 1$ and a^2 is positive and sufficiently close to zero, a^1, b^1, b^2 and $s(\ )$ can be chosen such that (12), (16), (25) and (32) are simultaneously satisfied. Furthermore, by (31), the square-bracketed expression in (27) is negative, thus the numerator of (27) can be made as close to zero (in absolute value) as desired by manipulating $\hat s'$.[8]

III. Predetermined Prices[9]

The models studied in this paper share the feature that along perfect foresight paths expectations are self-fulfilling from time zero onwards. However, no requirement was imposed in order that their associated prices be in accord with what people expected them to be before time zero. Taken the model of Section 1, for example,[10] and assume for ease of exposition that the young in period (-1) held point expectations over the price of land at time zero. Denothing the latter q_0^e it is then clear that as long as $q_0 \neq q_0^e$ the expectations of the old at time zero will not be fulfilled. Since our examples of multiple equilibria depend on our assuming that q_0 is free to vary, it follows that except when $q_0^e = q_0$ all other perfect-foresight paths are characterized by the fact that the expectations of at least some group of people who are alive during some subset of $[0, \infty)$—specifically, the old at time zero—are not going to be fulfilled.

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[8] Notice that so far the only constraint on $s(\ )$ has been that $s(\hat i) = \hat p \hat k / \hat w$; so we are free to pick the slope of s at $\hat i$ (which we have denoted by $\hat s'$).

[9] I am thankful to Robert Solow whose comments on a previous draft inspired the remarks of this Section.

The validity of a model is closely connected with the type of phenomena one wishes to explain; that is particularly true in the present case. Thus if we were interested in examing the implications of perfect foresight when *everybody's* expectations are fulfilled then it may be reasonable to require $q_0^e = q_0$, and even to extend the definition to cover all the generations down to minus infinity. However, if the objective is to set up a model where expectations are consistent with equilibrium from time zero to the indefinite future (*i.e.*, where expectations are rational over that time interval), q_0^e has no role to play in the definition of perfect foresight (in our model) because it does not enter into demand or supply functions in any of the markets operating from time zero to plus infinity. This is the reason why our results are particularly relevant for the literature of rational expectations (see, *e.g.*, the theoretical papers of Sargent and Wallace [19, 20], Black [3], Brock [4], Calvo [7], Lucas [13, 14], and the econometric studies of Sargent [18] and Barro [2]).

A natural question that arises in this connection is, if what we have in mind is a world of rational individuals, who, in the case of multiple equilibria, would be aware of the volatility of the future price of land, would it not be plausible to assume that they will try to secure its resale value by setting up a forward market for land? And if so, wouldn't that eliminate our type of nonuniqueness which, after all, seems to be linked to the fact that the price that the old get for their land at time zero is not predetermined.

To answer the question we have to change the model a little bit because as it stands the young at time t would not have anabody to whom to sell their land in the forward market. For simplicity, let us assume that parents of individuals to be born at time $t + 1$ can perform land transactions at time t on behalf of their children. These transactions are binding and the actual exchange of land for output is carried out in $t + 1$; parents are assumed to have rational expectations and to make those transactions in order to maximize the utility of their children. Besides this element of altruism, each person's utility is assumed, as before, to depend on his two-periods consumptions. Let ${}_tq_{t+1}$ be the price of $(t+1)$-land in its forward market in period t in terms of output in $t+1$. Thus a young person in period t can buy land at the price q_t and sell it tomorrow at the price ${}_tq_{t+1}$ if he makes use of the forward market for land at t.

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[10] In this part only the land model will be discussed, but it will be clear that the same points could also be made in terms of the model of Section 2.

For the sake of definiteness, let us further assume that at time zero all the land has been sold in the forward market in period (-1) and that each young person at time zero is committed to buy the same quantity of land. Thus the price of land that each young person is obliged to pay at time zero is prespecified and equals ${}_{-1}q_0$.

First note that in equilibrium we must have

$$ {}_tq_{t+1} = q_{t+1}, \quad t = 0, 1, \ldots \tag{35} $$

because if ${}_tq_{t+1} < q_{t+1}({}_tq_{t+1} > q_{t+1})$ for some t, parents at t will tend to buy (sell) an infinite amount of land in the forward market at t on behalf of their children since the latter can gainfully undo any part of that operation by selling (buying) in the spot market for land in $t+1$, a situation that cannot be consistent with equilibrium. This conclusion is not true for ${}_{-1}q_0$ and q_0 at time zero because the forward market at (-1) is already closed; q_0 is therefore free to take any value.

For the sake of definiteness and without loss of generality let us just examine perfect foresight paths where all the land is sold in the forward market at every point in time and (22) holds (or, more accurately, that preferences are homothetic). Equation (6) now becomes

$$ s(i_t)[\bar{w} - {}_{t-1}q_t + q_t] = q_t, \quad t = 0, 1, 2, \ldots \tag{36} $$

because the net worth of an individual of generation t in period t is his wage minus the value of land that he is committed to buy, ${}_{t-1}q_t$, plus the (spot) value of the land he has acquired, q_t (recall that by our previous assumptions individuals of each generation are identical in all respects and that both the units of land and number of individuals per generation are equal to one by normalization). Furthermore

$$ 1 + i_t = \frac{{}_tq_{t+1} + \bar{R}}{q_t}, \quad t = 0, 1, 2, \ldots \tag{37} $$

By (35)-(37), the equilibrium condition can be stated

$$ s\left(\frac{q_1 + \bar{R}}{q_0} - 1\right)[\bar{w} - {}_{-1}q_0 + q_0] = q_0 \tag{38a} $$

$$ s\left(\frac{q_{t+1} + \bar{R}}{q_t} - 1\right)\bar{w} = q_t, \quad t = 1, 2 \tag{38b} $$

Equation (38b) is identical to (6) except that the latter also holds for $t = 0$.

Cases of nonuniqueness can now easily be constructed. Suppose (8) holds with strict inequalities (which as argued in Section 1 is sufficient to ensure local stability of $\hat{q}$ for (6) and hence for (38b)) and let us also assume that $_{-1}q_0 = \hat{q}$. Clearly $q_t = \hat{q}$ is a solution of (38). However, since (8) implies $s'(\hat{r}) \neq 0$ we can appeal to the Implicit Function Theorem to prove that there exists some neighborhood of $\hat{q}$, N, and a function ψ

$$q_1 = \psi(q_0), \quad q_0 \in \mathrm{N} \tag{39}$$

where $\psi(\hat{q}) = \hat{q}$, such that (38a) is satisfied when q_1 is given by (39). This fact together with the local stability of (38b) implied by (8) can readily be used to show that there exists a neighborhood around $\hat{q}$ such that if q_0 belongs to that neighborhood there exists a path of q's satisfying (38) for all t that converges to $\hat{q}$. Therefore, *the presence of forward markets for land does not rule out the existence of a continuum of perfect foresight paths.*

It is interesting to note that in this example of nonuniqueness the old at time zero will get the price that they expected for their land, $\hat{q} = {_{-1}q_0}$, but if $q_0 \neq \hat{q}$ they would not have made the right decision on behalf of their children. It is only in that sense that their expectations (in period -1) would be unfulfilled.

IV. FINAL REMARKS

4.1. The main point of this paper is that in infinite-horizon competitive models with perfect foresight it is possible to find cases where one cannot even locally determine equilibrium prices and interest rates. This was shown to be so in the context of a one-and two-sector versions of the Samuelson-Diamond model where all behavioral equations can be derived from explicit utility and profit maximization.[11] Furthermore, in each of our examples there is only one homogeneous "capital" thus showing that the above indeterminacy is independent of the existence of "reswitching of techniques" and associated phenomena that has occupied the attention of capital theory almost since its inception.

The infinite-horizon assumption plays a crucial role in our results. To see this, notice that if we assume, instead, a finite number (T, say) of generations the price of assets in period $T + 1$ will be zero because at that time nobody would be interested in accumulating wealth. The difference equations for the systems—(6) and (22)—will, however, stay the same for all $t \leq T$. Since, as pointed out above, q or p at $T + 1$ equals zero, it is, therefore, clear that equilibrium solutions will, in general, be isolated from one another. However, it would be erroneous to conclude that no problem would arise if we are ready to assume a finite horizon. As indicated above, uniqueness is, in principle, recovered but a new, puzzling, difficulty might pop up with finite horizon when conditions for the type of indeterminacy of this paper prevail. To show this in a simple manner, suppose (6) gives rise to a linear difference equation and (8) holds with strict inequalities (see footnote 3 for an example). Hence, the unique steady state would be globally stable, giving rise to nonuniqueness when $T \to \infty$. Under these circumstances it is easy to see that for finite T, equilibrium q_0 will be very sensitive to changes in the terminal conditions as T grows large. Of course, if people are absolutely sure about what happens in period $T + 1$, no problem arises; but a slight divergence of opinion on the terminal conditions would result in widely different computations of equilibrium q_0 if T is sufficiently large; that is to say, today's equilibrium prices become strongly dependent on what people think will happen in the distant future. I find this "anti-turnpike" property as disturbing, if not more, than the nonuniqueness found in the previous sections.[12]

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[11] This is, in fact, the main "contribution," as against "point," of the paper because the latter has already been discovered in [15] and [22] with an ad-hoc savings function.

4.2. It is important to notice that the indeterminacy disclosed by the examples is not necessarily connected with Koopmans-Phelps "dynamic" inefficiency. As a matter of fact, since in our examples population is constant and the steady-state rate of interest is positive one can show that each converging perfect-foresight path is dynamically efficient.

Another relevant point is that in our perfect-foresight paths the price of the productive asset equals the present discounted value of its future returns, showing, in particular, that uniqueness would not be recovered by creating an infinitely-lived asset-holding firm. We will prove this fact for the model of Section 1.

By (5) we have

$$q_{t+1} = q_t(1 + i_t) - \bar{R};$$

therefore,

$$q_0 = \frac{q_t}{\prod_{s=0}^{t-1}(1 + i_s)} + \bar{R} \sum_{r=0}^{t-1} \frac{1}{\prod_{s=0}^{r}(1 + i_s)}$$

Now, in our examples

$$\lim_{t \to \infty} i_t = \hat{i} > 0$$

$$\lim_{t \to \infty} q_t = \hat{q}, \qquad 0 < \hat{q} < \infty$$

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[12] I am thankful to Rolf Mantel for bringing up this point to my attention.

Hence taking the limit in the expression for q_0 when t goes to infinity, we get

$$q = \bar{R} \sum_{t=0}^{\infty} \frac{1}{\prod_{s=0}^{t} (1+i_s)},$$

which is what we wanted to prove.

4.3. More research is certainly needed before we pass final judgement on the relevance of the indeterminacy problem. One of the most interesting issues would be to study the likelihood of this phenomenon when individuals live for several periods as in Cass and Yaari [8], particularly when their life span becomes very large. I have no firm conjecture to offer at this point, but since savings will always be to some extent affected by current wages and capital rentals, the following remarks on ad-hoc savings functions may help in guiding future research.

Consider the continuous-time version of the production side of Section 2's economy and assume, again, that k is at its sationary level $\hat{k}$. Suppose the *flow* of (gross) savings at time t can be expressed as

$$s_w(i_t)\, w_t + s_\pi(i_t)\, r_t \hat{k} \tag{40}$$

where now i_t is the instantaneous interest rate at time t and $s_{w,\pi}$ are arbitrary differentiable functions. Clearly,

$$i_t = \frac{\dot{p}_t + r_t - \delta p_t}{p_t} \tag{41}$$

where δ is now the (instantaneous) depreciation rate on capital. In equilibrium, gross savings = gross investment = $p\delta\hat{k}$; thus, using (40) and (41)

$$s_w\left(\frac{\dot{p}_t + r_t - \delta p_t}{p_t}\right) w_t + s_\pi\left(\frac{\dot{p}_t + r_t - \delta p_t}{p_t}\right) r_t \hat{k} = p_t \delta \hat{k} \tag{42}$$

which, recalling (18), is a differential equation in t. By the now familiar method of analysis, indeterminacy cases can be spotted by studying the stability of (42) at a steady state.

It is easy to show (the details will be left to the interested reader) that assuming $s'_{w,\pi} > 0$ we have indeterminacy if: (a) s'_w is set sufficiently close to zero when $\Delta > 0$ (*i.e.*, when investment goods are more labor intensive than consumption goods), or (b) s'_π is set sufficiently close to zero when $\Delta < 0$ (*i.e.*, when the previous factor-intensity assumption is reversed). As an interesting consequence, notice that in the Ricardo-Marx case where $s_w = 0$, indeterminacy would occur if capitalists' "animal spirits" are somewhat, but not too much, rekindled by the expectation of a higher rate of interest, and the investment sector is the most capital-intensive. (However, if s_π is totally insensitive to i, uniqueness is recovered as long as $\Delta \neq 0$.)

By (42) it is clear that if $s_{w,\pi}$ are insensitive to i, uniqueness will prevail except in some borderline, and hence odd, circumstances. Consequently, the above cases are independent from, and are thus complementary to, the ones presented in [15] and [22], where indeterminacy prevails despite assuming a constant savings propensity. In order to facilitate comparison it will be useful to examine their assumption in the same context as above.

Let us interpret δ as the (positive) constant rate of population growth. Let us assume

$$p_t(k_t + \delta k_t) + \dot{p}_t k_t = s(y_t + \dot{p}_t k_t) \tag{43}$$

where $0 < s < 1$ and y is factor income in terms of consumption. Thus, "for any individual, his income, and his saving, includes capital gains" (Mirrlees [15, p. 4]). As before, we will study the model when $k_t = \hat{k}$ and, therefore, the output of the consumption sector is a constant $\hat{c}$, and that of the investment sector equals $\delta \hat{k}$; hence

$$y_t = \hat{c} + p_t \delta \hat{k}. \tag{44}$$

In view of (43) and (44) we have

$$\dot{p}_t(1 - s) = s\frac{\hat{c}}{\hat{k}} - (1 - s)\, p_t \delta \tag{45}$$

The stationary point of (45) is clearly stable and hence indeterminacy results.

4.4. Nonuniqueness is by far not an inevitable consequence of perfect foresight as the papers of Shell and Stiglitz [23], Sargent and Wallace [19], Brock [4] and myself [7] demonstrate. Therefore, I think it would be incorrect to claim that we have found a fundamental flaw in the Walras-Fischer-Rational-Expectations approach. In my view the above examples simply alert us of the fact that given a model (even standard simple models) one cannot *assume* local uniqueness without taking the trouble of proving it and, therefore, they shed some doubts on the general applicability of methods that rely on finding closed-form solutions like in [11, 13].

In a sense, our examples could be taken as "good news" for the believers in Rational Expectations because they indicate that it is in no way an empty hypothesis: it definitely rules out a certain set of models and parameters. This, of course, should not be confused with the statement that such a hypothesis is *always* appropriate because, after all, our examples have been given in the context of extremely conventional economies.

References

1. K. Arrow and F. Hahn, "General Competitive Analysis," Holden–Day, San Francisco, 1971.
2. R. J. Barro, Unanticipated money growth and unemployment in the United States, *Amer. Econ. Rev.* **67** (1977), 101–115.
3. F. Black, Uniqueness of price level in monetary growth models with rational expectations, *J. Econ. Theory* **7** (1974), 53–65.
4. W. A. Brock, Money and growth: The case of long-run perfect foresight, *Internat. Econ. Rev.* **15** (1974), 750–777.
5. E. Burmeister, Ch. Caton, A. R. Dobell, and S. Ross, The "Saddle Point Property" and the structure of heterogeneous capital good models, *Econometrica* **41** (1973), 79–96.
6. P. Cagan, The monetary dynamics of hyperinflation, *in* "Studies in the Quantity Theory of Money" (M. Friedman, ed.), 25–120, Univ of Chicago Press, Chicago, 1956.
7. G. Calvo, "On Models of Money and Perfect Foresight," No. 75-7615, The Economics Workshops, Columbia University, November 1975; *Internat. Econ. Rev.*, in press.
8. D. Cass and M. Yaari, Individual saving, aggregate capital accumulation, and efficient growth, *in* "Essays on the Theory of Optimal Growth" (K. Shell, ed.), 233–268, The M.I.T. Press, Cambridge, Mass., 1967.
9. W. M. Corden, The two-sector growth model with fixed coefficients, *Rev. Econ. Stud.* **33** (1965), 161–168.
10. P. Diamond, "National Debt in a Neoclassical Growth Model," *Amer. Econ. Rev.* **55** (1965), 1126–1150.
11. R. Dornbusch, "Expectations and Exchange Rate Dynamics," *J. Polit. Econ.* **84** (1976), 1161–1176.
12. S. Fischer, long-term contracts, rational expectations, and the optimal money supply rule, *J. Polit. Econ.* **85** (1977), 191–206.
13. R. E. Lucas, Jr., Expectations and the neutrality of money, *J. Econ. Theory* **4** (1972), 103–124.
14. R. E. Lucas, Jr., An equilibrium model of the business cycle, *J. Polit. Econ.* **83** (1975), 1113–1144.
15. J. Mirrlees, Indeterminate growth theory, Mimeo, Nuffield College, Oxford University, August 1975.
16. E. S. Phelps and J. B. Taylor, Stabilizing powers of monetary policy under rational expectations, *J. Polit. Econ.* **85** (1977), 163–190.
17. P. Samuelson, An exact consumption–loan model of interest with or without the social contrivance of money, *J. Polit. Econ.* **64** (1958), 467–482.
18. T. J. Sargent, A classical macroeconometric model of the United States, *J. Polit. Econ.* **84** (1976), 207–237.
19. T. J. Sargent and N. Wallace, The stability of models of money and perfect foresight, *Econometrica* **41** (1973), 1043–1048.
20. T. J. Sargent and N. Wallace, "Rational" expectations, the optimal monetary instrument, and the optimal money supply rule, *J. Polit. Econ.* **83** (1975), 241–254.
21. K. Shell, Notes on the economics of infinity, *J. Polit. Econ.* **79** (1971), 1102–1011.
22. K. Shell, M. Sidrauski, and J. E. Stiglitz, Capital gains, income and saving, *Rev. Econ. Stud.* **36** (1969), 15–26.
23. K. Shell and J. E. Stiglitz, The allocation of investment in a dynamic economy, *Quart. J. Econ.* **81** (1967), 592–609.

24. Y. Shinkai, On equilibrium growth of capital and labor, *Internat. Econ. Rev.* **1** (1960), 107–111.
25. W. Stolper and P. Samuelson, Protection and real wages, *Rev. Econ. Stud.* **30** (1941), 105–118.
26. J. B. Taylor, Conditions for unique solutions in stochastic macroeconomic models with rational expectations, *Econometrica* **45** (1977), 1377–1386.

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On Pareto-Improving Price Changes*

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1. INTRODUCTION

The main result presented here is a set of necessary and sufficient conditions for the existence of price changes which will lead to a Pareto improvement for consumers relative to an initial set of demands. It is assumed that all preferences can be represented by continuous non-satiated strictly quasiconcave utility functions. This result, together with the characterization of the set of Pareto-improving directions of price changes, may prove useful in a variety of contexts. Here it is applied to second-best (optimal taxation) problems.

In the next section the basic model is presented. Section 3 reviews the contribution of Diamond and Mirrlees [3] to this problem. Section 4 presents the main theorem. Section 5 considers special cases of this result as well as an alternative way of modelling the problem. In Section 6, applications are discussed.

2. THE MODEL

The modelling of the consumer sector follows Hahn [6] which in turn is based on Debreu [2]. There are H households each having continuous non-satiated strictly quasiconcave utility functions, $u_h = u_h(x_h)$, where x_h is an n-dimensional consumption vector, $x_h \in X_h$. For simplicity, let $X_h = R_+^n$ which, by the Debreu assumptions [2, p. 84], requires each household to have a strictly positive endowment, $\bar{x}_h \in R_+^n$. As this assumption on endowments

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is only used to establish the continuity of the demand functions, weaker assumptions could be adopted. Using Hahn's notation: $z_h = x_h - \bar{x}_h$, $x = \Sigma_h x_h$, $\bar{x} = \Sigma_h \bar{x}_h$, and $\hat{x} = (x_1, \ldots, x_H)$. In this model there are no lump-sum incomes nor are there individual entitlements to profits.

The set of all consumer allocations which can be achieved by the use of a price mechanism is d where

$$d = \{\hat{x} \mid x_h = x_h(q) \text{ for all } h \text{ and some } q > 0\}.$$

The expression $x_h(q)$ is the solution to the household's utility maximization problem:

$$\max_{x_h} u_h(x_h) \text{ subject to } qz_h \leq 0$$

where q is a price vector.

The indirect utility functions are $v_h(q) = u_h(x_h(q))$, which are defined for non-negative, non-zero prices. Properties of these functions differ somewhat from the properties of indirect utility functions with positive lump-sum income. It is straightforward to establish that $v_h(q)$ is continuous, quasi-convex, and homogeneous of degree zero in q. Thus,

$$N_h(\bar{q}) = \{q \mid v_h(q) \leqslant v_h(\bar{q})\}$$

is a closed, convex cone. There may be "thick" indirect indifference curves, but they will not occur in the absence of kinks on the direct indifference curves. A global minimum of $v_h(q)$ is attained by any price vector which supports the consumption of the endowment.

Definition 1. Given $\hat{x}(q)$ and $\delta > 0$, Δq is a set of Pareto-improving price changes if $\| \hat{x}(q + \Delta q) - \hat{x}(q) \| < \delta$ and $v_h(q + \Delta q) \geq v_h(q)$ all h with $v_h(q + \Delta q) > v_h(q)$ some h.

The question considered here is: Suppose that demand changes are restricted to be within a δ-neighborhood of $\hat{x}(q)$, what further assumptions must be placed on the demand functions in order to ensure that for all $\hat{x} \in d$ and all $\delta > 0$ there exist Pareto-improving price changes?

Definition 2. Given $\hat{x}(q)$, t is a Pareto-improving *direction* of price changes if $\forall \delta > 0$, $\exists \lambda > 0$ such that $\| \hat{x}(q + \lambda t) - \hat{x}(q) \| < \delta$ and $v_h(q + \lambda t) \geq v_h(q)$ all h with $v_h(q + \lambda t) > v_h(q)$ some h.

For an initial set of demands, Theorem 1 demonstrates that if Pareto-improving price changes exist for all $\delta > 0$, then these price changes may all be chosen to be in the same direction.

Theorem 1. *Given $\hat{x}(q)$, there exist Pareto-improving price changes for all $\delta > 0$ if and only if there exists a Pareto-improving direction of price changes.*

Proof. (a) If t is a Pareto-improving direction of price changes, then $\Delta q(\delta) = \lambda(\delta)\, t$ are Pareto-improving price changes.

(b) Suppose Pareto-improving price changes exist for all $\delta > 0$. Since $N_h(q)$ is a closed, convex cone and since demands are continuous functions of prices [1, pp. 101-102], by choosing $\boldsymbol{\delta}$ sufficiently close to zero, $v_h(q + \lambda \Delta q(\boldsymbol{\delta})) \geq v_h(q)$ for all $\lambda \in [0, 1]$. It is sufficient to show that $\Delta q(\boldsymbol{\delta})$ is a Pareto-improving direction of price changes. Clearly, $\Delta q(\boldsymbol{\delta})$ is Pareto-improving for all $\delta \geq \boldsymbol{\delta}$. Consider $\delta < \boldsymbol{\delta}$. Since demands are continuous functions of prices, there exists $\lambda \in (0, 1]$ such that $0 < \| \hat{x}(q + \lambda \Delta q(\boldsymbol{\delta})) - \hat{x}(q) \| < \delta$. The first inequality ensures that $v_h(q + \lambda \Delta q(\boldsymbol{\delta})) > v_h(q)$ for some h.

If the indirect utility functions are locally non-satiated in the neighborhood of q, then the λ employed in the last step of Theorem 1 may be chosen arbitrarily close to zero; small changes in demands are attainable with small price changes.

3. THE DIAMOND-MIRRLEES CRITERION

Rather than proceeding directly to the conditions which are necessary and sufficient for the existence of Pareto-improving price changes, it will be instructive to motivate the final results by considering a global version of the sufficiency criterion established by Diamond and Mirrlees [3].

ASSUMPTION A. (i) For all $\hat{x} \in d$ except $\hat{x} = \hat{x}(q)$ for $q \in E$, either (a) $z_i > 0$ and $z_{hi} \geq 0$ for all h and some i with $q_i > 0$ or (b) $z_i < 0$ and $z_{hi} \leqslant 0$ for all h and some i. (ii) For $\hat{x} = \hat{x}(q)$ with $q \in E$, $z_{hi} = 0$ for all h and some i.

In the statement of Assumption A, i indexes commodities while E is the set of price vectors which yield exchange equilibria. E is non-empty [2, pp. 83–84; 3, p. 22].

The point of the assumption is straightforward. When $q \notin E$, if there is some good which is in positive (negative) net demand by each household, then a Pareto-improving change can be found by lowering (raising) that good's price holding all other relative prices constant. A.(ii) is, in fact, redundant as it is implied by A.(i) together with the continuity of the demand functions.[1]

THEOREM 2. *Assumption A is a sufficient condition for the existence of*

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[1] Hahn [6, p. 97] suggested the use of *A*. (i) for all q. This is inconsistent with the description of the economy as there can be no aggregate household net demand or supply for any good when $q \in E$.

Pareto-improving price changes for all $\hat{x}(q)$ and all $\delta > 0$ when all individuals have continuous non-satiated strictly quasiconcave utility functions.

The proof of this Theorem will be established by a series of lemmas.

LEMMA 1. For all $\hat{x} \in d$ *except* $\hat{x} = \hat{x}(q)$ *for* $q \in E$, *A implies that there exists a Pareto-improving direction of price changes.*

Proof. [6, p. 98].

LEMMA 2. *For* $\hat{x} = \hat{x}(q)$ *with* $q \in E$ *and* $n = 2$, *A implies that there exists a Pareto-improving direction of price changes.*

Proof. Since $n = 2$ this is the degenerate case of complete autarchy: all $v_h(q)$ are at their global minima. Consequently, any vector t whose components are not all equal is a Pareto-improving direction of price changes.

LEMMA 3. *For* $\hat{x} = \hat{x}(q)$ *with* $q \in E$, *A implies that there exists a Pareto-improving direction of price changes.*

Proof. By assumption the economy is autarchic in at least one good, say good one. By Hicks' aggregation theorem [8, pp. 312-313], the other goods can be treated as one commodity if their relative prices are held constant. The result then follows from Lemma 2.

Proof of Theorem 2. Theorem 2 is an immediate consequence of Theorem 1 and Lemmas 1 to 3.

4. NECESSARY AND SUFFICIENT CONDITIONS

The geometry of the problem is quite simple. Let $T(q)$ be the hyperplane through the origin orthogonal to q; $T(q)$ is the net demands budget hyperplane for all consumers. Let $Z^*(q) = \bigcup_h z_h(q)$. Clearly, $Z^*(q) \subset T(q)$. Suppose there exists another hyperplane through the origin, $H(q)$, distinct from $T(q)$ such that $Z^*(q)$ is contained in one of the closed halfspaces formed by $H(q)$. When $H(q)$ exists it is possible to change the consumer prices and leave all the original net demands budget feasible by rotating $T(q)$ around $H(q) \cap T(q)$. As a consequence, all consumers are no worse off as a result of the price changes. Individuals whose net demands do not lie on $H(q)$ will change their demands in response to the price changes; by revealed preference they are better off. The same result will occur for consumers whose net demands lie on $H(q)$ if their indirect utility functions have "thin" indifference curves. However, it is possible for all net demands to lie on $H(q)$ and for all direct utility functions to be kinked at these demands. Then there need not be any demand change in response to the price changes. However, since the indirect utility functions are continuous, by changing prices sufficiently someone will eventually change their demands, resulting in a Pareto improvement.

The Diamond and Mirrlees condition can also be interpreted in this fashion. They require $H(q)$ to be spanned by at most $n - 1$ vectors in $\{e_1, ..., e_n\}$, the n unit-length coordinate direction vectors. $H(q)$ is coincident with the coordinate system generated by some strict subset of the n commodities. If $Z^*(q) \subset H(q)$, then there is autarchy in at least one good; Assumption A.(ii) would be satisfied.

ASSUMPTION B. For all $\hat{x} \in \mathcal{A}$, there exists a hyperplane, $H(q)$, containing the origin and distinct from $T(q)$ such that $Z^*(q)$ is contained in one of its closed halfspaces.

THEOREM 3. *Assumption B is a necessary and sufficient condition for the existence of Pareto-improving price changes for all $\hat{x}(q)$ and all $\delta > 0$ when all individuals have continuous non-satiated strictly quasiconcave preferences.*

Proof. (a) For sufficiency, define a new economy isomorphic to the original economy. Choose the new coordinate system so that $H(q)$ is spanned by some subset of the e_i's. Now each "commodity" is a bundle of the original goods in some fixed proportions; preferences in the new space are defined in the natural way. By construction, Assumption A is satisfied in the new economy. Hence, Theorem 2 and the isomorphism establishes sufficiency.

(b) To demonstrate necessity, assume that no $H(q)$ as defined in B exists. Given $\delta > 0$, consider q' so that $0 < \| \hat{x}(q') - \hat{x}(q) \| < \delta$. Let $H'(q) = T(q) \cap T(q')$. It must be the case that there is at least one $z_h(q)$ in one of the open halfspaces (relative to $T(q)$) formed by $H'(q)$ and at least one $z_{h'}(q)$ in the other open halfspace, otherwise any $\bar{H}(q)$ such that $\bar{H}(q) \cap T(q) = H'(q)$ would satisfy B. Thus, only one of $z_h(q)$ and $z_{h'}(q)$ is budget feasible at q'. Without loss of generality, suppose $z_h(q)$ is budget feasible at q'. Clearly, $q' \cdot z_h(q) < 0$ so u_h increases. Since $z_{h'}(q)$ is not feasible at q', $u_{h'}$ necessarily decreases for small δ. As δ was arbitrary and the only restriction on q' was that $0 < \| \hat{x}(q') - \hat{x}(q) \| < \delta$, the demonstration is complete.

The construction in the sufficiency part of the proof suggests a simple interpretation of Assumption B. For all $q > 0$, Assumption B requires the existence of some "Hicksian composite good" which is weakly in net demand (or supply) by all consumers.

Guesnerie's work [5] suggests an alternative way of expressing this assumption. Let $\Lambda(q)$ be the smallest closed convex cone with vertex at the origin containing $Z^*(q)$. Assumption B is equivalent to Assumption C.

ASSUMPTION C. For all $\hat{x} \in d$, $\operatorname{Int} A(q) \cap \operatorname{Int}(-A(q)) \neq \emptyset$, where Int $A(q)$ is the interior of $A(q)$ relative to $T(q)$.

To simplify the exposition, for the remainder of this section assume that all the direct utility functions are smooth; any finite price changes will change each household's demands. Guesnerie's work can be used to characterize the Pareto-improving directions of price changes. Because of the homogeneity of the demand functions, the direction of all price changes can be taken to be vectors in $T(q)$. Let $P(q)$ be the negative polar cone of $A(q)$. Assumption C is equivalent to requiring $P(q)$ to be non-vacuous for all $q > 0$ and $P(q) \setminus \{0\}$ is the set of Pareto-improving directions of price changes for $\hat{x}(q)$. $P(q)$ is the set of all price changes in $T(q)$ which ensure that the cost of all H initial consumer demands do not increase in value. Thus, any non-zero element of $P(q)$ supports a revealed preference argument.[2] As a consequence, all Pareto-improving directions of price changes are easily calculated from a knowledge of the initial demand vectors.

5. SPECIAL CASES AND EXTENSIONS

While it is not too difficult to check whether Assumption B (or C) is satisfied using programming techniques, there are a number of special cases which are particularly easy to identify. One example is the Diamond and Mirrlees case of some good being in net demand or supply by all households.

Let $Z(q)$ be the $H \times n$ matrix of net demands when prices are q. Consider the system of linear equations (1).

$$Z(q) \cdot a = 0. \tag{1}$$

Solutions to (1) exist as $a = q$ is a solution.

THEOREM 4. *For $x(q)$, if the rank of $Z(q) \leq n - 2$, then a Pareto-improving direction of price changes exists.*

Proof. The rank condition is equivalent to requiring the dimension of the kernel of (1) to be at least two. When this is the case, one can always rotate the net demands budget hyperplane in such a way that all $z_h(q)$ and the origin are still feasible. Theorem 4 is then a corollary to Theorem 3.

The rotation of the budget hyperplane used in the proof of Theorem 4 corresponds to the case discussed in the last section when all demands are actually on $H(q)$. A sufficient condition for the rank condition to hold is for there to be at least two more goods than households. There would then be $H + 1$ relative prices to vary, but only H households' utilities to raise.[3]

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[2] Harris [7] uses arguments of this form.

[3] Diewert [4] has similar theorems.

Should the initial price vector result in an exchange equilibrium, Theorem 2 demonstrates that autarchy in some good is a sufficient condition for the existence of Pareto-improving price changes. This corresponds to having a column of zeroes in $Z(q)$ which combined with the dependence of the rows of $Z(q)$ yields the rank condition.

In the definition of Pareto-improving price changes, the new set of net demands were required to be in a prespecified neighborhood of the initial demands. Prices were not required to be in some arbitrarily small neighborhood of the initial prices. However, as noted, if the direct utility functions are smooth, when Pareto-improving price changes exist, these changes can be chosen to be arbitrarily small. Furthermore, in this situation any Pareto-improving price changes make all consumers better off. This strict improvement follows from the fact that if the price normalization used above is adopted, then all indirect utility functions will be strictly quasiconvex on this restricted domain.

Instead of working with finite changes, it is possible to instead assume that the indirect utility functions are differentiable [5] and work with differential price changes.

DEFINITION 3. Given $\mathscr{X}(q)$, t is a Pareto-improving direction of differential price changes if $\nabla v_h(q) \cdot t \geq 0$ all h with $\nabla v_h(q) \cdot t > 0$ some h.

In Definition 3 the gradient vectors are used to evaluate the desirability of differential price changes. Since the gradient vectors are proportional to the negative of the net demand vectors by Roy's Theorem, the set of (normalized) strictly Pareto-improving directions of differential price changes are Int $P(q)$. Int $P(q)$ is the interior of the positive polar cone to the cone generated by the gradient vectors, $-A(q)$. However, for finite changes the set of strictly Pareto-improving directions of price changes is $P(q)\backslash\{0\}$. Thus, a direction of price changes which is strictly Pareto-improving in the differentiable case is also strictly Pareto-improving for small finite changes. However, a differentiable change in some direction may be viewed by some consumer as no improvement while a finite change in the same direction is an improvement. The differentiable definition can thus be viewed as an approximation to the finite version.[4]

6. APPLICATIONS

One application of Theorem 3 is to the production efficiency theorems considered in the second-best literature [3]. In its simplest form, all production is under the control of a government but all consumer allocations must be in d, the demands generated by some prices q with no lump-sum incomes. The question is: "If the social welfare function is individualistic, is production efficiency desirable at a welfare optimum?"

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[4] The differential version of Theorem 3 can be established as a special case of the results obtained by Plott [11]. An extended discussion of the approximation to finite changes introduced by the use of gradient vectors may be found in Matthews [9].

Suppose the second-best optimum is inefficient, then the existence of Pareto-improving price changes would lead to new demands which are still feasible contradicting the assumption that production efficiency was not desirable. Consequently, when Assumption B is satisfied, production efficiency is a feature of the second-best optimum.

Establishing the desirability of production efficiency is only part of the problem. Generally there will be a number of feasible allocations which are production efficient and it is still necessary to choose between them. Guesnerie [5] analyzed this problem from a tax reform viewpoint. Starting with an initial feasible allocation which is production efficient, is it possible to find a set of Pareto-improving price changes which result in demands which are still feasible? To answer this question, it is necessary to find the set of Pareto-improving price changes and the set of changes in consumer prices which lead to feasible changes in demands. Assumption B ensures that the first set is non-empty. Furthermore, this set is $P(q)\backslash\{0\}$. It is noteworthy that it is not sufficient to know whether Pareto-improving price changes exist, it is also necessary to know what directions lead to the improvement.

Many optimal taxation discussions assume that it is possible to use a common lump-sum poll subsidy or tax [10]. This trivially ensures the desirability of production efficiency. However, this extra instrument may be of no use in answering the second question posed above. Since the initial allocation is production efficient, a poll subsidy could result in demands which are not feasible, yet it may be possible to find a Pareto improvement by changing relative prices.

7. CONCLUSION

Necessary and sufficient conditions for the existence of Pareto-improving price changes when consumers have continuous non-satiated strictly quasi-concave preferences have been characterized here. The existence of such changes has been central to the recent work of Diewert [4], Guesnerie [5], Harris [7], and others working on taxation problems. Undoubtedly such changes will remain a central concern of this literature.

REFERENCES

1. K. J. ARROW AND F. H. HAHN, "General Competitive Analysis," Holden–Day, San Francisco, 1971.

2. G. Debreu, "Theory of Value," Wiley, New York, 1959.
3. P. A. Diamond and J. A. Mirrlees, Optimal taxation and public production, *Amer. Econ. Rev.* **61** (1971), 8–27, 261–278.
4. W. E. Diewert, Optimal tax perturbations, *J. Publ. Econ.* **10** (1978), 139–177.
5. R. Guesnerie, On the direction of tax reform, *J. Publ. Econ.* **7** (1977), 179–202.
6. F. H. Hahn, On optimum taxation, *J. Econ. Theory* **6** (1973), 96–106.
7. R. Harris, Efficient commodity taxation, *J. Publ. Econ.*, in press.
8. J. R. Hicks, "Value and Capital," 2nd ed., Oxford Univ. Press, Oxford, 1946.
9. S. A. Matthews, "Directional Cores in Simple Dynamic Games," California Institute of Technology, Pasadena, Calif., 1978.
10. J. A. Mirrlees, The theory of optimal taxation, Nuffield College, Oxford University, 1977.
11. C. R. Plott, A method for finding 'acceptable proposals' in group decision processes, *Papers on Non-Market Decision Making* **2** (1967), 45–59.

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A Probabilistic Model of Location Choice with Neighborhood Effects*

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This paper tries to develop a probabilistic model of location choice by many types of individuals who interact among themselves in the presence of neighborhood effects. Equilibrium is defined as an allocation of individuals among locations such that for every type of individuals the selection probability for each location is equal to the proportion of individuals actually choosing the location in question. The existence of equilibrium is proved under usual continuity assumptions. The equilibrium is shown to be unique and stable in the presence of neighborhood effects, if the degree of externality is sufficiently small.

In this paper we attempt to develop a probabilistic model of location choice by many types of individuals who interact among themselves in the presence of neighborhood effects. Our work may be viewed as a generalization of Schelling's model of segregation between two groups into a situation with many types of individuals choosing among many alternative locations.[1] By applying McFadden's theory of qualitative choice behavior,[2] we assume that the selection probabilities of each type of individuals for various locations are functions of the representative utilities obtained from all possible locations. Neighborhood effects are taken into account by assuming that the representative utility obtained by each type of individuals is a function of the numbers of individuals of all types who are actually choosing that location. Note that in this paper we deal with neighborhood effects *within* each location, but not those *across* locations.

First, equilibrium is defined as an allocation of individuals among locations such that for every type of individuals the selection probability for each

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[1] Schelling [6] and [7]. Especially, see his bounded-neighborhood model [7, p. 167–186].

[2] See Marschak [2] and McFadden [3].

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location is equal to the proportion of individuals actually choosing the location in question. The existence of equilibrium is guaranteed under usual continuity assumptions, and its uniqueness and stability are examined under some additional assumptions. The equilibrium is shown to be unique and stable in the presence of nighborhood effects, if the degree of neighborhood effects is sufficiently small.

Finally, we take up a simple but important case with two types of individuals (e.g., blacks and whites) and two locations (e.g., a city and its suburbs), and show that even if the two groups do not like each other, the "integration" equilibrium is stable if the degree of negative externality between them is sufficiently small relative to the variance of the distribution of utility among the individuals of each type.

1. THE MODEL AND EXISTENCE

Each individual is assumed to choose among n alternative locations, which can be considered as n different neighborhoods in a city, or n different cities in a region. There are m types of individuals with a certain number of individuals for each type, where any two different types can be easily distinguished by observation.

For type k ($k = 1,\ldots, m$), the utility obtained from location i, say U^{ki}, is assumed to be randomly distributed over the population of individuals of type k, and the selection probability of type k for location i, say P^{ki}, can be defined as the probability that an individual drawn randomly from the population of type k will find location i most attractive:

$$P^{ki} = \text{Prob}(U^{ki} > U^{kj} \text{ for all } j \neq i) \qquad (k = 1,\ldots, m;\ i = 1,\ldots, n). \tag{1}$$

Under certain assumptions,[3] P^{ki} can be expressed as a function of "representative" utilities, $u^{k1},\ldots, u^{kn}$ ($k = 1,\ldots, m$), where u^{kj} is the representative utility of type k for location j and is equal to the non-stochastic component of U^{kj}. Then we have

$$P^{ki}(u^{k1},\ldots, u^{kn}) > 0 \text{ and } \sum_{i=1}^{n} P^{ki}(u^{k1},\ldots, u^{kn}) = 1 \qquad \text{for all } u^{k1},\ldots, u^{kn}\ (k = 1,\ldots, m). \tag{2}$$

Let us now consider "neighborhood effects" resulting from the interaction of various types of individuals at each location. Denoting by N_i^k the number of individuals of type k who actually choose location i, we assume that the representative utility of type k for location i, u^{ki}, depends on the numbers of individuals of all types at location i;

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[3] McFadden [3, p. 108].

$$u^{ki} = u^{ki}(N_i^1,\dots, N_i^m) \quad (k = 1,\dots, m;\ i = 1,\dots, n). \tag{3}$$

If, for example, individuals of type k like (or dislike) individuals of type j at location i, then $\partial u^{ki}/\partial N_i^j > 0$ (or < 0). Increasing returns (agglomeration economies) and decreasing returns (congestion) can be represented by the derivative of $u^{ki}(N_i^1 + \cdots + N_i^m)$ being positive and negative, respectively.

In the present context, we define equilibrium as an allocation of individuals among n alternative locations such that the selection probability of type k for location i, P^{ki}, is equal to the proportion of indivduals of type k who actuall choose location i, that is, N_i^k/L^k, where L^k is the total number of individuals of type k, which is given as a positive constant. Here we treat the number of individuals as a continuous variable in order to avoid any complexities arising from a discrete number of individuals. Thus, equilibrium is defined as an allocation of individuals $N_i^k \geqslant 0$ such that

$$\sum_{i=1}^{n} N_i^k = L^k > 0 \quad (k = 1,\dots, m), \tag{4}$$

$$N_i^k = P^{ki}(u^{k1}(N_1^1,\dots, N_1^m),\dots, u^{kn}(N_n^1,\dots, N_n^m))\, L^k \quad (k = 1,\dots, m;\ i = 1,\dots, n). \tag{5}$$

Then we can prove the existence of an equilibrium:

THEOREM 1. *If $u^{ki}(N_i^1,\dots, N_i^m)$ is continuous in N_i^h for $0 \leqslant N_i^h \leqslant L = \sum_{k=1}^{m} L^k$ $(h = 1,\dots, m)$, and $P^{ki}(u^{k1},\dots, u^{kn})$ is continuous in $u^{k1},\dots, u^{kn}$ and satisfies* (2), *for all $k = 1,\dots, m$ and all $i = 1,\dots, n$, then there exists an equilibrium.*

Proof. Consider a simplex defined by

$$N_i^k \geqslant 0 \text{ and } \sum_k \sum_i N_i^k = L > 0. \tag{6}$$

Then we have a continuous function $P^{ki}L^k$, which maps from the simplex (6) to itself. Thus, Brouwer's fixed point theorem ensures the existence of a fixed point $N_i^{k*} \geqslant 0$ such that $N_i^{k*} = P^{ki}(u^{k1}(N_1^{1*},\dots, N_1^{m*}),\dots, u^{kn}(N_n^{1*},\dots, N_n^{m*}))\, L^k$ for all k and all i. (4) is also satisfied, since $\sum_i N_i^{k*} = \sum_i P^{ki}(\ )\, L^k = L^k$ in view of (2). Q.E.D.

2. UNIQUENESS

Let us further assume that for each type k, P^{ki} will decrease (or not increase) when any one of u^{kj}'s other than u^{ki} increases:

$$P^{ki}_j = \partial P^{ki}/\partial u^{kj} \leqslant 0 \ (i \neq j). \tag{7}$$

This means that all locations are substitutes for (or independent of) each other, and location i becomes less (or no more) attractive compared to location j whose utility increases. It is easy to see that (7) implies that

$$P^{ki}_i = \partial P^{ki}/\partial u^{ki} = -\sum_{j\neq i} (\partial P^{kj}/\partial u^{ki}) \geqslant 0 \ (k = 1,\dots, m;\ i = 1,\dots, n), \tag{8}$$

because $\sum_{j=1}^{n} (\partial P^{kj}/\partial u^{ki}) = 0$ from (2).[4]

Define $m \times n$ dimensional vectors N and f as

$$N = (N_1^1,\dots, N_n^1, N_1^2,\dots, N_n^2,\dots, N_1^m,\dots, N_n^m), \tag{9}$$

$$f \equiv (f^{11},\dots, f^{1n}, f^{21},\dots, f^{2n},\dots, f^{m1},\dots, f^{mn}), \tag{10}$$

where

$$f^{ki} \equiv N_i^k - P^{ki}(\ )\, L^k \ (k = 1,\dots, m;\ i = 1,\dots, n). \tag{11}$$

Then it is clear that the equilibrium N^* is given by solving the system $f(N) = 0$. Also define

$$u^{ki}_h = \partial u^{ki}/\partial N_i^h \ (k, h = 1,\dots, m;\ i = 1,\dots, n). \tag{12}$$

We shall prove that the equilibrium is unique if the "degree" of externality is sufficiently small:

THEOREM 2. *If* $P^{ki}(u^{k1},\dots, u^{kn})$ *satisfies* (2) *and* (7) *for all* $u^{k1},\dots, u^{kn}$ ($k = 1,\dots, m$; $i = 1,\dots, n$) *and*

$$\sum_{k=1}^{m} (P^{kj}_j \mid u^{kj}_h \mid L^k) < \tfrac{1}{2} \ \textit{for}\ 0 \leqslant N_i^k \leqslant L^k \ (k = 1 \cdots m;\ i, j = 1 \cdots n), \tag{13}$$

then the solution to the equilibrium system $f(N) = 0$ *is unique.*

Proof. Since P^{ki} and u^{ki} are all differentiable, we have a differentiable mapping $f: \Omega \to R^{m\times n}$, where Ω is a closed rectangular region $\Omega = \{N \mid 0 \leqslant N_i^k \leqslant L^k\}$. The Jacobian matrix of the mapping f is

$$J(N) = I + A, \tag{14}$$

---

[4] The properties (7) and (8) are actually satisfied by McFadden's selection probabilities $P^{ki} = e^{u^{ki}}/(\sum_{j=1}^{n} e^{u^{kj}})$ (see [3]) and also by Marschak's $P^{ki} = u^{ki}/(\sum_{j=1}^{n} u^{kj})$ (see [2]).

and

$$A = \begin{vmatrix} A_{11} \cdots A_{1m} \\ \vdots \quad A_{kh} \quad \vdots \\ A_{m1} \cdots A_{mm} \end{vmatrix}, \tag{15}$$

where

$$A_{kh} = \begin{bmatrix} -P_1^{k1}u_h^{k1}L^k & \cdots & -P_n^{k1}u_h^{kn}L^k \\ \vdots & -P_j^{ki}u_h^{kj}L^k & \vdots \\ -P_1^{kn}u_h^{k1}L^k & \cdots & -P_n^{kn}u_h^{kn}L^k \end{bmatrix} \quad (k, h = 1,\ldots, m). \tag{16}$$

It can be shown that $I + A$ is a dominant diagonal matrix with positive diagonal elements for all N in Ω, because for any column, that is for any h and j,

$$1 - P_j^{hj}u_h^{hj}L^h \geq 1 - P_j^{hj} \mid u_h^{hj} \mid L^h > \sum_{i \neq j} \mid P_j^{hi} \mid\mid u_h^{hj} \mid L^h$$

$$+ \sum_{k \neq h} \left(\sum_{i=1}^{n} \mid P_j^{ki} \mid\mid u_h^{kj} \mid L^k \right), \tag{17}$$

where the last inequality follows from (13) as

$$1 > 2P_j^{hj} \mid u_h^{hj} \mid L^h + \sum_{k \neq h} (2P_j^{kj} \mid u_h^{kj} \mid L^k)$$

$$= P_j^{hj} \mid u_h^{hj} \mid L^h + \sum_{i \neq j} \mid P_j^{hi} \mid\mid u_h^{hi} \mid L^h + \sum_{k \neq h} \left(\sum_{i=1}^{n} \mid P_j^{ki} \mid\mid u_h^{kj} \mid L^k \right), \tag{18}$$

in view of $P_j^{kj} = \sum_{i \neq j} \mid P_j^{ki} \mid$ from (2). Thus, according to Gale and Nikaido,[5] the mapping f is univalent in Ω, and as a result, the solution to $f(N) = 0$ is unique. Q.E.D.

3. STABILITY

Suppose that the system is out of equilibrium such that the number of individuals of type k who like location i best is not equal to the number of individuals of type k who actually choose location i for at least one k and one i. In such disequilibrium situations, some kind of dynamic adjustment must take place over time. We take up two alternative adjustment processes, that is, discrete adjustment and continuous adjustment.

First, let us assume a one-period time lag for individuals to adjust their actual behavior to their desired one. Then we may describe the adjustment process by the following difference equation system:

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[5] Gale and Nikaido [1, p. 84].

$$N_i^k(t+1) = P^{ki}[u^{k1}(N_1^1(t),..., N_1^m(t)),..., u^{kn}(N_n^1(t),..., N_n^m(t))]\, L^k$$
$$(k = 1,..., m;\ i = 1,..., n), \tag{19}$$

where

$$\sum_{i=1}^{n} N_i^k(t) = L^k\ (k = 1,..., m;\ i = 1,..., n). \tag{20}$$

By expressing $N_n^k(t)$ as $N_n^k(t) = L^k - \sum_{i=1}^{n-1} N_i^k(t)$ from (20), we shall deal with $m \times (n-1)$ variables:

$$N = (N_1^1,..., N_{n-1}^1, N_1^2,..., N_{n-1}^2,..., N_1^m,..., N_{n-1}^m). \tag{21}$$

We can now show that the equilibrium N^* is locally stable, if the degree of externality is sufficiently small:

THEOREM 3. *If* $P^{ki}(u^{k1},..., u^{kn})$ *satisfies* (2) *and* (7) *at the equilibrium* N^* *and*

$$\sum_{k=1}^{m} (2P_j^{kj} \mid u_h^{kj} \mid + P_n^{kn} \mid u_h^{kn} \mid)\, L^k < 1\ (h = 1,..., m;\ j = 1,..., n-1) \tag{22}$$

at the equilibrium N^*, *then the equilibrium is locally stable, according to the adjustment process* (19).

Proof. Linearizing (19) around the equilibrium, together with (20), leads to

$$N(t+1) - N^* = B(N(t) - N^*), \tag{23}$$

where

$$B = \begin{vmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & B_{kh} & \vdots \\ B_{m1} & \cdots & B_{mm} \end{vmatrix}, \tag{24}$$

and

$$B_{kh} = \begin{bmatrix} (P_1^{k1}u_h^{k1} - P_n^{k1}u_h^{kn})\, L^k & \cdots & (P_{n-1}^{k1}u_h^{kn-1} - P_n^{k1}u_h^{kn})\, L^k \\ \vdots & (P_j^{ki}u_h^{kj} - P_n^{ki}u_h^{kn})\, L^k & \vdots \\ (P_1^{kn-1}u_h^{k1} - P_n^{kn-1}u_h^{kn})\, L^k & \cdots & (P_{n-1}^{kn-1}u_h^{kn-1} - P_n^{kn-1}u_h^{kn})\, L^k \end{bmatrix} \tag{25}$$

$$(k, h = 1,..., m).$$

If λ is a characteristic root of B, we know that[6]

$$
\begin{aligned}
|\lambda| &\leq \max_{h,j}\left(\sum_{k=1}^{m}\sum_{i=1}^{n-1}|P_j^{ki}u_h^{kj} - P_n^{ki}u_h^{kn}|\,L^k\right) \leq \max_{h,j}\left[\sum_{k=1}^{m}\left(\sum_{i=1}^{n-1}(|P_j^{ki}||u_h^{kj}|\right.\right.\\
&\qquad \left.\left. + |P_n^{ki}||u_h^{kn}|)\right)L^k\right]\\
&= \max_{h,j}\left[\sum_{k=1}^{m}(2P_j^{kj}|u_h^{kj}| + P_j^{kn}|u_h^{kj}| + P_n^{kn}|u_h^{kn}|)\,L^k\right]\\
&\leq \max_{h,j}\left[\sum_{k=1}^{m}(2P_j^{kj}|u_h^{kj}| + P_n^{kn}|u_h^{kn}|)\,L^k\right],
\end{aligned}
\tag{26}
$$

for $h = 1,\dots, m$ and $j = 1,\dots, n-1$. Thus it follows from (22) that $|\lambda| < 1$.

Q.E.D.

Alternatively, we may treat time as continuous and introduce gradual adjustment through time. A natural counterpart of the system (19) in the context of continuous time can be given by the following differential equation system:

$$
\begin{aligned}
\dot{N}_i^k(t) &= P^{ki}[u^{k1}(N_1^1(t),\dots, N_1^m(t)),\dots, u^{kn}(N_n^1(t),\dots, N_n^m(t))]\,L^k - N_i^k(t)\\
&\qquad (k = 1,\dots, m;\ k = 1,\dots, n),
\end{aligned}
\tag{27}
$$

where $\dot{N}_i^k(t) = dN_i^k(t)/dt$. Note that if the initial condition $N_i^k(0)$ satisfies (20) at time 0, (20) will be satisfied for all $t \geq 0$, since

$$
\sum_{i=1}^{n}\dot{N}_i^k(t) = \sum_{i=1}^{n}(P^{ki}(t)\,L^k - N_i^k(t)) = L^k - \sum_{i=1}^{n}N_i^k(t) = 0 \text{ for all } t \geq 0. \tag{28}
$$

It can be shown that the adjustment process (27) is quasi-stable,[7] if the degree of externality is sufficiently small.

THEOREM 4. *If* $P^{ki}(u^{k1},\dots, u^{kn})$ *satisfies* (2) *and* (7) *for all* $u^{k1},\dots, u^{kn}$, *and*

$$
\sum_{k=1}^{m}(P_j^{kj}|u_h^{kj}| + P_n^{kn}|u_h^{kn}|)\,L^k < 1 \quad (h = 1,\dots, m;\ j = 1,\dots, n-1) \tag{29}
$$

for all $N_i^k \geq 0$ *such that* $\sum_{i=1}^{n} N_i^k = L^k$, *then the adjustment process* (27) *is quasi-stable.*

---

[6] McKenzie [4, p. 49].

[7] For the concept of quasi-stability, see Uzawa [8, p. 618].

Proof. Define a distance function $V(t)$ as

$$V(t) = \sum_{(k,i)\in I_t^+} [P^{ki}(u^{k1}(N_1^1(t)\cdots N_1^m(t)),\ldots,u^{kn}(N_n^1(t)\cdots N_n^m(t)))\,L^k - N_i^k(t)], \tag{30}$$

where $I_t^+ = \{(k,i) | P^{ki}(t)\,L^k - N_i^k(t) > 0$, or $P^{ki}(t)\,L^k - N_i^k(t) = 0$ and $d(P^{ki}(t)\,L^k - N_i^k(t))/dt > 0\}$. From the fact that $\sum_{k=1}^m \sum_{i=1}^n (P^{ki}L^k - N_i^k) = 0$, it follows that

$$V(t) = -\sum_{(k,i)\in I_t^-} [P^{ki}(u^{k1}(N_1^1(t)\cdots N_1^m(t)),\ldots,u^{kn}(N_n^1(t)\cdots N_n^m(t)))\,L^k - N_i^k(t)], \tag{31}$$

where $I_t^- = \{(k,i) | P^{ki}(t)\,L^k - N_i^k(t) < 0$, or $P^{ki}(t)\,L^k - N_i^k(t) = 0$ and $d(P^{ki}(t)\,L^k - N_i^k(t))/dt < 0\}$. It is clear that $V(t) = 0$ if and only if the system is in equilibrium, and otherwise $V(t) > 0$. In order to show that $V(t)$ will decrease over time, define $W(t) = \overline{\lim}_{h\to 0}(1/h)(V(t+h) - V(t))$. Then it follows that[8]

$$\begin{aligned}
W(t) &= \sum_{(k,i)\in I^+} \frac{d[P^{ki}(t)\,L^k - N_i^k(t)]}{dt} = \sum_{h=1}^{m}\sum_{j=1}^{n}\sum_{(k,i)\in I_t^+} \frac{\partial(P^{ki}L^k - N_i^k)}{\partial N_j^h}\,\dot{N}_j^h \\
&= \sum_{(h,j)\in I_t^+}\sum_{(k,i)\in I_t^+} \frac{\partial(P^{ki}L^k - N_i^k)}{\partial N_j^h}\,\dot{N}_j^h \\
&\quad - \sum_{(h,j)\in I_t^-}\sum_{(k,i)\in I_t^-} \frac{\partial(P^{ki}L^k - N_i^k)}{\partial N_j^h}\,\dot{N}_j^h \\
&= \sum_{(h,j)\in I_t^+}\Bigl[\sum_{(k,i)\in I_t^+} (P_j^{ki}u_h^{kj} - P_n^{ki}u_h^{kn})\,L^k - 1\Bigr]\dot{N}_j^h \\
&\quad - \sum_{(h,j)\in I_t^-}\Bigl[\sum_{(k,i)\in I_t^-} (P_j^{ki}u_h^{kj} - P_n^{ki}u_h^{kn})\,L^k - 1\Bigr]\dot{N}_j^h,
\end{aligned} \tag{32}$$

which is negative except in equilibrium, because we have $\dot{N}_j^h > 0$ for at least one $(h,j)\in I_t^+$ and $\dot{N}_j^h < 0$ for at least one $(h,j)\in I_t^-$ out of equilibrium, and the expressions in the parentheses are always negative from (29) since for $(h,j;\,k,i)\in I_t^+$ or I_t^-,

$$\sum_{(k,i)} (P_j^{ki}u_h^{kj} - P_n^{ki}u_h^{kn})\,L^k \leq \sum_{k=1}^{m} (P_j^{kj}\,|\,u_h^{kj}\,| + P_n^{kn}\,|\,u_h^{kn}\,|)\,L^k < 1. \tag{33}$$

Clearly $W(t) = 0$ in equilibrium, because I_t^+ and I_t^- become null sets. Thus, according to Uzawa,[9] the process (27) is quasi-stable. Q.E.D.

---

[8] Special care should be taken to find $W(t)$. See Negishi [5, p. 205].

[9] Uzawa [8, p. 620–621].

It is well known that quasi-stability implies global stability if equilibrium is unique.[10] Thus, Theorems 2 and 4 yield the following corollary:

COROLLARY 1. *If $P^{ki}(u^{k1},\ldots,u^{kn})$ satisfies* (2) *and* (7) *for all $u^{k1},\ldots,u^{kn}$, and*

$$\sum_{k=1}^{m} \left(2P_j^{ki}\,|\,u_h^{ki}\,| + P_n^{kn}\,|\,u_h^{kn}\,|\right) L^k < 1 \text{ for } 0 < N_i^k < L^k$$

$$(k, h = 1 \cdots m;\ i, j = 1 \cdots n), \tag{34}$$

then the equilibrium is globally stable, according to the process (27).

4. THE TWO-TYPE-TWO-LOCATION CASE

Consider a special case with two types of individuals (e.g., blacks and whites) choosing between two locations (e.g., the inner city and suburban neighborhoods). For each type of individuals, the representative utility is assumed to be positively related to the ratio of the number of individuals of their own type to that of the other type at each location:[11]

$$u^{11} = u^{11}(N_1^1/N_1^2),\ u^{12} = u^{12}(N_2^1/N_2^2),\ u^{21} = u^{21}(N_1^2/N_1^1),\ u^{22} = u^{22}(N_2^2/N_2^1), \tag{35}$$

where $u^{ki\prime}(\ ) > 0$ for all k, $i = 1, 2$.

We shall focus on the stability analysis and take up the continuous adjustment process (27). In view of the constraints $N_1^1 + N_2^1 = L^1$ and $N_1^2 + N_2^2 = L^2$, we find

$$\begin{aligned} \dot{N}_1^1 &= P^{11}[u^{11}(N_1^1/N_1^2),\ u^{12}((L^1 - N_1^1)/(L^2 - N_1^2))]\, L^1 - N_1^1 \\ \dot{N}_1^2 &= P^{21}[u^{21}(N_1^2/N_1^1),\ u^{22}((L^2 - N_1^2)/(L^1 - N_1^1))]\, L^2 - N_1^2. \end{aligned} \tag{36}$$

The Jacobian matrix of the system will then be

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \tag{37}$$

---

[10] More generally, if the set of equilibria is countable, quasi-stability is equivalent to global stability. See Uzawa [8, p. 619].

[11] This is essentially the same assumption as Schelling's in his bounded-neighborhood model [7, p. 167–186].

where

$$
\begin{aligned}
b_{11} &= [P_1^{11}u^{11\prime}(1/N_1^2) + P_2^{11}u^{12\prime}(-1/(L^2 - N_1^2))]\,L^1 - 1,\\
b_{12} &= [P_1^{11}u^{11\prime}(-N_1^1/(N_1^2)^2) + P_2^{11}u^{12\prime}((L^1 - N_1^1)/(L^2 - N_1^2)^2)]\,L^1,\\
b_{21} &= [P_1^{21}u^{21\prime}(-N_1^2/(N_1^1)^2) + P_2^{21}u^{22\prime}((L^2 - N_1^2)/(L^1 - N_1^1)^2)]\,L^2,\\
b_{22} &= [P_1^{21}u^{21\prime}(1/N_1^1) + P_2^{21}u^{22\prime}(-1/(L^1 - N_1^1))]\,L^2 - 1.
\end{aligned}
\tag{38}
$$

In order to illustrate local stability conditions, let us assume that everything is identical and symmetric at the equilibrium so that we may write

$$u^{ki\prime} = u',\ L^k = L,\ P_i^{ki} = p \text{ for all } k, i = 1, 2. \tag{39}$$

Then we find

$$b_{11} = 4pu' - 1,\ b_{12} = -4pu',\ b_{21} = -4pu',\ b_{22} = 4pu' - 1, \tag{40}$$

and a necessary and sufficient condition for local stability is

$$u' < 1/(8p). \tag{41}$$

That is to say, the "integration" equilibrium is locally stable, if and only if the degree of externality u' is sufficiently small, more specifically, smaller than $1/(8p)$.

Furthermore, it can be shown that if the utilities from the two locations are independently, identically and logistically distributed, then the expression p is inversely related to the standard deviation of the distribution of the utility. Assuming that U^{k1} and U^{k2} are independently and identically distributed, we may write their joint distribution function as the product of marginal probability functions

$$f(U^{k1})\,f(U^{k2})\ (k = 1, 2). \tag{42}$$

Then we can find

$$P^{ki} = \text{Prob}(U^{ki} > U^{kj};\, j \neq i) = \int_{-\infty}^{\infty} f(U^{ki})\Big[\int_{-\infty}^{U^{ki}} f(U^{kj})\,dU^{kj}\Big]\,dU^{ki}$$

$$(k, i, j = 1, 2). \tag{43}$$

For the sake of illustration, consider the logistic distribution case:

$$f(U) = \frac{e^{-(U-u)/\tau}}{\tau[1 + e^{-(U-u)/\tau}]^2}, \tag{44}$$

where

$$\tau = \sqrt{3}\sigma/\pi, \text{ and } u = E(U). \tag{45}$$

In this case, since everything is symmetric at the equilibrium, it can be proved that[12]

$$p \equiv P_i^{ki} = -\frac{1}{\tau^2}\int_{-\infty}^{\infty}\frac{e^{-(U-u)/\tau}-e^{-2(U-u)/\tau}}{[1+e^{-(U-u)/\tau}]^4}\,dU = \frac{1}{\tau}\frac{(1/3)-e^{-(U-u)/\tau}}{2[1+e^{-(U-u)/\tau}]^3}\Bigg|_{-\infty}^{\infty}$$

$$= \frac{1}{6\tau} \equiv \frac{\pi}{6\sqrt{3}\sigma}. \tag{46}$$

Combining (41) and (46), we obtain

$$u' < \frac{3\sqrt{3}}{4\pi}\sigma, \tag{47}$$

and conclude that the integration equilibrium is locally stable if and only if the degree of externality is sufficiently small relative to the degree of dispersion of the distribution of the utility. Figure 1 illustrates the stable case, whereas Figure 2 shows the unstable case.

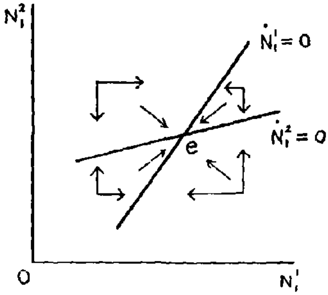

FIGURE 1

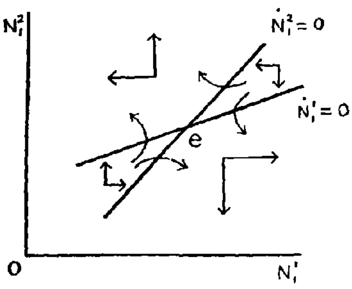

FIGURE 2

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[12] This manipulation is due to Perry Shapiro.

REFERENCES

1. D. GALE AND H. NIKAIDO, The Jacobian matrix and global univalence of mappings, *Math. Ann.* **159** (1965), 81–93.
2. J. MARSCHAK, Binary-choice constraints and random utility indicators, *in* "Mathematical Methods in the Social Sciences" (K. J. Arrow, S. Karlin, and P. Suppes, Eds.), pp. 312–329, Stanford Univ. Press, Stanford, Calif., 1960.
3. D. MCFADDEN, Conditional logit analysis of qualitative choice behavior, *in* "Frontiers of Econometrics" (P. Zarembka, Ed.), pp. 105–142, Academic Press, New York, 1974.
4. L. MCKENZIE, Matrices with dominant diagonals and economic theory, *in* "Mathematical Methods in the Social Sciences" (K. J. Arrow, S. Karlin, and P. Suppes, Eds.), pp. 47–62, Stanford Univ. Press, Stanford, Calif., 1960.
5. T. NEGISHI, "General Equilibrium Theory and International Trade," North–Holland, Amsterdam; American Elsevier, New York, 1972.
6. T. C. SCHELLING, Models of segregation, *Amer. Econ. Rev.* (2) **56** (1969), 488–493.
7. T. C. SCHELLING, Dynamic models of segregation, *J. Math. Sociol.* **1** (1971), 143-186.
8. H. UZAWA, The stability of dynamic processes, *Econometrica*, **29** (1961), 617–631.

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Satisfactory Mechanisms for Environments with Consumption Lower Bounds*

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1. INTRODUCTION

This paper concerns the design of decision-making mechanisms having desirable properties for a certain class of economic environments. The special feature of these environments on which we concentrate is that, although some transfer of resources may be possible among the agents, the extent to which any agent can lose resources is bounded. Such a restriction is well known in general equilibrium theory, where the lower boundedness of consumption sets limits transfers to those that are physically feasible. Another interpretation is that the agents in the model are representatives[1] of segments of the population, and that the limitations on transfers among them arises through the institutional structure.[2]

Without bounds on transfers, Groves and Loeb [6] have given a class of mechanisms that make efficient choices, in a sense to be described below, in an equilibrium in which each player is following a dominant strategy. In this way, much of the strategic or manipulative aspect of the free rider

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[1] This is discussed in detail by Johansen [7]. We believe that the primary practical reason for treating the case of bounded transfers is that these representatives may have means which are much smaller than the intensities of the aggregate willingness-to-pay that are typical of their constituents.

[2] Indeed, in the one example of which we are aware where these methods were applied in a practical situation, it was precisely these considerations that were most important in the design of the decision-making procedures. The case is that of program selection by the stations in the Public Broadcasting System. Each of the 135 station managers who "bid" for programs has a limited programming budget but represents a large population. The differences in aggregate willingnesses-to-pay among alternative possible combinations of programs would surely swamp these budgets. Nevertheless a method of "social" decision-making is necessary which allows the station managers the flexibility necessary to accurately reflect the tastes of their potential viewing audiences.

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problem has been overcome.[3] Green and Laffont [2] have shown that the class of mechanisms described by Groves and Loeb contains all those with these properties.

In this paper we first extend the use of these mechanisms to systems with transfers bounded from below, maintaining all the other aspects of the model described above. Individuals' strategic options are studied. The main result is that one of the mechanisms of Groves and Loeb, a special one, previously developed by Clarke [1] and Vickrey [8], retains all its desirable properties despite the bounds on transfers. Other members of the original successful class lose this property in the present context. Sufficient conditions for a member of the Groves–Loeb class to remain efficient are presented.

Finally, we offer some remarks on the willingness of individuals to participate in this procedure, on extensions to more complex choice situations and on their unbiasedness.

2. THE MODEL

We concentrate on the case of a decision, d, to be taken between two alternatives which can be denoted 0 ("reject proposed project") and 1 ("accept proposed project").

Let x_i be the amount of the private good consumed by individual i who is assumed to have the utility function, defined for $x_i \geqslant 0$, given by:

$$\begin{aligned} u_i(x_i, d) &= x_i && \text{if } d = 0 \\ &= x_i + v_i && \text{if } d = 1. \end{aligned}$$

The initial endowment of the ith individual is denoted $\bar{x}_i$; it is assumed to be non-negative.

Since there are no successful dominant strategy mechanisms always having balanced budgets (see Green and Laffont [3]) we must also consider the net transfer of the decision maker (government) in defining feasibility and efficiency. Thus, a *feasible social state* is defined by:

$$z = \left[d, x_1, \dots, x_n, \sum_i (\bar{x}_i - x_i)\right]$$

such that $x_i \geqslant 0$ for all i.

The last entry represents the net transfer from the individuals to the decision maker. It is natural to suppose that the utility of the decision maker is strictly increasing in this quantity, as revenues are usable for other purposes or may be substituted for other taxes being collected.[4]

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[3] Of course some problems do remain. Hurwicz [1970] has shown that it is impossible to maintain feasibility in two person environments. This has been extended by Green and Laffont [3], Green, Kohlberg, and Laffont [4], and Walker [9].

Without fear of confusion we can write the utility of individual i in social state z as $u_i(z) = u_i(d, x_i)$.

The utility of the decision maker is then:

$$u_0(z) = \sum_i (\bar{x}_i - x_i).$$

A social state z is said to be *efficient relative to* $u_0(z)$, (or "efficient" for brevity) if for any other feasible social state z' such that $u_i(z') \geqslant u_i(z)$, $i = 0, 1,..., n$, we have that $u_i(z') = u_i(z)$, $i = 0, 1,..., n$. Efficiency is just Pareto optimality for the set of agents expanded to include the decision maker.

It is easy to see that there may be some efficient states for which $d = 0$ and $\sum v_i > 0$, or $d = 1$ and $\sum v_i < 0$. For example, consider the two-person economy defined by:

$$\bar{x}_1 = 0, \qquad v_1 = 5$$

$$\bar{x}_2 = 10, \qquad v_2 = -1.$$

The social state $z = (0, 0, 10, 0)$ which produces utilities

$$u_0 = 0, \qquad u_1 = 0, \qquad u_2 = 10$$

is Pareto optimal. This can be seen because if $u_0 \geqslant 0$, then $x_1 + x_2 \leqslant 10$ is required. If $u_2 \geqslant 10$, then $d = 0$ requires $x_2 \geqslant 10$ and $d = 1$ requires $x_2 \geqslant 11$. In the former instance $x_1 \leqslant 0$ and z is the only feasible social state; in the latter, there are no feasible social states since $x_1 \leqslant -1$ would be implied.

Let $w_i \in R$ be the strategy of agent i, $i = 1,..., n$.

We define a *mechanism* as a mapping, f, from R^n to social states.

$$w = (w_1, ..., w_n) \rightarrow f(w) = z.$$

We use the notation:

$$w_{-i} = (w_1, ..., w_{i-1}, w_{i+1}, ..., w_n),$$

$$(w_{-i}, \xi) = (w_1, ..., w_{i-1}, \xi, w_{i+1}, ..., w_n).$$

---

[4] Of course, if individuals were to recognize this dependence, the favorable incentive properties of the mechanisms to be studied would be destroyed (see, however, Green, Kohlberg, and Laffont [4]).

A strategy w_i is called *admissible* if, for every w_{-i},

$$f(w_{-i}, w_i) = \left[d, x_1, \ldots, x_n, \sum (\bar{x}_i - x_i)\right]$$

is such that $x_i \geq 0$.

An admissible strategy w_i, is called *admissible dominant* (or just *dominant* for brevity) if, for every other admissible strategy, w_i'

$$u_i[f(w_{-i}, w_i)] \geq u_i[f(w_{-i}, w_i')]$$

for all w_{-i}.

If $w_i = v_i$ is a *dominant* strategy for every *i*, then the mechanism will be said to be *strongly individually incentive compatible.* Note, however, that v_i may not be admissible for some mechanisms and some individuals. There may nevertheless exist other dominant strategies.

A mechanism will be called *satisfactory* if

(i) for every i there exists an admissible dominant strategy, w_i^*, and

(ii) $f(w^*) = f(w_1^*, \ldots, w_n^*)$ is an efficient state.

In Green and Laffont [2] it was shown that, for economies without bounds on consumption of the private good, the class of all mechanisms that are strongly individually incentive compatible and satisfactory is given by:

$$f(w) = \left[d, x_1, \ldots, x_n, \sum_i (\bar{x}_i - x_i)\right] \tag{2.1}$$

where

$$\begin{aligned} d &= 0 && \text{iff} \quad \sum w_i < 0 \\ x_i &= \bar{x}_i + \sum_{j \neq i} w_j + h_i(w_{-i}) && \text{if} \quad d = 1 \\ x_i &= \bar{x}_i + h_i(w_{-i}) && \text{if} \quad d = 0 \end{aligned}$$

where the $h_i(\cdot)$ are an arbitrary set of real-valued functions defined on R^{n-1}.

3. The Pivotal Mechanism

In this section we consider the special case in which, for all i.

$$h_i(w_{-i}) = \min\left(-\sum_{j \neq i} w_j, 0\right). \tag{3.1}$$

This defines the *pivotal mechanism* (see Green and Laffont [3], Clarke [1] and Vickrey [8]). Under this mechanism we always have $x_i \leq \bar{x}_i$. The strategy w_i is pivotal when $w_i \cdot \sum_{j \neq i} w_j < 0$ and $| w_i | > | \sum_{j \neq i} w_j |$, because the presence of individual i changes the social decision. In such a case $x_i \neq \bar{x}_i$.

LEMMA 1. *For the mechanism defined by* (3.1), *the strategies*

$$\begin{aligned} \hat{w}_i(\bar{x}_i, v_i) &= v_i - \max(v_i - \bar{x}_i, 0) \qquad & v_i \geq 0 \\ &= v_i - \min(v_i + \bar{x}_i, 0) \qquad & v_i < 0 \end{aligned}$$

are the unique and dominant strategies.

Proof. Consider first the case in which $v_i \geq 0$. Note that $u_i(w_{-i}, w_i)$ is defined for all w_{-i}.

However, for $| \tilde{w}_i | > \bar{x}_i$, $u_i(w_{-i}, \tilde{w}_i)$ is not defined whenever w_{-i} is such that

$$| \tilde{w}_i | > \Big| \sum_{j \neq i} w_j \Big| > \bar{x}_i$$

and

$$\tilde{w}_i \cdot \sum_{j \neq i} w_j < 0$$

because then

$$x_i = \bar{x}_i - \Big| \sum_{j \neq i} w_j \Big| < 0.$$

For $| \tilde{w}_i | \leq \bar{x}_i$, $u_i(w_{-i}, \tilde{w}_i)$ is defined for all w_{-i}, and hence these values of $\tilde{w}_i$ are admissible strategies.

If $v_i < \bar{x}_i$, then $\hat{w}_i = v_i$ and the proof in Green and Laffont [2, Theorem 3] establishes the dominance of $\hat{w}_i$.

If $v_i > \bar{x}_i$, so that $\hat{w}_i = \bar{x}_i$, and $\tilde{w}_i$ is any other admissible strategy, we have the following cases: the value of w_{-i} is such that

(i) $\hat{w}_i$ is pivotal but $\tilde{w}_i$ is not,

(ii) $\hat{w}_i$ and $\tilde{w}_i$ are both pivotal,

(iii) neither $\hat{w}_i$ nor $\tilde{w}_i$ are pivotal.

Case (*i*). Since $\hat{w}_i = \bar{x}_i > 0$ is pivotal, it must be that $\sum_{j \neq i} w_j < 0$, and $\bar{x}_i > -\sum_{j \neq i} w_j$. Thus:

$$u_i[f(w_{-i}, \hat{w}_i)] = v_i + \bar{x}_i + \sum_{j \neq i} w_j$$

and

$$u_i[f(w_{-i}, \tilde{w}_i)] = \bar{x}_i.$$

Using $v_i > \bar{x}_i > -\sum_{j \neq i} w_j$, the former outcome dominates the latter.

Case (ii). We must have $\hat{w}_i > \tilde{w}_i > 0$, $\sum_{j\neq i} w_j < 0$, $\tilde{w}_i > -\sum_{j\neq i} w_j$. Therefore the outcome associated with $(w_{-i}, \hat{w}_i)$ is the same as that associated with $(w_{-i}, \tilde{w}_i)$.

Case (iii). Similarly, the two outcomes are identical.

The case of $v_i < 0$ is analogous and therefore $\hat{w}_i$ is the unique dominant strategy. Q.E.D.

Our definition of the pivotal mechanism includes a rule by which "ties" are broken, namely, if $\sum_i w_i = 0$ then $d = 1$ is the decision chosen. In unconstrained environments, the choice of tie breaking rules is clearly irrelevant since $v_i = w_i$ and thus either $d = 0$ or $d = 1$ results in an efficient state. The present context differs from this one, however, in that inefficiency may result if the tie is broken in the "wrong" way. Consider the following example.

$$\bar{x}_1 = 5 \qquad v_1 = -6$$

$$\bar{x}_2 = 10 \qquad v_2 = +5.$$

By virtue of Lemma 1, $w_1 = -5$, $w_2 = +5$ are the strategies played. Hence, under the pivotal mechanism for instance,

$$z = (1, 5, 5, 5)$$

is the resulting state. The associated utilities are -1, 10, 5 for the two agents and the decision maker, respectively. However, the state

$$z' = (0, 0, 10, 5)$$

gives rise to the utilities 0, 10, 5, which obviously dominates z (relative to $u_0 = 5$). In some sense, however, the likelihood of such cases of exact ties is negligible. This motivates the following definition:

A mechanism is said to be *essentially satisfactory* if

(i) it is satisfactory, or

(ii) it fails to be satisfactory only at environments for which $\sum w_i$ is zero, where w is the vector of dominant strategies.

THEOREM 1. *The pivotal mechanism is essentially satisfactory.*

Proof. By virtue of the lemma it suffices to prove that $f(\hat{w}_1, ..., \hat{w}_n)$ is efficient, or else $\sum \hat{w}_i = 0$.

Let

$$f(\hat{w}_1, ..., \hat{w}_n) = \left[d, x_1, ..., x_n, \sum (\bar{x}_i - x_i)\right].$$

Consider any feasible social state $z' = [d', x_1', \ldots, x_n', \sum (\bar{x}_i - x_i')]$. If $d' = d$, then it is clear that we cannot have $x_i' \geq x_i$ and $u_0' \geq u_0$ without having $x_i' = x_i$ for all i. Thus if $f(\hat{w}_1, \ldots, \hat{w}_n)$ can be dominated, it must be by a social state such that $d' \neq d$.

Consider first the case of $d = 0$:

Let

$$\begin{aligned} S_+ &= \{i \mid 0 < \hat{w}_i = v_i \leq \bar{x}_i\} \\ S_- &= \{i \mid 0 > \hat{w}_i = v_i \text{ and } |v_i| \leq \bar{x}_i\} \\ T_+ &= \{i \mid 0 < \hat{w}_i = \bar{x}_i < v_i\} \\ T_- &= \{i \mid 0 > \hat{w}_i = -\bar{x}_i > v_i\}. \end{aligned}$$

Let

$$t_i = \bar{x}_i - x_i, \qquad i = 1, \ldots, n$$

be the transfer under the pivotal mechanism.

In order that z' dominate $f(\hat{w}_1, \ldots, \hat{w}_n)$, we require that for all $i = 1, \ldots, n$,

$$v_i + x_i' \geq x_i$$

or

$$x_i' \geq x_i - v_i = \bar{x}_i - t_i - v_i .$$

For the four groups of individuals above this means

$$\begin{aligned} i \in S_+ &\to x_i' \geq \bar{x}_i - t_i - \hat{w}_i \\ i \in S_- &\to x_i' \geq \bar{x}_i - t_i - \hat{w}_i \\ i \in T_+ &\to x_i' \geq \bar{x}_i - t_i - v_i \\ i \in T_- &\to x_i' \geq \bar{x}_i - t_i - v_i \geq \bar{x}_i - t_i - \hat{w}_i . \end{aligned}$$

Note that for $i \in T_+$, the relevant constraint is really $x_i' \geq 0$ since $\bar{x}_i < v_i$ and $-t_i \leq 0$, so that the right hand side of the inequality above is negative.

Therefore the minimal amount of the private good necessary to sustain a Pareto superior point is

$$\begin{aligned} P &= \sum_{S_+} (\bar{x}_i - t_i - \hat{w}_i) + \sum_{S_-} (\bar{x}_i - t_i - \hat{w}_i) \\ &\quad + \sum_{T_+} 0 + \sum_{T_-} (\bar{x}_i - t_i - \hat{w}_i) + \sum_i t_i \\ &= \sum_{S_+} \bar{x}_i + \sum_{S_-} \bar{x}_i + \sum_{T_-} \bar{x}_i + \sum_{T_+} t_i - \sum_{S_+ \cup S_- \cup T_-} \hat{w}_i . \end{aligned}$$

Now, since $d = 0$ was the decision with strategies $\hat{w}_i$, $i = 1,\ldots,n$, we have

$$0 > \sum_i \hat{w}_i = \sum_{S_+ \cup S_- \cup T_-} \hat{w}_i + \sum_{T_+} \hat{w}_i$$

or

$$-\sum_{S_+ \cup S_- \cup T_-} \hat{w}_i > \sum_{T_+} \hat{w}_i \geq \sum_{T_+} \bar{x}_i .$$

Substituting above

$$P \leq \sum_i \bar{x}_i + \sum_{T_+} t_i \leq \sum_i \bar{x}_i$$

since $t_i \leq 0$ for all i, under the pivotal mechanism.

This inequality establishes the infeasibility of a Pareto superior allocation with $d = 0$.

With $d = 1$ and $\sum_i \hat{w}_i > 0$, the proof is analogous. When $\sum_i \hat{w}_i = 0$, the second part of the definition of essential satisfactoriness applies. Q.E.D.

4. THE GENERAL CASE

In order to discuss the results above in the context of mechanisms other than the pivotal mechanism, we will proceed in two stages. First we ascertain the class of mechanisms for which dominant strategies exist. Then, we study those which produce Pareto optimal results.

For simplicity we concentrate on the symmetric case, where $h_i(\cdot)$ is the same function for all i and is a symmetric function of w_{-i}.[5] We denote this common function by $h(\cdot)$. In order that dominant strategies exist for the mechanism defined by h, it is first necessary that it have a non-empty set of admissible strategies for all possible levels of the endowment.

Writing

$$\begin{aligned} -t_i(w_{-i}, w_i) &= \sum_{j \neq i} w_j + h(w_{-i}) && \text{if } \sum w_j \geq 0 \\ &= h(w_{-i}) && \text{if } \sum w_j < 0 \end{aligned}$$

we can define

$$m(h, w_i) = \min_{w_{-i}} -t_i(w_{-i}, w_i)$$

Since $\bar{x}_i \geq 0$, it is required that

$$\max_{w_i} m(h, w_i) \geq 0$$

---

[5] None of our results depend on this restriction used for simplicity of notation.

or that

$$\min_{w_{-i}} -t_i(w_{-i}, w_i) \geq 0 \qquad \text{for some choice of } w_i .$$

Lemma 2. *If for some w_{-i}, $h(w_{-i}) < \min(-\sum_{j\neq i} w_j, 0)$ then the mechanism defined by $h(\cdot)$ has no admissible strategies for $\bar{x}_i$ sufficiently small.*

Proof. The proof is immediate for if $h(\bar{w}_{-i}) < \min(-\sum_{j\neq i} \bar{w}_j, 0)$ then $t_i(w_i, \bar{w}_{-i})$ takes on one of the two values $-\sum_{j\neq i} w_j - h(\bar{w}_{-i})$ or $-h(\bar{w}_{-i})$ according to the sign of $\sum w_j$, and each of these is strictly positive.

For $h(\cdot)$ satisfying

$$h(w_{-i}) \geq \min\Bigl(-\sum_{j\neq i} w_j, 0\Bigr)$$

admissible strategies exist. Before we analyze the dominance of one strategy within this set, we characterize the admissible set according to the following:

Lemma 3. *For any mechanism defined by a function $h(\cdot)$, the set of admissible strategies for each $\bar{x}_i > 0$ is an interval. Moreover,*

(i) *it is a closed interval if $h(\cdot)$ is continuous,*

(ii) *it contains zero if $h(\cdot) \geq \min(-\sum_{j\neq i} w_j, 0)$.*

Proof. Suppose w and w' are admissible strategies and $w > w'$. The following inequalities must then hold:

(i) $h(w_{-i}) \geq -\bar{x}_i$ for $\sum_{j\neq i} w_j < -w$

(ii) $h(w_{-i}) \geq -\sum_{j\neq i} w_j - \bar{x}_i$ for $\sum_{j\neq i} w_j \geq -w$

(iii) $h(w_{-i}) \geq -\bar{x}_i$ for $\sum_{j\neq i} w_j < -w'$

(iv) $h(w_{-i}) \geq -\sum_{j\neq i} w_j - \bar{x}_i$ for $\sum_{j\neq i} w_j \geq -w'$.

Since (iii) implies (i) and (ii) implies (iv), then for w'' such that $w' < w'' < w$ we will have that

$$h(w_{-i}) \geq -\sum_{j\neq i} w_j - \bar{x}_i \qquad \text{for} \quad \sum_{j\neq i} w_j \geq -w$$

and therefore for

$$\sum_{j\neq i} w_j \geq -w''$$

and

$$h(w_{-i}) \geq -\bar{x}_i \qquad \text{for} \quad \sum_{j\neq i} w_j < -w'$$

and therefore for

$$\sum_{j\neq i} w_j < -w''.$$

The closedness of the interval of admissible strategies follows from the closedness of the requirement $x_i \geq 0$ and the continuity of $h(\cdot)$.

The admissibility of zero for the function $h(\cdot) = \min(-\sum_{j \neq i} w_j, 0)$ follows from the definition.

This characterizes the admissible strategies for every endowment. Because of this one can find the dominant admissible strategy directly. It will be the closest point to the true willingness-to-pay within the admissible set. From now on, we will assume that $h(\cdot)$ is continuous.

THEOREM 2. *Let $[-\alpha, \beta]$ be the interval of admissible strategies for an individual with endowment $\bar{x}$ and preferences v, when the mechanism is defined by $h(\cdot)$.*

Let w^* solve $\min_{w \in [-\alpha, \beta]} |v - w|$. Then w^* is the dominant strategy.

Proof. We consider individual i but drop the index when there is no risk of confusion.

Take the case $v < -\alpha < \beta$ and suppose $-\alpha < w < \beta$. Consider

$$u[f(-\alpha, w_{-i})] - u[f(w, w_{-i})] = 0 \qquad \text{if } \sum_{j \neq i} w_j \notin [-w, \alpha]$$

$$= \bar{x} - t(-\alpha, w_{-i}) - v - \bar{x} + t(w, w_{-i})$$

or

$$= -v - \sum_{j \neq i} w_j \qquad \text{if } \sum_{j \neq i} w_j \in [-w, \alpha]$$

but $v < -\alpha$ and $\sum_{j \neq i} w_j \leq \alpha$ implies $-v - \sum_{j \neq i} w_j > 0$ and hence setting $w = -\alpha$ dominates any other admissible strategy. Other cases can be treated analogously.

We have seen that we must place some restrictions on the arbitrary h—it must be continuous and bounded below by the pivotal mechanism—in order to insure the existence of a dominant admissible strategy for all agents. It is nevertheless still not true that the class of all mechanisms defined by functions h with these characteristics are satisfactory. That is, there are some dominant strategy mechanisms which may, in some cases, produce inefficient outcomes in an essential way. An example is the following.

There are two individuals. For each i, the function $h(w_j)$ for $j \neq i$ is defined by

$$\begin{aligned} h(w_j) &= -2(w_j + 1) & w_j &< -1 \\ &= 0 & w_j &\geq -1. \end{aligned}$$

This function clearly satisfies our requirements. One can verify that the set A of admissible strategies for each agent, as a function of his endowment, $\bar{x}_i$, is given by

$$\begin{aligned} A(\bar{x}_i) = {} & [-\infty, +\bar{x}_i] && \text{if } \bar{x}_i \geq 1 \\ & [-\infty, +\infty] && \text{if } \bar{x}_i < 1. \end{aligned}$$

Let

$$\begin{aligned} \bar{x}_1 &= 0, & v_1 &= 5 \\ x_2 &= 10, & v_2 &= -3. \end{aligned}$$

Thus

$$\begin{aligned} w_1 &= 0 \\ w_2 &= -3 \end{aligned}$$

will be the dominant admissible strategies of the agents. The project will be rejected and a transfer of $+4$ will be given to agent 1. The social state attained is therefore

$$z = (0, 4, 10, -4)$$

and the utilities attained are just the indicated consumption levels.

However, consider the alternative social state

$$z = (1, 0, 14, -4).$$

The utilities are now 5, 11 and -4 for the two agents and the government respectively. This clearly dominates the equilibrium attained by the mechanism.

It is therefore clear that only some of the satisfactory mechanisms for economies with unrestricted transfers of the private good continue to have this property when the non-negativity of consumption is required. We can then ask, naturally, for the class of mechanisms that are satisfactory in this case.

Conditions sufficient to insure the successfulness of a dominant-strategy inducing mechanism are not hard to derive. It does not seem possible, however, to state a necessary and sufficient condition on the function h, in a readily interpretable form.

By virtue of Lemmas 2 and 3 the endpoints of the interval defining the admissible strategies are both at least equal to the level of endowment in absolute value. If h is above the pivotal h function, then strategies that exceed $\bar{x}$ in absolute value are allowable. A sufficient condition that a mechanism be satisfactory when consumption is bounded below by zero is that it never provide any individual a subsidy that is greater than the absolute value of the difference between his strategy and his endowment. It may be seen that the pivotal mechanism, which never provides any subsidies, satisfies this criterion.

Using the notation paralleling that of Theorem 1, let

$$\begin{aligned} S_{+} &= \{i \mid 0 < w_i < v_i\} \\ S_{-} &= \{i \mid 0 > w_i > v_i\} \\ T_{+} &= \{i \mid 0 < w_i = v_i\} \\ T_{-} &= \{i \mid 0 > w_i = v_i\}. \end{aligned}$$

Suppose the strategies $(w_1^*, \ldots, w_n^*)$ are played and the mechanism rejects the project, resulting in the social state

$$z = \left(0, x_1^*, \ldots, x_n^*, \sum \bar{x}_i - \sum x_i^*\right)$$

and that

$$z' = \left(1, x_1', \ldots, x_n', \sum \bar{x}_i - \sum x_i'\right)$$

dominates z. This would require:

$$\begin{aligned} x_i' &\geq \sum x_i^* \\ x_i' &\geq x_i^* - v_i \\ x_i' &\geq 0 \end{aligned}$$

for each i. Hence the amount of the private good necessary to sustain z', P, would have to satisfy

$$\begin{aligned} P &= \sum x_i' + \sum \bar{x} - \sum x_i^* \\ &\geq \sum_{S_+ \cup S_- \cup T_-} (x_i^* - w_i^*) + \sum \bar{x}_i - \sum x_i^* + \sum_{T_+} x_i' \\ &\geq -\sum_{T_+} x_i^* - \sum w_i^* + \sum_{T_+} w_i^* + \sum \bar{x}_i \\ &\geq \sum \bar{x}_i + \sum_{T_+} [(w_i^* - \bar{x}_i) - (x_i^* - \bar{x}_i)] \end{aligned}$$

since $\sum w_i^* < 0$.

Thus, if for every individual whose strategy is constrained from above, and who therefore responds with the upper endpoint of the set of admissible strategies, the maximum possible subsidy is less than the difference between this strategy and the endowment, the last sum will be positive and a contradiction to the non-optimality of z is derived.

By a parallel argument for the case of $\sum w_i^* < 0$, we arrive at the condition

$$P \geq \sum \bar{x}_i + \sum_{T_-} [(-w_i - \bar{x}_i) - (x_i^* - \bar{x}_i)]$$

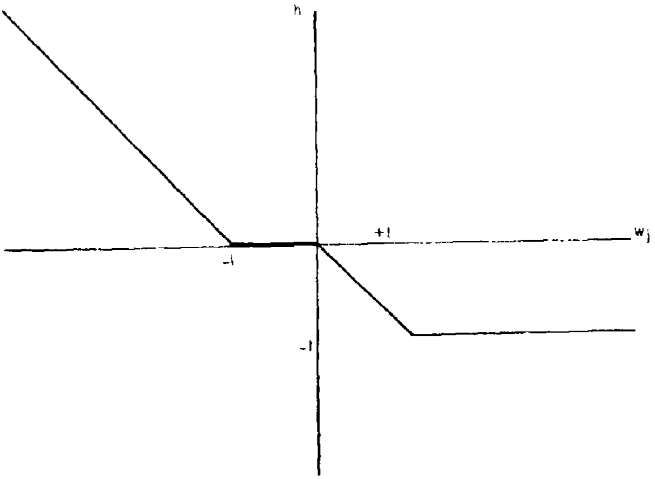
which has the interpretation given above.

Since this is a sufficient condition, one can extend the class of satisfactory mechanisms by checking that they satisfy it. For example, the pivotal mechanism plus any constant transfer K will be successful since the maximum subsidy is K and the admissible strategies are $[-\bar{x}_i - K, \bar{x}_i + K]$, so that members of T_+ will choose $\bar{x}_i + K$ and members of T_- will choose $-\bar{x}_i - \bar{K}$.

It is not the case, however, that any mechanism admitting the possibility of a subsidy greater than $|w_i| - \bar{x}_i$, will fail to be a member of the satisfactory class. To see this, we consider an example closely related to that used above.

Let

$$\begin{aligned} h &= -w_j - 1 && w_j \leq -1 \\ &= 0 && -1 \leq w_j \leq 0 \\ &= -w_j && 0 \leq w_j \leq 1 \\ &= -1 && 1 \leq w_j . \end{aligned}$$



The mechanism will have dominant admissible strategies by virtue of Lemma 3 and Theorem 2. The admissible strategy correspondence is given by

$$\begin{aligned} A(\bar{x}_i) &= [-\bar{x}_i, +\bar{x}_i] && \text{if } |\bar{x}_i| \leq 1 \\ &= [-\infty, +\infty] && \text{if } |\bar{x}_i| > 1. \end{aligned}$$

We will now show that this mechanism is essentially satisfactory for all two-person economies even though when

$$\bar{x}_1 = 0, \qquad v_1 = 5$$
$$\bar{x}_2 = 10, \qquad v_2 = -3$$

we have that the project is rejected, since $w_1 = 0$, $w_2 = -3$, and agent 1 is in T_+ with $h(w_2) = 2 > 0 = w_1 - \bar{x}_1$. This will demonstrate that the nonexistence of such situations is not a necessary condition for satisfactoriness.

In the demonstration of satisfactoriness to follow we use two conditions which are each sufficient to insure that a particular social state is efficient. They have both been proven and discussed above.

Let

$$z = \left(d, x_1, \ldots, x_n, \sum \bar{x}_i - \sum x_i\right).$$

If

(*) $d = 1$ and $\sum v_i \geq 0$, or $d = 0$ and $\sum v_i \leq 0$, then z is efficient relative to $\sum \bar{x}_i - x_i$.

(**) If z is such that for each i in T_+, T_-, $x_i - \bar{x}_i = w_i - \bar{x}_i$, then z is efficient relative to $\sum \bar{x}_i - x_i$.

We consider three cases, which, by symmetry, are exhaustive:

(I) $\bar{x}_1 \geq 1, \bar{x}_2 \geq 1$

(II) $\bar{x}_1 < 1, \bar{x}_2 \geq 1$

(III) $\bar{x}_1 < 1, \bar{x}_2 < 1$.

Case (I). Here, $w_1 = v_1$ and $w_2 = v_2$ are both admissible. Thus the resulting social state will satisfy the hypothesis in condition (*) and will be efficient.

Case (II). We will have $w_1 \leq \bar{x}_1$, and $w_2 = v_2$. Only individual 1 can be in T_+ or T_-. Take the case in which he is in T_+, so that $v_1 > \bar{x}_1$, and $w_1 = \bar{x}_1$ is his dominant admissible strategy. If $w_2 \geq -\bar{x}_1$, condition (*) applies and the result is Pareto optimal. If $w_1 < \bar{x}_2$, the project will be rejected even though $v_1 + v_2$ may be positive. However, $w_2 < 0$ and rejection of the project implies that the subsidy received by agent 1 is exactly $h(w_2)$. If there exists a feasible superior situation, it means that the quantity of the private good consumed by agent 1, $h(w_1) + \bar{x}_2$ exceeds the amount of compensation it is necessary to give agent 2, $-v_2$, for the change in the decision towards acceptance of the project. Since in this case $w_2 = v_2$, the nonoptimality of the social state selected means

$$h(w_2) + \bar{x}_1 \geq -w_2 .$$

But the left hand side of this inequality is either $-w_2 - 1$ or zero, according to whether w_2 is above or below -1.

Consider first the situation where $w_2 < -1$. Hence we have

$$-w_2 - 1 + \bar{x}_1 < -w_2$$

or

$$\bar{x}_1 < 1$$

as the condition for inefficiency—but this contradicts the hypothesis of case (II).

If $w_2 \geq -1$, then we need $\bar{x}_1 < -w_2$ for nonoptimality. But since $w_1 = \bar{x}_1$ $(1 \in T_+)$ and $w_1 + w_2 < 0$ $(d = 0)$, this condition cannot be satisfied. Therefore a superior social state cannot exist in this case.

Case (III). Here $|w_1| < \bar{x}_1 < 1$ and $|w_2| < \bar{x}_2 < 1$. But for this strategy, the *h* function coincides with the pivotal mechanism. Therefore no positive subsidies can ever arise and condition (**) applies to insure optimality of the resulting social state.

In summary, the nonexistence of potential situations in which subsidies in excess of $|w_i| - \bar{x}_i$ are paid is a sufficient, but not a necessary condition for the satisfactoriness of social decision mechanisms with bounded consumption sets. A more precise delineation of the class of satisfactory mechanisms is at present an open problem.

5. REMARKS

The pivotal mechanism has been shown to be quite useful even in environments with bounded consumption in that it can elicit dominant strategies that produce efficient outcomes. It has, in addition, two further properties which are highly desirable. We will comment on these below.

Our definition of efficiency allowed for negative transfers to the decision maker, even though we restricted individual agents to have positive net holdings of the transferable resource. This was justified by the prospect that deficits might be covered out of "general revenues." If this were impossible, or irrelevant, one would want to insure a non-negative level of revenue generation. It is clear that this property is satisfied by the pivotal mechanism, which never gives a positive transfer to any agent. Further, any h-function which is greater than the pivotal h-function at any point runs the risk of subsidizing someone. By virtue of Lemma 2, therefore, the pivotal mechanism is the only one in the satisfactory class which can insure that the decision-makers transfer is never negative.

Another property of social decision mechanisms that is often required is "individual rationality"—that no agent can ever suffer a decrease in utility vis à vis his initial position. Clearly, no member of the satisfactory class has this property: if v_i is negative and $\sum_{j \neq i} w_j$ is larger in absolute value, but positive, then agent i will be hurt, and uncompensated, by the project's acceptance under the pivotal mechanism. Even if the transfer to such an agent were positive in another mechanism, he would lose if $|v_i|$ were sufficiently large.

There is another concept which is related in spirit to "individual rationality" but distinct from it. There are many situations, especially those involving public goods, where the appropriate benchmark from which to measure utility change is ill-defined. For instance, "$d = 1$" may represent the cancellation of an ongoing project. In a non-coercive society, however, one option open to any agent is to decline participation in the mechanism. He should be able to avoid all transfers in this way, and the decision will reflect the strategies of the remaining agents.

The pivotal mechanism has the property that no agent would decline participation in this way. To see this, observe that the strategy of non-participation in this mechanism is precisely equivalent to setting $w_i = 0$. If this is dominated by any other value of w_i, non-participation is dominated as well. (In cases of indifference, no bias in the decision results in any event.) Other members of the successful class possess this property as well, but since the pivotal mechanism has all the attributes we are looking for, we have not attempted to characterize the set of those possessing this feature in more detail.

Finally, and unfortunately, we must point out the failure of these satisfactory mechanisms in situations with three or more possible public decisions.

Let $K = \{0, 1, 2\}$ and consider an agent, say 1, for whom $v_1 = (0, 4, 6)$ describes his willingness-to-pay for each of the three public projects in K, and $\bar{x}_1 = 5$. For the pivotal mechanism generalized to the case of vector-valued responses representing evaluations of $k \in K$, one can verify straightforwardly that the admissible strategy space is

$$\{w_i(\cdot) \mid \max_K w_i(k) - \min_K w_i(k) \leq \bar{x}_i\}.$$

The problem with using the pivotal mechanism is not that an inefficient outcome might be selected, but rather that dominant strategies might fail to exist. In the present instance, consider strategies of the form

$$w_1 = (0, w, 5)$$

where $0 < w < 5$, which are the obvious candidates for dominant strategies.

To prove that no strategy of this form is dominant, consider three cases:

I. $w < 3$

II. $4 \geq w \geq 3$

III. $w > 4$

and let

$$\left(0, \sum_{j\neq 1} w_j(1), \sum_{j\neq 1} w_j(2)\right)$$

be the (normalized) response of all the other agents.

Strategies in case I fail to be dominant when, for example, $\sum_{j\neq 1} w_j(1) = -3$ and $\sum_{j\neq 1} w_j(2) = -10$. The project $K = 0$ would be adopted, yielding a payoff of zero, whereas $\tilde{w} = (0, 3 + \epsilon, 5)$ would produce a payoff of one. Case III can be eliminated in a similar fashion.

In case II, the matter is more delicate: when $w < 4$ a counterexample similar to that in case I can be arranged by taking $-4 < \sum_{j\neq i} w_j(1) < -w$. The payoff to the strategy $w = (0, 3 + \epsilon, 5)$ would be superior.

When $w = 4$, let $\sum_{j\neq 1} w_j(1) = -3\frac{1}{2}$ and $\sum_{j\neq 1} w_j(2) = -4\frac{3}{4}$. Thus,

$$\sum_i w_i(1) = \tfrac{1}{2} \quad \text{and} \quad \sum_i w_i(2) = \tfrac{1}{4}$$

so that $d = 1$ and the net payoff to agent 1 is $+\frac{1}{2}$. By playing $w = (0, 3, 5)$, the decision $d = 2$ would be taken and a net payoff of $1\frac{1}{4}$ would be obtained.

Thus, the lack of a dominant strategy within the admissible strategy space—even though there would be a dominant strategy for a consumer with a large enough endowment—precluded the existence of satisfactory mechanisms in these environments.

REFERENCES

1. E. H. CLARKE, Multipart pricing of public goods, *Public Choice* (1971), 19–33.
2. J. GREEN AND J. J. LAFFONT, Characterization of satisfactory mechanisms for the revelation of preferences for public goods, *Econometrica* **45** (1977), 427–438.
3. J. GREEN AND J. J. LAFFONT, "Révélation des Préférences pour les Biens Publics: Première partie, Caractérisation des Mécanismes Satisfaisants" (E. Malvinaud, Ed.), pp. 83–103, Cahiers du Seminaire d'Économetrie No. 19, Centre National de la Recherche Scientifique, Paris, 1977.
4. J. GREEN, E. KOHLBERG AND J. J. LAFFONT, Partial equilibrium approach to the free-rider problem, *J. Public Econ.* **6** (1976), 375–394.
5. T. GROVES, Incentives in teams, *Econometrica* **41** (1973), 617–631.
6. T. GROVES AND M. LOEB, Incentives and public inputs, *J. Public Econ.* **4** (1975), 211–226.
7. L. JOHANSEN, The theory of public goods: Misplaced emphasis, *J. Public Econ.* **7** (1977), 147–152.
8. W. VICKREY, Counterspeculation, auctions and competitive sealed tenders, *J. Finance* **16** (1961), 1–17.
9. M. WALKER, "On the Impossibility of a Dominant Strategy Mechanism to Optimally Decide Public Goods Questions," Department of Economics Working Paper No. 160, S.U.N.Y. at Stony Brook, Stony Brook, New York, 1976.

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Income Inequality and Income Mobility*

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1. INTRODUCTION

The usual indices of inequality are derived from observations on income, wealth etc. corresponding to a particular point or period of time. It has been frequently argued that inequality values by themselves do not accurately reflect the differences between individuals, since the true situation depends to a large extent on how the relative positions of individuals vary over time. Thus, it has been argued, "static" measures of inequality should be supplemented by "dynamic" measures of changes through time, which we shall call measures of mobility.

Studies which have proposed ways of quantifying these dynamic changes broadly fall into two categories: those which use elementary statistics, such as the correlation coefficient; and those which make more sophisticated suggestions based on transition matrices and other simple stochastic specifications of dynamic processes. Shorrocks [9] provides a number of references and discusses some of the issues involved in deriving an index of mobility from transition matrices. Particular consideration is given to the interval of time between observations, since a relationship is expected between the amount of observed movement and the length of time over which movement can take place; in a short space of time there is little opportunity for movement, even if the society is inherently very mobile.

These earlier attempts to define an index of mobility are mainly concerned with stock variables, interpreted in a wide sense to include social status and occupation as well as wealth and the assets of firms. Once attention is turned to flow variables, such as income, it becomes apparent that there is another important consideration. Observed variations in income depend not only on the interval between observations, but also on the length of the

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accounting period chosen for incomes. Data availability and custom dictate that the period selected is normally one year, although shorter intervals, a week or a month, are occasionally used. If the accounting period were extended from, say, one month to one year, variations in monthly incomes (previously classified as dynamic changes) become subsumed within the annual income figure. Some of the *dynamic* changes are therefore incorporated in the *static* inequality value, and the distinction between the static and dynamic aspects becomes very blurred. Similarly, as we pass from annual to lifetime income inequality, intra-lifetime income mobility is lost in the process of aggregation.

However, the effects of income variations over time do not disappear altogether: they are reflected in the changes recorded in the inequality value. Those occupying the highest and lowest positions in the income hierarchy rarely remain there forever. So the aggregation of incomes over time tends to improve the relative position of those temporarily found at the bottom of the distribution, and the situation of those at the top tends to deteriorate. For this reason it is commonly supposed that inequality falls as the accounting period is lengthened. Empirical confirmation of this relationship requires longitudinal income data samples, of which very few exist. However, the little evidence available agrees with expectations.[1] For example, Soltow [10] traced the annual incomes of a sample of Norwegians over the period 1928-1960. The Gini coefficient for the 33 years combined was 0.134 compared to an average value of 0.183 for the separate years. Using US data, Kohen *et al.* [3] found that the Gini coefficient for family income and earnings of young men (aged 16-24) fell by 4.7-7.4 % when cumulated over two years, and by 9.2-10.8 % when cumulated over three. For middle-aged men (45-59 years old), aggregating incomes over two years caused the Gini to decline by about 4 %.[2]

There are reasonable grounds, therefore, for supposing that the existence of mobility causes inequality to decline as the accounting interval grows. Furthermore, intuition suggests that the extent to which inequality declines will be directly related to the frequency and magnitude of relative income variations. If the income structure exhibits little mobility, relative incomes will be left more or less unaltered over time and there will be no pronounced egalitarian trend as the measurement period increases. In contrast, inequality may be expected to decrease significantly in a very (income) mobile society. The main purpose of this paper is to exploit this relationship between mobility and inequality, to derive an index of mobility for flow variables. In essence, mobility is measured by the extent to which the income distribution is equalised as the accounting period is extended.

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[1] The evidence from a sample of Delaware tax payers reported by Kravis [6, p. 272] is somewhat ambiguous. Cumulating incomes forward through time, the Gini coefficient tended to *increase*, but cumulating backwards from the last year it tended to fall.

[2] Kohen et al [3], Tables 3 and 4. The data for middle-aged men refer to the years 1966-67 and those for young men to 1966-68. The inequality reduction is calculated as a percentage of the mean of the annual Gini coefficients in the relevant years.

In the next section, the relationship between recorded inequality and the length of the accounting period is considered in more detail. For a wide class of inequality measures, including all those derived from quasi-concave social welfare functions using the approach of Atkinson [1], Sen [8] and others, it is shown that inequality must tend to decline as the measurement interval increases.[3] However, this result does not hold for all inequality measures satisfying the Dalton-Pigou Principle of Transfers.

In section 3 we define an index of mobility (or, more accurately, a family of mobility indices) which can be applied to flow variables.[4] This index has little similarity with mobility measures proposed in other studies, although the notion of a completely immobile structure, where the relative positions of all individuals do not change and which is associated with a zero mobility value, is retained.

Section 4 presents some simple computations of mobility values based on the figures published in Kohen *et al.* [3]. A few concluding remarks, directed in particular at the social welfare implications, are given in section 5.

Before we begin to discuss the issues outlined above, it may be opportune to make one brief comment on the concept of mobility. In this paper income mobility is defined to be changes in the relative incomes of individuals through time. The term therefore refers to observed movements in income, which depend on both the opportunities for movement and the response of individuals to the opportunities offered. In other applications, such as labour mobility, the concept appears to refer only to the latter: mobility is related to the amount of "friction" in the system—the speed of response to economic incentives. As far as income movements are concerned, it seems premature to attempt to identify these components and distinguish the role of friction from the opportunities available. However, for those readers who are used to regarding mobility as the lack of friction, the broader sense in which mobility is used in this paper should be borne in mind.

2. INCOME INEQUALITY AND THE ACCOUNTING PERIOD

Consider a population consisting of n individuals (or households) and let Y_i denote the total income received by individual i during a particular time interval. Measures of income inequality are defined over the income distribution vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ whose mean $\mu = \Sigma_i Y_i/n$ is assumed to be positive. In this paper, attention is primarily directed at the set of inequality measures which can be written in the form

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[3] Essentially the same result is discussed in Kolm [5, Section IX]. In his terminology, this paper expoits the "weighted sub-additivity" property of convex inequality indices.

[4] We concentrate here on income mobility, but the general approach can be equally well applied to other flow variables. For example, an examination of the concentration of sales among firms could be used to indicate the extent of industrial mobility.

$$I[Y] = g\left(\frac{\mathbf{Y}}{\mu}\right) \text{ where } g(\cdot) \text{ is strictly convex} \tag{1}$$

A variety of measures satisfying (1) can be derived from social welfare functions, using the approach adopted by Atkinson [1], Dasgupta, Sen, and Starrett [2] and others. Given a social welfare function

$$W = F(Y_1, Y_2, \ldots, Y_n) = F(\mathbf{Y}) \tag{2}$$

the equally distributed equivalent income Y_e is that amount of income which, if received by all individuals, generates the same level of social welfare as the distribution $\mathbf{Y}$.[5] It is therefore defined implicitly by

$$F(Y_e, Y_e, \ldots, Y_e) = F(\mathbf{Y}) \tag{3}$$

If $F(.)$ is continuous, strictly monotonic and symmetric, Y_e can be written explicitly

$$Y_e = f(\mathbf{Y}) > 0 \tag{4}$$

where $f(.)$ is also continuous, strictly monotonic and symmetric. The inequality index is then

$$I = 1 - \frac{Y_e}{\mu} = 1 - \frac{f(\mathbf{Y})}{\mu} \tag{5}$$

We impose two further restrictions on the swf: $F(.)$ is assumed to be strictly quasi-concave and homothetic. The former implies that $f(.)$ is strictly quasi-concave, and this ensures that I takes values in the range $[0, 1)$. If the swf is homothetic, $f(.)$ is linearly homogeneous; both of these correspond to the assumption that the inequality measure is *mean independent* — I does not change when all incomes increase or decrease in the same proportion.[6] Thus (5) becomes

$$I = 1 - f\left(\frac{\mathbf{Y}}{\mu}\right) \tag{6}$$

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[5] The earlier paper by Kolm [4, p. 186] calls Y_e the "equal equivalent" of the distribution $\mathbf{Y}$.

[6] Since I is mean-independent, the same inequality value would be recorded if the index were calculated from the average flow rates of income over the accounting period, rather than the actual income receipts.

It remains to be shown that $g(x) = 1 - f(x)$ *is strictly convex: Define* $\sigma > 0$ such that $f(\mathbf{x}) = \sigma f(\mathbf{x}') = f(\sigma \mathbf{x}')$. Then

$$\begin{aligned} & g(\lambda \mathbf{x} + (1-\lambda)\mathbf{x}') \\ & = 1 - f(\lambda \mathbf{x} + (1-\lambda)\mathbf{x}') \\ & = 1 - \frac{\sigma\lambda + 1 - \lambda}{\sigma} f\left(\frac{\sigma\lambda}{\sigma\lambda + 1 - \lambda}\mathbf{x} + \frac{1-\lambda}{\sigma\lambda + 1 - \lambda}\sigma\mathbf{x}'\right) \\ & \qquad \text{by homogeneity of } f(.) \\ & < 1 - \frac{\sigma\lambda + 1 - \lambda}{\sigma}\left[\frac{\sigma\lambda}{\sigma\lambda + 1 - \lambda} f(\mathbf{x}) + \frac{1-\lambda}{\sigma\lambda + 1 - \lambda} f(\sigma\mathbf{x}')\right] \\ & \qquad \text{using strict quasi-concavity} \\ & = \lambda[1 - f(\mathbf{x})] + (1-\lambda)\left[1 - \frac{1}{\sigma} f(\sigma\mathbf{x}')\right] \end{aligned}$$

Hence

$$g(\lambda \mathbf{x} + (1-\lambda)\mathbf{x}') \leq \lambda g(\mathbf{x}) + (1-\lambda) g(\mathbf{x}') \tag{7}$$

with equality holding if and only if $\mathbf{x} = \sigma\mathbf{x}'$. But since $g(.)$ is defined over the unit simplex, this can only occur when $\mathbf{x} = \mathbf{x}'$.

The properties of the index (6) are well known. We note here only that it satisfies the Principle of Transfers, that it can be interpreted as the relative welfare loss, and that its minimum value of zero is attained if and only if all incomes are equal. For most of the results that follow, it is not necessary to assume that the inequality measure has been derived from a swf: that the index satisfies (1) is all that is required.[7] This enables us to consider measures which can take values greater than 1.

Consider a time interval $[t_0, t_m)$ which is partitioned into m sub-periods of the form $[t_{k-1}, t_k)$, where $t_0 < t_1 < \cdots < t_{m-1} < t_m$. Let $Y_i(t_{k-1}, t_k)$ denote the income received by individual i over the time interval $[t_{k-1}, t_k)$. Then

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[7] The class of indices satisfying (1) contains a wide variety of mean-independent inequality measures in common use. It includes all members of the Atkinson family,

$$I = 1 - \left[\frac{1}{n}\Sigma_i\left(\frac{Y_i}{\mu}\right)^{1-\epsilon}\right]^{1/(1-\epsilon)} \qquad \epsilon > 0$$

and any measure which can be written

$$I = \frac{1}{n}\Sigma_i\,\phi\left(\frac{Y_i}{\mu}\right) \qquad \text{where } \phi''(\cdot) > 0.$$

The square of the coefficient of variation and the Theil coefficient both belong to this latter group.

$\mu(t_{k-1}, t_k) > 0$ is the corresponding mean income and $I[Y(t_{k-1}, t_k)]$ is the associated inequality value. For the aggregate time period we have

$$\mathbf{Y}(t_0, t_m) = \sum_{k=1}^{m} \mathbf{Y}(t_{k-1}, t_k) \tag{8}$$

and

$$\mu(t_0, t_m) = \sum_{k=1}^{m} \mu(t_{k-1}, t_k) \tag{9}$$

We now wish to compare the inequality $I[Y(t_0, t_m)]$, recorded for the aggregate period, with the index values $I[Y(t_{k-1}, t_k)]$ calculated for the m sub-periods.

THEOREM 1. *For any inequality index satisfying* (1)

(a) $I[Y(t_0, t_m)] \leqslant \sum_{k=1}^{m} w_k I[Y(t_{k-1}, t_k)]$ *where* $w_k = \mu(t_{k-1}, t_k)/\mu(t_0, t_m)$, *the proportion of aggregate income received in sub-period k.*

(b) *Equality holds in* (a) *if and only if* $\mathbf{Y}(t_{k-1}, t_k)/\mu(t_{k-1}, t_k)$ *is independent of k (or, in other words, iff the* relative *income of each individual does not change over time).*

Proof (a)

$$\begin{aligned} I[Y(t_0, t_m)] &= g\left(\frac{\sum_k \mathbf{Y}(t_{k-1}, t_k)}{\mu(t_0, t_m)}\right) \\ &= g\left(\sum_k w_k \frac{\mathbf{Y}(t_{k-1}, t_k)}{\mu(t_{k-1}, t_k)}\right) \\ &\leqslant \sum_k w_k g\left(\frac{\mathbf{Y}(t_{k-1}, t_k)}{\mu(t_{k-1}, t_k)}\right) \quad \text{by convexity of } g(.) \\ &= \sum_k w_k I[Y(t_{k-1}, t_k)] \end{aligned}$$

(b) Since $w_k > 0$ for all k, strict convexity of $g(.)$ ensures that equality holds in (a) if and only if $\mathbf{Y}(t_{k-1}, t_k)/\mu(t_{k-1}, t_k)$ is the same in each sub-period.

This result is a formal statement of the notion that income inequality falls as we extend the time period over which incomes are defined.[8] To examine the implications of the theorem in more detail, it is convenient to consider first the situation arising when inequality is the same in each of the sub-periods. We shall also assume, for simplicity, that the sub-periods are one year in length and the aggregate period is therefore m-years.

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[8] We are only interested here in the aggregation of incomes over time. But the problem is formally identical to that of aggregating income from different sources (earnings, transfers, dividends, interest receipts, etc.). So Theorem 1 can also be applied to this kind of income breakdown.

Annual Inequality Constant

Since $I[Y(t_{k-1}, t_k)]$ is independent of k, part (a) of Theorem 1 becomes

$$I[Y(t_0, t_m)] \leq I[Y(t_{k-1}, t_k)] \tag{10}$$

Thus the income inequality computed for the whole m-year period *cannot exceed* that recorded in each of the single years. Moreover, by part (b) of the theorem, aggregating over consecutive years *must reduce* the inequality value unless the relative annual incomes of all individuals remain constant over time.

This result clearly relies heavily on the form of the inequality index chosen in (1). For those inequality measures which do not satisfy that requirement, two cases are worth distinguishing. Some measures, notably the Gini coefficient, violate (1) because the corresponding $g(.)$ function is convex, but not strictly convex. For these Theorem 1(a) and equation (10) still hold, but not Theorem 1(b). Examples can be constructed in which relative annual incomes are not constant over time, but m-year inequality is the same as the annual value.[9] Other measures violate (1) because they do not correspond to a $g(.)$ function which is even convex. These may be perfectly acceptable inequality measures (in that they satisfy the Principle of Transfers) and yet m-year inequality may exceed the inequality value in every individual year.[10]

Annual Inequality Varies

In general, income inequality will vary over consecutive time periods. Cross-section estimates of income inequality within age groups, for example,

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[9] For the Gini coefficient this would occur if changes in relative incomes are sufficiently small that all individuals maintain their same income *rank* in all periods. Of course, it is extremely unlikely that income variations over time will not alter the order of individuals in the income hierarchy.

[10] For example, consider a 3 person population with income shares y_i, ordered so that $y_1 < y_2 < y_3$, and the index

$$I = \tfrac{1}{2}(1 - 3y_1)(3y_3 - 1) \tag{11}$$

If (0, 3, 3), (1, 1, 4) are two annual distributions, the 2-year distribution (1, 4, 7) is such that

$$I[1, 4, 7] > I[0, 3, 3] = I[1, 1, 4].$$

In geometric terms, Theorem (1a) does not hold for the index (11) because the iso-inequality contours on the unit simplex do not enclose convex sets.

tend to show that inequality increases with age, at least up to the age of retirement. Thus we shall normally have to be content with a weaker statement of the relationship between income inequality and the length of the observation period. Theorem 1 indicates that m-year inequality will not exceed a weighted average of the annual inequality values, where the weights are proportional to the mean incomes in the corresponding years (and sum to unity); and m-year inequality will be less than this weighted average unless the relative incomes of all individuals remain constant over time. In particular,

$$I[Y(t_0, t_m)] \leq \max_k I[Y(t_{k-1}, t_k)] \qquad k = 1, \ldots, m \qquad (12)$$

so m-year inequality will never exceed the maximum inequality value recorded in the separate years. This will also be true of inequality measures corresponding to convex, but not strictly convex, $g(.)$ functions in (1). But (12) need not hold for all inequality measures satisfying the Principle of Transfers, as the example using the index (11) demonstrates.[11]

3. AN INDEX OF MOBILITY

We now turn to the problem of defining an index of mobility which reflects the variations in income over time. Any attempt to condense data on the income profiles of a group of individuals into a single statistic clearly involves a great loss of information. Single number measures of static inequality are now generally recognised to have serious drawbacks for exactly this reason and the measurement of mobility is an even more complex subject, since it involves the extra dimension of time. A single valued index of mobility will never, therefore, do justice to all the various aspects that warrant consideration. Nevertheless, such an index may provide a useful tool in the analysis of income variations and in indicating the limitations of such an approach.

It is convenient to begin by identifying the situation in which there is no income mobility. We shall assume that this state of *complete immobility* (or *perfect rigidity*) occurs when the income of every individual is constant over time; in other words, when the income profiles are horizontal. This can only

[11] Theorem 1 also casts some light on the relationship between annual and lifetime income inequality. For a cohort of individuals and a suitable index, the information suggests that lifetime inequality will tend to be lower than the annual figure. There is some confusion on this particular issue, since lifetime income is often defined as the total income receipts over the lifespan, i.e. average annual income times the length of life. The negative correlation between mortality and income can then lead to greater dispersion in lifetime incomes compared with incomes in any one year. However there is no conflict with Theorem 1, because the population of individuals does not remain the same. If the population were maintained at a constant size, by assigning zero income to those who have died, the results of Theorem 1 would still apply.

happen when there is no time trend in the mean income of the group. If mean income is changing, it seems natural to extend the concept to include all cases where *relative* incomes remain constant. Thus income profiles may not be horizontal, but they are all scalar multiples of one another.

DEFINITION. There is *complete immobility* when the relative incomes of all individuals remain constant through time.

A completely immobile income structure provides the benchmark with which other sets of income profiles can be compared. As income profiles deviate further from this extreme, income mobility increases. The feature of the completely immobile case on which we focus attention, is the fact that the inequality value recorded by a (mean independent) index will be the same for any time interval chosen for income. Moreover, for inequality measures satisfying (1), this property *only* holds when relative incomes are constant over time: if incomes are not "completely immobile", inequality tends to decline as the length of the measurement period increases.

THEOREM 2. *For inequality measures satisfying* (1), *observed inequality is independent of the measurement period if and only if there is complete immobility.*

Proof. this follows directly from Theorem 1.

Let us now suppose, for the sake of simplification, that (relative) income inequality is constant during m consecutive years, but the income rankings of some individuals change over time. If these changes in relative positions are infrequent and fairly small, the income pattern will deviate only slightly from complete immobility. Relative incomes over the whole m-years will be similar, but not identical, to those observed in the single years. The inequality recorded for the m-year period will be approximately, but not exactly, the same as the annual value. In fact, we know from Theorem 1 that inequality over the longer period will be slightly less than annual inequality: aggregation over time periods makes the distribution more egalitarian.

If changes in the income positions of individuals occur more frequently, or are greater in magnitude, this egalitarian trend will become more pronounced, and m-year income inequality will deviate further from the annual figure. Thus we anticipate that the frequency and magnitude of relative income changes over time will be reflected in the difference between long and short period inequality.

If the short period corresponds to one year, the "long" period may be anything from two years upwards. Given income observations in consecutive years, we can take a succession of long periods, initially aggregating over the first two years, then the first three years and so on. The relationship between the inequality of incomes aggregated over the first m years and the value of m, is illustrated in figure 1 for different income structures.

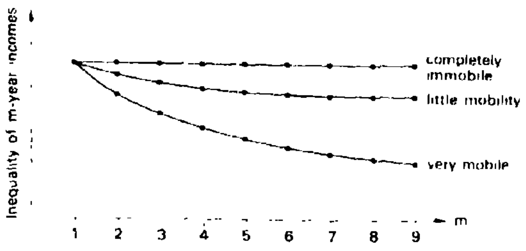

FIGURE 1

In a completely immobile system, inequality is independent of the length of the observation period and produces the horizontal line. In all other cases m period inequality is less than annual inequality, so the curves lie below the horizontal. Equalisation is more pronounced in a very mobile society, and the corresponding curve lies below that of the society with little (income) mobility.

The curves are drawn downward sloping, suggesting that equalisation continues as the "long" time period is extended. Empirically, this may be the most likely relationship, but it need not necessarily be true. Consider the following society consisting of two individuals, A and B.

Year	1	2	3	1 + 2	1 + 2 + 3
income A	1	2	1	3	4
income B	2	1	2	3	5
maximum/minimum	2	2	2	1	1.25

With only two individuals, any inequality measure will be an increasing function of the ratio of maximum to minimum income. Year 2 receipts equalise incomes over the two year period, and cause inequality to drop. But incomes in year 3 move the distribution back towards that observed in year 1, so inequality over the three year period exceeds that over the first two years.

When income inequlity in consecutive years is not constant, Theorem 1 suggests that the appropriate comparison is between the m-year index value and the weighted average of the annual values, with the weights chosen proportional to the mean annual incomes. Define the ratio

$$R = \frac{I[Y(t_0, t_m)]}{\sum_{k=1}^{m} w_k I[Y(t_{k-1}, t_k)]} \tag{13}$$

where

$$w_k = \mu(t_{k-1}, t_k)/\mu(t_0, t_m)$$

When R is plotted against the aggregate interval length m, the curves are still expected to correspond to those depicted in figure 1: a horizontal line for a completely immobile system, and a downward shift of the graph as relative income changes become more frequent and/or larger in magnitude. Thus, for the *same* aggregation period m, more mobile structures record a lower value of R. We shall take R to be a measure of the *rigidity* of an income system, and $M = 1 - R$ to be an index of mobility.

Under this definition, mobility is regarded as the degree to which equalisation occurs as the observation period is extended. This viewpoint captures the prime importance of mobility for economists. When income distributions are compared, it is often noted that the inequality observed in annual incomes may not be a good guide to the inequality of lifetime incomes, which is a more appropriate indication of the differences in opportunities faced by different individuals. A country in which individuals are given great opportunities for movement within the distribution in the long run, may have high annual inequality and may compare badly with other countries on an annual basis, even though it has lower lifetime inequality. The measure of mobility proposed here provides a way of identifying those countries which appear to exhibit considerable inequality (high annual I), but which would compare more favorably over a longer period (high mobility M, or low rigidity, R). If a country A has both greater annual inequality I and greater rigidity R than country B, it will be more unequal than B whatever period is chosen for comparison. But if A exhibits more mobility, this may be sufficient to offset the higher annual inequality and change the rankings as the observation period is extended.

Similarly, when measuring the extent of monopoly power in an industry by the concentration of revenues, an industry which has higher annual concentration and higher rigidity can be said unambiguously to be more concentrated. This is not the case when the higher annual concentration figure is associated with an industry which is more mobile, and the positions of firms in the sales hierarchy are constantly changing.

Let us now consider the properties of the proposed measure of rigidity given by (13). To begin with, apart from one case discussed below, R is defined for all sets of income profiles and lies in the interval [0, 1]. From Theorem 1, R takes its maximum value of 1 if and only if there is complete immobility. The other extreme, $R = 0$, arises when $I(Y(t_0, t_m)) = 0$; in other words, when incomes are exactly equalised over the whole time interval of m periods. If this happens, the income structure will be called *completely mobile*.[12] The incomes given earlier for individuals A and B provide a simple example of a completely mobile structure over a two year period.

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[12] This differs from the concept of *perfect mobility* frequently used in the literature on occupational and social mobility. The definition of perfect mobility originates with Prais [7] and his analysis of intergenerational social mobility. He proposed that the term

A slight problem occurs if we suppose that all individuals have identical incomes in each period ($Y_i(t_{k-1}, t_k) = \mu(t_{k-1}, t_k)$ for all i, k). Since relative incomes are constant over time, this situation is one of complete immobility. But all incomes are equal over the m-period interval, so it may also be called "completely mobile". The apparent contradiction is due to the fact that the denominator and numerator in (13) are both zero, so R is undefined. This itself need not cause too much alarm. It may be argued that the concept of mobility has little relevance in a society where all incomes are always the same; and, in any case, it is hardly likely to arise in practice. It is more disturbing to find that we can construct two sequences of income structures both converging to this extreme situation, one of which is a sequence of completely immobile structures and the other a sequence of completely mobile structures. As the two sequences approach the limiting case, they also converge towards each other, but still record mobility values at the opposite ends of the scale. This suggests that the computation of mobility values may be more sensitive to sampling errors and data problems when annual income inequality is rather low.

If the inequality measure is based on a social welfare function, the rigidity index R can be given an interesting social welfare interpretation. Substituting (5) into (13) we obtain

$$R = \frac{\mu(t_0, t_m) - Y_e(t_0, t_m)}{\sum_{k=1}^{m} \{\mu(t_{k-1}, t_k) - Y_e(t_{k-1}, t_k)\}} \tag{14}$$

The numerator indicates the *true welfare loss* (TWL) per capita over the period (t_0, t_m)—the amount of income per person which could have been saved if the same level of social welfare had been achieved with an egalitarian distribution. The terms summed in the denominator indicate the welfare loss per capita in each of the separate sub-periods. The denominator is therefore the amount of income per person which would have been saved if incomes had been equalised in *each sub-period* whilst maintaining the same levels of social welfare. This will be called the *apparent welfare loss* (AWL). If there are any changes in relative incomes over time, R will be less than one and the apparent welfare loss will exceed the true welfare loss.[13] Thus any

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should be used when the social class occupied by children is independent of that occupied by the parent. However, a number of problems arise, notably the fact that a structure can exhibit "more than perfect" mobility. See Shorrocks [9].

[13] The rigidity index given by (14) is therefore the ratio of TWL to AWL. The mobility value

$$M = 1 - R = \frac{AWL - TWL}{AWL}$$

might be described as *relative welfare illusion*.

attempt to estimate the welfare cost of inequality by calculating the cost year by year and summing over the years will usually exaggerate the true figure. Estimates of the welfare loss due to inequality therefore tend to be biassed upwards if they are computed from short-run data.[14]

Another explanation of this result is as follows. Suppose that within each period, the incomes of individuals are replaced by the corresponding equally distributed equivalent income. Thus some individuals are given money to bring them up to Y_e and others have their incomes reduced to this figure. On average, indivduals lose from this process of redistribution, since Y_e never exceeds the mean. Looking back over m periods of redistribution, each person finds that his income has been (on average) reduced to the sum of the equally distributed values in the separate periods. It is then discovered that too money has been taken away: m-period social welfare is less than the level of social welfare corresponding to the m-period incomes before redistribution. This is because

$$Y_e(t_0, t_m) \geq \sum_{k=1}^{m} Y_e(t_{k-1}, t_k) \tag{15}$$

(unless the relative incomes of all individuals remain constant over time). The "excessive" annual redistribution is directly related to the amount of mobility $M = 1 - R$; this can be seen by rearranging (14) to give

$$Y_e(t_0, t_m) - \sum_{k=1}^{m} Y_e(t_{k-1}, t_k) = \frac{M}{1-M}[\mu(t_0, t_m) - Y_e(t_0, t_m)] \tag{16}$$

At the beginning of this section, we commented on the inadequacy of any single number representation of mobility. Although the rigidity index (13) (and associated mobility value $M = 1 - R$) gives a single number, it first requires two choices to be made: an inequality index selected from those satisfying (1) and m, the number of time intervals over which aggregation is to take place. This provides us with two important degrees of freedom and it is more appropriate to regard (13) as specifying a set of mobility values corresponding to different choices of $I[Y]$ and m. Given the opportunities afforded by these choices, it may be possible to preserve the major features of mobility when condensing a set of income profiles into a number of mobility statistics. The loss of information may therefore not be unduly severe.

A suitable selection of inequality measures from those satisfying (1) will enable us to vary the emphasis placed on different portions of the distribution. As regards the choice of m, the best procedure is to compute R for $m = 2$ up

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[14] Short run estimates of welfare losses due to inequality can be made consistent with the true long run value by reducing the short run estimate by the factor R. The short run estimate of welfare loss in period k, corrected for this upward bias, thus becomes

$$R[\mu(t_{k-1}, t_k) - Y_e(t_{k-1}, t_k)]$$

to the largest value of m that the data allow, plotting the results graphically as in figure 1. In this way it may be possible to make inferences concerning the sources of mobility. For example, suppose we were to compare two groups, one of which had large variations in transitory income, whilst the other experienced substantial changes in permanent incomes (but small transitory changes). Year-to-year income variations might appear to be rather similar. Yet their "rigidity curves" may be expected to be radically different. If incomes changes are purely due to transitory effects, relative incomes will rapidly approach their permanent values and there will then be no substantial further equalisation. The rigidity curve will therefore tend to become horizontal after the first few years. This contrasts with the group with more mobility in permanent incomes, whose rigidity curve will continue to decline as the aggregation period is extended. So the curves may appear as in figure 2.

Calculating values of R over different aggregation periods may thus be all that is required to make the important distinction between these alternative types of income variations.

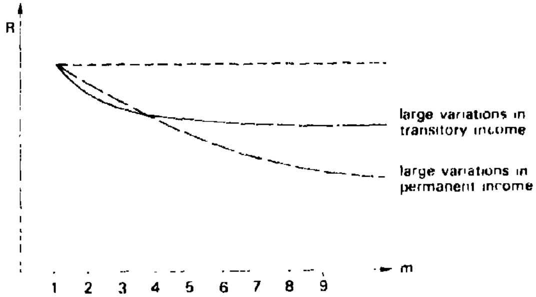

FIGURE 2

4. Empirical Estimates of Income Rigidity

Kohen *et al.* [3] provide data on mean income and income inequality for a sample of US males from two age groups. For the younger age group (16-24 years old) the Gini coefficient is given for the incomes of both whites and blacks in the years 1966, 1967 and 1968, and also for incomes cumulated over the periods 1966-67, 1966-68 and 1967-68. Data on the sample of middle-aged men (45-59 years old) is only available for the single years 1966 and 1967, and for the combined period 1966-67. Weighting the annual Gini coefficients by the mean income of the group in the corresponding years, and comparing this figure with the Gini coefficient for incomes cumulated over the longer period, we arrive at the rigidity values R as defined by (13). These are given in Table I, below.

TABLE I

Income Rigidity R for Males, USA 1966-68[a]

Group	Sample size	Period 1966–67	 1966–68	 1967–68
Males aged 16-24				
whites	1590	0.928	0.896	0.931
blacks	164	0.950	0.916	0.935
Males aged 45-59				
whites	6345	0.968	–	–
blacks	672	0.967		–

[a] Computed from Kohen *et al.* [3, Tables 3, 4, 5].

The data are far from ideal: the maximum period over which incomes are cumulated is small (3 years) and we are restricted to just one inequality measure. Nevertheless, it is possible to draw some tentative conclusions from the figures presented in the table. Mobility appears to increase as the overall timespan is extended, as evidenced by the lower rigidity value for the 3-year interval 1966-68 compared to either of the 2-year periods. This is consistent with our expectation that the longer period gives more opportunities for relative income movements. Within the group of young men, the rigidity values for whites lie below the corresponding figures for blacks, suggesting greater mobility for whites. In the absence of income data cumulated over a longer period, however, it is impossible to determine whether this indicates more genuine long-term mobility or reflects a greater variability in short-run transitory income. Finally, the figures for the older age group suggest that relative income changes are much less pronounced than at the earlier stage in their lives, and that, by this age, the different pattern found for whites and blacks has vanished. It remains to be seen whether these conclusions still apply when data is available for other inequality measures and longer periods of time.

5. MOBILITY AND SOCIAL WELFARE

The relationship between mobility and social welfare has already been touched upon in section 3 above. It was suggested that mobility has a positive influence on social welfare in the sense that if two societies have identical income distributions in each year, then social welfare will be greater for the society which exhibits more mobility. (The longer run inequality will be less for this society). Thus it might be said that mobility is unambiguously "good": from the social welfare viewpoint we would always prefer more mobility, *ceteris paribus*. But this conclusion rests on certain assumptions about the way in which inequality is measured: that the same index should be used for m-period incomes as was applied to single period incomes; and that the appropriate income concept is the total income received during the accounting period.

Concerning the first of these points, it is not obvious that two identical vectors of income receipts, observed over different periods of time, should necessarily generate the same inequality value. In other words, that

$$\mathbf{Y}(t, T) = \mathbf{Y}'(t', T') \qquad T - t \neq T' - t'$$

implies

$$I[Y(t, T)] = I[Y'(t', T')]$$

Justification for this assumption can, however, be obtained by exploiting the similarity between aggregation over populations and aggregation over time. It has been suggested that combining populations with identical distributions results in an aggregate distribution with the same degree of inequality as the constituent populations. By analogy with this principle of *population replication*,[15] we shall invoke the principle of *time repetition*.

Time Repetition. If the incomes of all individuals are repeated in r consecutive time intervals of the same length θ, the inequality over the combined period of length $r\theta$ is the same as that recorded in each of the shorter periods.

$$\text{i.e. if } \mathbf{Y}(0, \theta) = \mathbf{Y}(\theta, 2\theta) = \cdots = \mathbf{Y}((r-1)\,\theta, r\theta)$$

then

$$I[Y(0, r\theta)] = I[Y((k-1)\,\theta, k\theta)] \qquad k = 1, \ldots, r$$

If time repetition holds, and the inequality measure is continuous and mean independent, it can be shown that

$$I[Y(t, T)] = I[Y'(t', T')]$$

whenever

$$\mathbf{Y}(t, T) = \lambda \mathbf{Y}'(t', T') \qquad \lambda > 0$$

Thus for mean independent measures satisfying time repetition, the inequality value will be independent of the accounting period.[16]

We turn now to the question of the income concept appropriate for inequality comparisons. Clearly we may want to discount incomes if the time interval is sufficiently long. This does not affect the analysis of the previous sections. Nor do problems arise if adjustments are made for price changes. But when these are all incorporated, there remains the presumption that individuals are indifferent between two income streams offering the same real present value. This might be true if capital markets were perfect (or if there was perfect substitutability of income between periods), but it seems likely that individuals are concerned with both the average rate of income receipts and the pattern of receipts over time. We may go further and suggest that individuals tend to prefer a constant income stream, or one which is growing steadily, to one which continually fluctuates. If this is a reasonable description of individual preferences, and if inequality values are supposed to reflect differences in the welfare levels of individuals, it is essential that some account be taken of the shape of the income profiles.

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[15] See Dasgupta *et al.* [2, p. 184].

[16] The proof is straightforward. Note that the assumption of mean independence is essential for this result. If we accept the intuitively plausible property of time repetition, but not mean independence, the inequality value computed from a set of incomes received at a constant rate will generally vary with the length of the accounting period.

No attempt is made here to discuss this issue in detail. But the impact on the social welfare implications of mobility can be gauged from a few brief remarks. To begin with, we need a basis for evaluating the different income streams faced by individuals. The obvious choice is a simple intertemporal welfare function

$$W_i = V_i(Y_i(t_0, t_1), Y_i(t_1, t_2), \ldots, Y_i(t_{m-1}, t_m))$$

which will indicate the kind of income stream the individual prefers, given some fixed present value level of income. Suppose, for the sake of argument, that individuals prefer the constant stream

$$Y_i^*(t_{k-1}, t_k) = \frac{t_k - t_{k-1}}{t_m - t_0} Y_i(t_0, t_m)$$

Then there will be some constant income flow rate y_i^c generating receipts

$$Y_i^c(t_{k-1}, t_k) = (t_k - t_{k-1}) y_i^c \qquad k = 1, \ldots, m$$

which gives the same level of welfare as the income stream he currently faces. For inequality comparisons between individuals, it is more appropriate to use these constant rate equivalent incomes, y_i^c, than the actual receipts, since the former provides some comparability between income streams which differ markedly in their patterns of fluctuations.

Replacing actual recorded incomes with this alternative income concept in the computation of inequality values introduces a new dimension into the discussion of mobility. No longer is mobility necessarily desirable. Changes in relative incomes still tend over time to equalise the distribution of total income receipts, and to this extent welfare is improved. But greater variability of incomes about the same average level is disliked by individuals who prefer a stable flow. So to the extent that mobility leads to more pronounced fluctuations and more uncertainty, it is not regarded as socially desirable. A more detailed examination of these two facets of mobility will provide a better understanding of the impact of income variability and the implications for social welfare.

REFERENCES

1. A. B. ATKINSON, On the measurement of inequality, *J. Econ. Theory* **2** (1970), 244–263.
2. P. S. DASGUPTA, A. K. SEN, AND D. STARRETT, Notes on the measurement of inequality, *J. Econ. Theory* **6** (1973), 180–187.
3. A. I. KOHEN, H. S. PARNES AND J. R. SHEA, Income instability among young and middle-aged men, *in* "The Personal Distribution of Income and Wealth" (J. D. Smith, Ed.), Conference on Research in Income and Wealth, Vol. 39, NBER, New York, 1965.
4. S.-CH. KOLM, The optimal production of social justice, *in* "Public Economics" (J. Margolis and H. Guitton, Eds.) Macmillan & Co., London, 1969.
5. S.-CH. KOLM, Unequal inequalities. II. *J. Econ. Theory* **13** (1976), 82–111.
6. I. B. KRAVIS, "The Structure of Income," Univ. of Pennsylvania Press, Philadelphia, 1962.
7. S. J. PRAIS, Measuring social mobility, *J. Roy. Statist. Soc. Ser. A.* **118** (1955), 56–66.
8. A. K. SEN, "On Economic Inequality," Oxford Univ. Press, London/New York, 1973.
9. A. F. SHORROCKS, The measurement of mobility, *Econometrica* **46** (1978), 1013–1024.
10. L. SOLTOW, "Towards Income Equality in Norway," Univ. of Wisconsin Press, Madison, 1965.

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Equilibria In Monopolistically Competitive Insurance Markets

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INTRODUCTION

Recent interest in the allocative properties of markets in the presence of imperfect information has been concerned with the nonexistence of an equilibrium.[1] One interesting example of this phenomenon is a model of an insurance market developed independently by Rothschild and Stiglitz [8] and Wilson [9]. The imperfect information aspect of this model is that sellers of insurance cannot differentiate between high- and low-risk buyers on the basis of observable personal characteristics. Insurance firms attempt to circumvent this problem by taking advantage of the fact that other things equal, higher-risk agents will prefer more extensive insurance coverage at given prices than their less risky counterparts. This means that self-selecting consumers revealing preferences over sets of insurance contracts may also be revealing information about their relative degree of risk.

Consider the simplest representation of the model. There are two types of buyers, high and low risk. Assume that the objective of each firm is to maximize expected profit. Then it seems reasonable to define equilibrium as:

> ...a set of contracts such that each firm at least breaks even (on the set of people who buy its contracts) and there exists no other contract which if offered will make a nonnegative profit for the company which offers it.[2]

Under the assumption that each firm can monitor the total insurance coverage purchased by its buyers it is shown in [8, 9] that equilibrium, as defined above, need not exist.

The major departure of this paper from previous work is to argue that the assumption that firms can costlessly monitor the total insurance coverage obtained by buyers is untenable. It is shown below that it is always profitable for some firms to guarantee their customers that they will not divulge the names of their buyers to other firms. Given that this incentive exists and is a

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[1] See Refs. [3, 4, 5, 7, 8, 9].

[2] This is the equilibrium concept used in [8].

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feasible strategy, I argue that the costless monitoring assumption should be dropped. It then becomes important to determine whether incentives ever exist for firms to divulge the identity of their customers. It is found that such incentives do exist so that in general the market is composed of two groups of firms; Those who are divulging purchase information and those who are not. The main effect on the market is a shrinking of the set of contracts firms will find profitable to offer. I term a market where firms cannot monitor consumers *total* purchases an informationally noncooperative market. The main purpose of the paper is to investigate the existence of equilibria in such markets.

After demonstrating, in the first section of the paper, that firms have a profit incentive not to cooperate with other firms about customer purchases attention is turned to the existence question. The main proposition of the second section generalizes a result of Rothschild and Stiglitz, showing that if firms are allowed to compete by price and quantity rationing there can *never* be a single price competitive equilibrium. Next *I* examine the market under the assumption that firms may ration the quantity of coverage sold to any buyer with the provision that all policies in the set offered by any particular firm must have the same implicit price or premium per unit of coverage. This assumption and the equilibrium concept quoted above is a minor generalization of the market studied by Rothschild and Stiglitz in there seminal paper. However, contrary to the result obtained by them under the monitoring assumption I find that a Nash-Cournot equilibrium always exists. The equilibrium contract set has some interesting properties. Although only a finite number of contracts are purchased, the equilibrium set contains a continuum of contracts that are necessary to support the equilibrium. The number of purchased contracts will equal the number of risk types so that an implicit separation of types occurs. Yet there will be pooling of agents of different risk classes on every contract except one. Suppose there are N risk types, $p^1,\ldots, p^N$. Let $p^i < p^{i+1}$ be the risk ranking. Then, in general, there will be N contracts purchased in equilibrium with each p^i-agent constructing a portfolio containing i contracts.

The importance of the assumptions made concerning firm information about total purchases is illuminated in a discussion of the monopolistic market when it is assumed firms have no information about buyer purchases other than what they have sold themselves. The only possible equilibrium contract set is the same set which produced an equilibrium under partial information. However, in the no-information case the informational structure of the market requires that some firm(s) have a monopoly on the supply of $N - 1$ of the N contracts purchased. Given this contract set and the supply strategy; there can be no offering of new contracts and therefore no entry of firms. But the existence of monopoly suppliers of some contracts prevents the existence of a strict Nash equilibrium. This opens the strong possibility that the market structure will be unstable. It is argued that the informational structure will be dominated by the previous structure where firms have *some* information about buyers purchases. These results serve to illuminate the extreme importance of the information firms hold about customer purchases. In the two polar cases firms either have complete information or *no* information. In the third case firms have partial information. This paper characterizes the existence question in the following way:

(a) If firms have *no* information there *never* exists a Nash equilibrium.

(b) If firms have *complete* information there sometimes exists a Nash equilibrium.

(c) If firms have *partial* information there always exists a Nash equilibrium.

In the third section the equilibrium concept is strengthened. The assumption that the policies offered by any particular firm all have the same average price per unit of coverage is relaxed. It is assumed that the set of policies offered by any firm may contain policies with different prices, but no firm offers policies with negative expected returns.[3] This assumption and the equilibrium concept adopted correspond to Wilson's $E1$ equilibrium. Again, contrary to the results reported by Wilson for the perfect monitoring case, I find that without perfect monitoring an $E1$ Nash–Cournot equilibrium always exists. In fact the same contract set achieved in the previous section is shown to support the more general equilibrium.

Spurred by the positive results of previous sections I attempt in Section 4 to relax the assumption that firms do not offer policies that are unprofitable. In previous work it was shown that a firm may be able to earn total positive profits by selling pairs of policies where the low-risk group earns enough profit to offset losses on the policy sold to high-risk agents. I am able to show that for the case of two risk types, the equilibrium set constructed in earlier sections is invulnerable to this kind of attack. That is, at least for the case of two risk groups a Nash equilibrium exists in the strongest sense. I conjecture that this result holds for many risk types, but must leave that an open question.

The fact that through much of the analysis consumer choice sets are nonconvex increases the complexity of the paper. These nonconvexities can lead one's intuition astray and as a result have forced me to make the presentation somewhat more formal than I originally desired. To lighten the reader's burden I have attempted to elucidate most important results graphically.

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[3] It is known from previous work [8, 9] that this assumption is not as innocent as it seems. Its consequences are examined in Section 4.

1. The Insurance Market

Each consumer in the market has the same two point random endowment $[w, w - L]$,

$$w > w - L > 0.$$

Let the finite set $I^* = (1, \ldots, I)$ be the indexing set of probability types. For each $i \in I^*$ there is a probability $p^i \in (0, 1)$ and a number $m^i \in (0, 1)$ such that m^i is the proportion of consumers in the market with probability p^i of having income $w - L$. Refer to such agents as p^i-agents. By convention if $i, j \in I^*$ and $i < j$ then $p^i < p^j$. Each agent is assumed to know his probability type.

An insurance contract is a couple $(\pi, T) \in R_+^2$, with π representing the premium a buyer must pay an insurance firm to have claim to T units of income contingent upon the occurence of $w - L$. Let

$$C = ((\pi, T) \in R_+^2 \mid w - \pi \geq 0, w - L - \pi + T \geqslant 0)$$

Then C is the set of insurance contracts that satisfy buyers' budget constraints. No firm would sell a contract not in C. Define $\hat{C}$ to be the compact subsets of C. For a given supply of contracts by all active firms there will exist an aggregate set of contracts from which each buyer may choose. In each instance the aggregate supply of contracts may be exceedingly difficult to describe in terms of the individual firm supplies. This is so because what contracts an agent is able to purchase will depend upon whether he can mix the contracts of different firms. Suppose that S_f is the supply of contracts by firm f. If each buyer is limited to purchasing one contract from at most one firm the aggregate policy set is

$$S = \bigcup_f S_f .$$

This is the case explored by Rothschild and Stiglitz and Wilson. If buyers are allowed to purchase any fraction of each firms policy offer the aggregate contract set is

$$S = \text{Convex hull} \sum_f S_f .$$

The latter is the closest formulation of firm behavior in this paper. Denote the aggregate contract set by S. Assume that agents are expected-utility maximizers. The following assumptions are made.

(A1) Each agent possesses the same strictly concave, and twice continuously differentiable utility function $U(\ )$ over income.

(A2) $(0, 0) \in S \subset \hat{C}$ for all S considered.

For each aggregate supply of policies S, buyers,

$$\operatorname*{Max}_{(\pi,z)\in S} V[\pi, z, p] = (1-p)\,U(w-\pi) + p\,U[w-L-\pi+z].$$

(D1) Let

$$O^i[S] = \{(\pi, T) \in S \mid V(\pi, T, p^i) \geq V(x, y, p^i) \text{ for all } (x, y) \in S\}.$$

In general to obtain an optimal contract an agent may have to purchase different contracts from many firms. The set of individual contracts purchased by p^i-agents plays an important role in the analysis.

(D2) Suppose $C^i \in O^i[S]$, define

$$A[C^i] = \Big\{A_j^i[C^i, S] \subset S \mid C^i = \sum_{C_\alpha \in A_j^i} C_\alpha\Big\}.$$

Define $A^i(S) = \{(\pi, T) \in S \mid (\pi, T) \in A_j^i[C^i, S]$ for some j and $C^i \in O^i[S]\}$.

In the analysis to follow $A^i[S]$ is the set of contracts purchased by some p^i-agent. If a policy is a member of $A^i[S]$ some p^i-agent purchases that contract when the aggregate policy set is S. $A_j^i[C^i, S]$ is a set of policies whose elements sum to C^i an optimal policy. For any optimal policy there may exist many such sets contained in the aggregate policy set. Therefore $A[C^i]$ need not be a one-element set.

Given the above definitions it should be clear that special care must be taken in describing demand functions.

If $O^i[S]$ is not unique p^i-agents will be indifferent between any two policies in the optimal set. Suppose a specific optimal policy C^i is chosen. If $A[C^i]$ is not unique type i agents will be indifferent between any two sets A_j^i, A_k^i belonging to $A[C^i]$. The approach taken here seems to be not only the most natural, but also the most reasonable. I assume that for any $O^i[S]$ p^i-agents are spread uniformly over the elements of the optimal set. Furthermore, for any contract $C^i \in O^i[S]$, the set of p^i-agents purchasing C^i will be uniformly spread over the elements of $A[C^i]$. With the addition of a few more fundamental assumptions this could be derived as a proposition using probability theory. I however simply introduce the following definition:

(D3). Let $\#S$ denote the number of elements in set S. Then the demand for contract (π, T) by p^i-agents when S is the aggregate policy set is

$$d_i[(\pi, T), S] = m^i \cdot \frac{\#\{A_j^i[C^i, S] \mid (\pi, T) \in A_j^i[C^i, S] \text{ for some } C^i \in O^i(S)\}}{\sum_{C^i \in O^i(S)} \#A[C^i]}.$$

The denominator of the fraction above represents the total number of ways optimal contracts can be purchased by p^i-agents. The numerator is the number of ways the policy (π, T) can be purchased in reaching an optimal policy. Clearly $d_i[(\pi, T)]$ is well defined and

$$d_i[(\pi, T), S] \le m^i \qquad \text{for all } (\pi, T) \text{ and } S.$$

As an example suppose that $\#O^i[S] = 1$ and that $A[C] = 2$ for $C = O^i(S)$. If $(\pi, T) \in A_1^i[C, S] \subset A[C]$ and $(\pi, T) \notin A_2^i[C, S]$; then

$$d_i[(\pi, T), S] = m^i \cdot \tfrac{1}{2}.$$

One-half of all p^i-agents purchase (π, T). Note also that if $S = \phi$ then $\#S = 0$.

(D4) The aggregate demand function $d\colon C \times \hat{C} \to R^I$ is defined

$$d[(\pi, T), S] = [d_1[(\pi, T), S], \ldots, d_I[(\pi, T), S]].$$

(D5) The aggregate expected profit $P\colon C \times \hat{C} \to R$ is

$$P[(\pi, T), S] = \sum_{i \in I^*} \frac{d_i[(\pi, T), S]}{\sum_{I^*} d_i} (\pi - p^i T)$$

if $d_i \neq 0$ some $i \in I^*$ and zero otherwise.

Expected profit is the sum of expected profits across risk types, with each types contribution weighted by its proportion of the total number of contracts purchased.

(A3) Suppose that N different firms offer the contract (π, T). The demand for each firms offer of (π, T) is

$$\frac{d[(\pi, T), S]}{N}.$$

This assumption is analogous to (D3) and has obvious consequences for the expected profits accruing to individual firms.

1.2 *Monopolistic Competition*

The market is characterized by freedom of entry. Firms are assumed to maximize expected profit. Each insurance firm knows the expected return $P[(\pi, T), S]$ for each contract (π, T) and $S \subset \hat{C}$.[4] The fundamental assumption concerning imperfect information is that no firm can discern any agents probability type. Suppose there are two p-types p^1 and p^2. If firms know each agents type, competition among firms would ensure that each p^i-agent received his most preferred contract among those that earn nonnegative expected profit when sold only to p^i-agents. For a p^i-agent this contract involves full coverage, $T = L$, at an actuarily fair premium $\pi^i = p^i T$. When sold only to the appropriate risk type these contracts have zero expected returns. Unfortunately, since they offer identical coverage and one has a lower premium, both types will prefer the less expensive contract. How will firms react to this problem? What is an appropriate concept of equilibrium, and what are the properties of equilibria if they exist? In what follows I attempt to provide answers to these questions. Using a Nashlike concept of equilibrium it was shown in [8, 9] that often no equilibrium exists. That result was achieved with a particular assumption about the institutional structure of the market. In this paper institutional structure is concerned with the choice variables firms use to compete; and with the flow of firm-specific information between firms. The choice variables firms use in the monopolistic market will be price and quantity of coverage. This paper differs from previous work in its assumption about the flow of consumer purchase information between firms. In previous analyses it was assumed that each insurance firm had complete knowledge about the *total* amount of coverage its buyers purchased. This meant that firms had the power to limit consumer purchases to a single contract. It is manifest that inexpensive enforcement of such an information structure necessarily requires explicit cooperation among *all* firms. Below it is shown that there is a strong incentive for some firms not to cooperate.

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[4] This assumption is extremely strong. It is made because I wish to study the Nash concept of equilibrium as in [8, 9].

(D6) An informationally noncooperative market (INCM) is a market where *some* of the active firms do not cooperate with *any* other firms by revealing the identity of those agents purchasing insurance from them. An informationally cooperative market (ICM) is a market where *all* firms reveal the identity of their customers.

An INCM is one where a firm cannot make the amount of coverage it offers a buyer conditional upon the amount purchased from other firms. Consequently buyers cannot be restricted to purchasing from only one seller. In an ICM firms can enforce such conditions.

In this section it is shown that there is a profit incentive for some firm not to freely divulge to other firms the identity of its customers. This result provides strong motivation for considering the allocational process in an INCM.

Consider the case with two risk groups p^1 (low risk) and p^2 (high risk). Figure 1 depicts the contract space. The lines $p^i T = \pi$ represent the set of actuarily fair contracts for the respective groups. Suppose that p^2-agents are allocated the contract (π^2, L). Note that this is the contract they would receive in competitive equilibrium with perfect information. Let *hh* be the indifference curve of a p^2-agent through (π^2, L). Consider the set of contracts

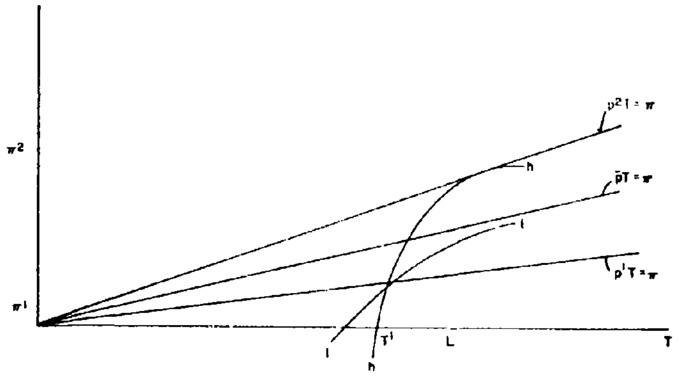

FIGURE 1

not strictly preferred to (π^2, L) by p^2-agents. On this set the contract most preferred by p^1-agents is (π^1, T^1).[5] This contract is actuarily fair, but offers less than complete insurance coverage. If there exists no contract below *hh* and *ll* that would attract both agent types and make nonnegative returns, then (π^2, L), (π^1, T^1) is an equilibrium in an ICM. Let $\bar{p} = m^1p^1 + m^2p^2$, the mean probability that any agent in the market will have endowment $w - L$. The equilibrium condition requires that there exist no contracts on the line $\pi - \bar{p}T = 0$ that are below the p^1-agents indifference curve *ll* through (π^1, T^1). If this condition is not satisfied some firm could offer a single contract pooling both types of agents. Under these circumstances there exists no equilibrium in the ICM (see [8, 9]).

Assume that the putative equilibrium exists. Consider the transparent fact that any p^2-agent could be made better off if allowed to purchase the contract (π^1, T^1) plus any contract of choice along $p^2T = \pi$. This means there is incentive for some firm to enter the market; offer the set of feasible contracts on $p^2T = \pi$, and agree not to inform other firms who has purchased its contracts. (By continuity of preferences the firm could instead offer a line segment of contracts slightly above $p^2T = \pi$ and have positive expected returns). Clearly p^2-agents could recontract by purchasing (π^1, T^1) and some contract $(\pi, L - T^1)$ on $p^2T = \pi$ to obtain L units of insurance at a lower total premium. The contract (π^1, T^1) would under these circumstances become unprofitable and apparently the set (π^2, L), (π^1, T^1) is no equilibrium after all. Whether this story is reasonable or not depends crucially upon the ability of the firm(s) selling (π^1, T^1) to force other firms to cooperate by revealing the identity of their customers. Under the assumption of an unregulated market with free entry it is extremely doubtful that this could be accomplished costlessly.

[5] It is necessary to assume here that p^2-agents do not demand (π^1, T^1) as is done in [8, 9].

The demonstration above provides strong support for the position that firms will not have knowledge about the complete coverage purchased by their customers. Note that this does not imply that firms must have no information about the purchases of each buyer, other than their own sales to the buyer. In some cases it may be profitable, even necessary, that a group of firms trade information. As an example suppose that two firms offer the market the same pooling contract $(\bar{\pi}, \bar{T})$ with $\bar{\pi}$ $\bar{p}\bar{T}$, $\bar{p}$ defined above. Each firm knows that the aggregate expected returns to this contract are zero if each agent in the economy purchases precisely one contract. In a great many cases high-risk types would attempt to purchase contracts from both firms. Under the usual assumption that both firms receive one-half of the low-risk buyers, profits will be negative for both firms. There is an obvious incentive for these firms to divulge information to each other about their buyers. Consider a cartel of such companies. Suppose cartel rules are that only firms providing information about their own customers can receive information from cartel members. All firms offering pooling contracts have positive incentives to join such cartels. In fact not joining the cartel and offering the pooling policy cannot be profit optimal behavior. This offers another explanation for the common practice of real world insurance firms to stipulate that buyers hold no other insurance policies.[6] Rothschild and Stiglitz come close to making this point when they assert:

> The person who buys two $1,000 accident insurance policies does not in general, have $2,000 worth of protection. If an accident occurs, all he gets... is the privilege of watching his companies squabble over the division of the $1,000 payment.

This situation will not occur for every policy and every firm. For example, suppose that the aforementioned pooling policy were the only contract offered on the market. If the contract has less than complete coverage high-risk types would desire more insurance at profitable prices. A firm selling this additional coverage would promise not to tell and pick up high-risk buyers. Any firm not making this guarantee would attract no customers. If it did, and informed on buyers, the sellers of pooling contracts would be able to refuse to sell high-risk agents coverage at the pooling price. The point to recognize is that in general, for a set of contracts on the market, there will be profit incentives for some firms to communicate firm-specific information, while other firms will have incentives to keep their information private.

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[6] The simplest and most plausible explanation is moral hazard. However, the point stressed above shows that the incentive exists, even in its absense.

2. The Nonviability of Competitive Equilibrium

In this section of the paper the set of policies offered by any firm is assumed to lie on a straight line emanating from the origin. The definition of equilibrium retains the Nash–Cournot spirit. The market is said to be in equilibrium when, given utility maximization by buyers, the expected return from the set of contracts offered by each firm is nonnegative; and given the aggregate set of contracts supplied there exists no alternative offer with positive expected profit. Recall that S_f is the supply of policies by firm f.

(D7) A MI monopolistic insurance equilibrium is a set of policies $\hat{S} \subset C$ such that

(i) $\sum_{(\pi, T) \in \hat{S}} P[(\pi, T), \hat{S}] \geq 0$.

(ii) $\pi = q_f T$ for all $(\pi, T) \in S_f$ with $q_f > 0$ all f.

(iii) There exists no $S \subset C$ with $\sum_{(\pi, T) \in S} P[(\pi, T), \hat{S} \cup S] > 0$

and $\pi = qT$ for all $(\pi, T) \in S$ and some $q > 0$.

If firms do not have complete purchase information, one form of competition is that where firms compete by price alone. Under such competition firms are assumed to offer insurance at a uniform price per unit of coverage. It is clear that if firms do not restrict quantities; in equilibrium all firms that sell insurance must be charging the same unit price. An equilibrium is simply a set of contracts S^*, and a unit price q^*, such that

$$\pi = q^* T \qquad \text{for all } (\pi, T) \in S^*$$

and

$$\sum_{(\pi, T) \in S^*} P[(\pi, T), S^*] \geq 0.$$

This supply strategy is not indicative of optimal firm behavior. The point that most critics are likely to stress about the price competition model is that firm supply behavior is too passive. Given the prices of insurance, consumers are assumed to maximize expected utility by purchasing their optimal amount of coverage subject to income constraints. Firms are assumed to simply sell insurance to whomever demands it in whatever amounts. This is an extremely naive assumption in face of the fact that firms are aware that expected profits are negative for some risk groups and positive for others, irrespective of the amount of coverage demanded. In short firms do not make supply decisions. More realistically it is obvious that any firm charging the price q would desire to sell an unlimited amount of insurance to all p'-agents,

$$p' < q,$$

and zero insurance to all p^j-agents,

$$p^j > q.$$

Since firms are unable to distinguish between agents such a supply decision could not be carried out. However, this does mean that the only alternative is the passive behavior assumed in the price competition model. It is easy to show that the amount of coverage demanded will be an increasing function of the probability of low endowment. Let $\hat{z}(q)$ be the maximum amount of coverage demanded among those with $z \leq L$. All of these coverages are demanded by agents whose probability is less than the price of a unit of coverage and therefore are profitable. For those $z > L$ probability is greater than unit price and those coverages lose profit. Therefore it would always be profitable to restrict coverage at any unit price to $\hat{z}(q)$. For an individual firm such a supply decision always decreases the expected losses from higher-risk agents, but does not affect the returns from lower risk buyers. Any firm adopting supply decisions of this sort will force other firms to do the same. Firms not restricting coverage have no reason to expect greater demand from low-risk buyers who have the same maximal choices from both types of firms, but can expect to gain more high-risk agents who desire unrestricted coverage. Of course if every firm adopts this practice buyers may adapt by attempting to purchase multiple contracts from many firms. The proof of Proposition 2 shows that monopolistic competition can solve this problem.

It seems that a supply strategy of offering consumers a unit price and a maximum amount of coverage at that price dominates price competition alone. This conclusion was stressed by Rothschild and Stiglitz in the case of an informationally cooperative market. The importance of Proposition 1 is that it generalizes this domination result to the case where firms cannot monitor customer purchases and it provides a clue as to why a monopolistic equilibrium exists.

(D8) For an economy with I risk types $p^1,\dots, p^I, p^i < p^{i+1}$, let $m(i, p)$ equal the proportion of agents with probability p in the population obtained by truncating the original economy by removing all p^j-agents with $j < i$.

So $m(1, p)$ is just the proportion of p-agents in the original economy, while $m(j, p)$ is the proportion of p-agents in the economy with $p^j,\dots, p^I$ types,

(D9) Let $\bar{p}(k) = \sum_{i=1}^{I} m(k, p^i)\, p^i$.

Given this definition $\bar{p}(k)$ is just the average probability of loss in the group composed of all $p^k,\dots, p^I$-types.

(D10) Let $Z(q, p)$ be the Walrasian demand of a p-agent when the unit price of insurance is constant at q.

Lemma 1. *$Z(q, p)$ is strictly increasing in p.*

Proof. A simple exercise in differentiation.

Lemma 2. *For $i = (1, 2, \ldots, I)$, $\bar{p}(i) < \bar{p}(i+1)$.*

Proof. Obvious.

Proposition 1. *In the insurance market no set of contracts S^* with $\pi = q^*Z$, $q^* > 0$, for all $(\pi, Z) \in S^*$ and each agent purchasing his Walrasian demand $Z(q^*, p)$, can be a M1 equilibrium if more than one p^i-type purchase positive coverage at q^*.*

Proof. Suppose such an equilibrium set exists. Assume there are I risk types in the market and that $I - [k - 1]$ of these groups purchase positive coverage. By Lemma 1 $Z(q^*, p^i) > 0$ for $i = k, \ldots, I$. Therefore the mean probability of those purchasing positive coverage is $\bar{p}(k)$. I claim that

$$q^* \geq \bar{p}(k)$$

with equality holding only if $k = I$. To see this consider the following reallocation of coverage among buyers at the same price q^*. Allocate L units of insurance to each p^i-agent $i = k, \ldots, I$ so that they all hold the contract

$$(\pi, Z) = (q^*L, L).$$
$$Z(q^*, p^i) \gtrless L \qquad \text{as } p^i \gtrless q^*.$$

Therefore in all cases except $p^i = q^*$ the transformation

$$Z(q^*, p^i) \to L$$

increases expected profits. Since S^* is an equilibrium set, the aggregate expected profits on S^* are nonnegative. This means that the expected profit on (π, Z) is strictly positive, unless $k = I$ in which case profits are zero. Now the contract

$$(\hat{\pi}, L) = (\bar{p}(k)\, L, L)$$

earns zero expected returns when sold to this same group of agents. This implies

$$q^* > \bar{p}(k) \qquad \text{when } k < I.$$

This means that there exist contracts $(\bar{\pi}, \bar{Z})$ with

$$q^* > \bar{\pi}/\bar{Z} \geqslant \bar{p}(k)$$

that will attract buyers if offered. The contract (π^*, Z^*) with

$$\pi^*/Z^* \qquad \bar{p}(k)$$

and

$$V[\pi^*, Z^*, P^k] \geq V(\lambda\pi^*, \lambda Z^*, p^k) \qquad \text{for all } \lambda \geq 0$$

is such a policy. Therefore, there is an incentive for some firm to offer insurance at price $\bar{p}(k)$ and restrict all buyers to at most Z^* units of coverage. Q.E.D.

No assumptions about the information firms hold with respect to buyers' total purchases are required in the above argument. This is because under any information conditions a single firm offering the price $\bar{p}(k)$ can prevent customers from purchasing more than one of its own contracts.

3. MONOPOLISTIC EQUILIBRIUM

The arguments of the previous section suggest that firms have incentives to compete with the following strategies.

(a) Offer a price at which consumers may buy.

(b) Set a maximum amount of coverage any agent can purchase at that price.

(c) Ration the number of contracts an agent can buy.

(d) Share customer identity information only if it is profitable.

It can now be shown that if firms compete in these ways, there always exists a MI equilibrium. The procedure is to construct the equilibrium contract set for the case with two groups. This is followed by a geometric discussion elucidating why the set is an equilibrium. After this I prove Proposition 2 which states that a monopolistic equilibrium exists for any finite number of p-types.

Consider the situation depicted in Fig. 2. The line $\pi = \bar{p}T$ represents the set of contracts offering zero expected returns if every agent in the economy is allocated the same contract. On this pooling line the contract α maximizes the utility of p^1-agents, as is indicated by the tangency of their indifference curve II at α. Suppose that a group of firms are offering the set of contracts 0α. Each firm offering 0α announces that each of its buyers is restricted to one contract in that set. If 0α are the only contracts offered all p^1-agents will demand α. All p^2-agents would prefer more coverage at the given price and would like to purchase from many of the firms. However, these firms have a profit incentive to prevent this, as outlined above, and no incentive to do otherwise. Therefore, p^2-agents will also demand α. Both types must be less than fully insured. Each would now be willing to purchase $L - x$ additional

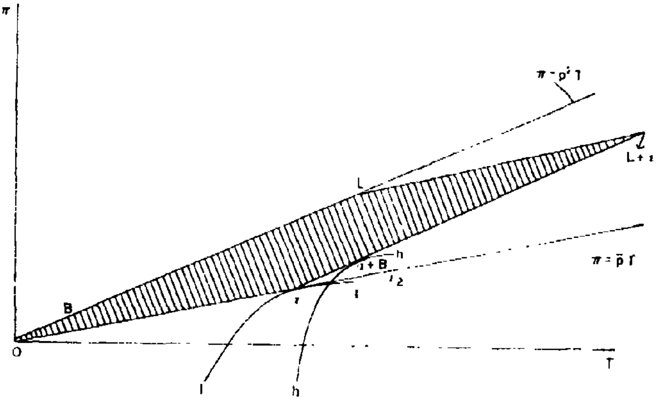

FIGURE 2

coverage at the respective actuarily fair price.[7] For p^1-agents this is infeasible. But there is an incentive for firms to offer the set of contracts $0L$. Then p^2-types will all buy the contract B, where the coverage at B is $L - x$. This allocation gives p^1-agents α and p^2-agents $\alpha + B$. The aggregate contract set is the shaded region $0L(L + \alpha)\alpha 0$. I claim that this contract set along with the special supply restraints on the contracts 0α is a monopolistic equilibrium. Let the set of contracts under discussion be termed S^*. By the definition of equilibrium S^* is not an equilibrium only if there exists an alternative set $S \subset \hat{C}$ all of whose policies have the same price, and the sum of the returns from the contracts in S is positive. Suppose there exists such an S. Suppose p^1-agents buy c_1 in S. Then since p^1-agents have their best contract on $\pi = \bar{p}T$ it is clear that c_1 is below $\pi = \bar{p}T$. This means that the average price of the policy c_1 is less than $\bar{p}$. In that case p^2-agents will be purchasing either c_1 or if possible another policy with the same average price and greater coverage. By dropping $\alpha + B$ and buying c_1, then recontracting along 0α and possibly $0L$ too, p^2-agents can obtain the same amount of total coverage L at a lower premium than before. But since c_1 and therefore any other policy in S have a lower unit price than $\bar{p}$ the profit from this offer must be negative whether both agents purchase c_1 or p^2-agents purchase more coverage at the given price. Alternatively, suppose that only p^2-agents purchase some $c_2 \in S$. If p^1-agents purchase a different policy at the same price the above argument applies. Purchase of c_2 by p^2-agents implies that the average price of c_2 is below p^2 and therefore c_2 is unprofitable. This exhausts the possibilities and shows that there exists no price and set of policies each carrying the given price that can destroy the equilibrium S^*.

The information about buyer purchases available to firms should be clarified. Those firms specializing in the supply of the contracts $0L$ not only have no incentive to join an information cartel, but are only able to sell because they refuse to join. Therefore only each buyer and the specific seller to that buyer know that a policy on $0L$ was purchased. Those firms specializing in the supply of the policies 0α must communicate buyer identities to each other if they are to guarantee the viability of the policies. But note with three or more risk types pooling policy cartels cannot make this information public. Otherwise firms offering lower-priced policies will refuse to sell to agents known to have purchased at higher prices. Therefore, in general, we may say that only firms offering pooling contracts will share purchase information; and they will do so only with firms offering the same pooling policy.

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[7] $\alpha = [\bar{p}x, x]$ and $B = [p^2(L - x), L - x]$.

A property of the equilibrium contract set that should be emphasized is the role played by the set of unpurchased contracts on 0α. These contracts are necessary to guarantee the existence of an equilibrium. If they are not offered it will not always be true that alternative contracts with prices below $\bar{p}(1)$ must always be unprofitable. Consider Fig. 2 and suppose that p^1 and p^2 are not very different so that the respective optimal contracts (α, α_2) on the pooling line are close. Suppose that the set of offered contracts is α and those on $0L$. Extend the line connecting $(\alpha + L)$ and α with slope p^2 through α from below. This line must cut the indifference curve of the low-risk agent so that there are policies on and above it, but below the pooling line that are strictly preferred by p^1-agents to α. Any policy $\underline{c}$ above the line and below *ll* satisfies the inequality:

$$V[c + \lambda B, p^2] < V[\alpha + B, p^2]$$

for all $\lambda > 0$.

If c and α are close enough it is also possible that

$$V[c + \alpha, p^2] < V[\alpha + B, p^2]$$

and the offer of c will attract only p^1-agents and be profitable. When the policies on 0α are offered this cannot happen.

One special case of interest is that where the subsidization cost incurred by p^1-agents at the pooling price $\bar{p}(1)$, is so great that their best contract at that price is $(0, 0)$. In equilibrium the only contract sold will be (p^2L, L). This is Akerlof's well-known lemon phenomenon where all the good risks are driven out of the market. It is also the special case where Proposition 1 does not hold.

Prior to providing the general proof of the existence of an equilibrium more definitions and a few simple lemmas are needed.

(D11) For an economy with I risk types let the I contracts

$$((\pi^i, T^i))_{i=(1,\dots,I)}$$

satisfy conditions (a) and (b).

(a) $\pi^i = \bar{p}(i)\, T^i$,

(b) $$V\left(\sum_{i=1}^{j} (\pi^i, T^i), p^j\right) \geq V\left(\sum_{i=1}^{j-1} (\pi^i, T^i) + (\pi, T), p^j\right)$$

for all $(\pi, T) \geq (0, 0)$ such that $\pi = \bar{p}(j)\, T$.

Condition (b) means that if a p^j-agent has purchased the contracts $[(\pi^1, T^1),\ldots, (\pi^{j-1}, T^{j-1})]$ and is given a choice of purchasing any contract (π, T) with $\pi = \bar{p}(j)\, T$ the optimal choice is (π^j, T^j) which may equal (0, 0).

(D12) Define the set of contracts S^* by

$$S^* = \left\{(\pi, T) \mid (\pi, T) = \sum_{i=1}^{I-1} \lambda^i(\pi^i, T^i) + \lambda^I(p^I L, L) \right.$$
$$\left. \text{for some } (\lambda^1,\ldots, \lambda^I) \text{ each } \lambda^i \in [0, 1]\right\}.$$

For $I = 2$ the set S^* is depicted in Fig. 2. S^* is presently shown to be the description of the equilibrium contract set for arbitrary, but finite, I. In the proof of Proposition 2 the reader should keep in mind the explicit assumptions that have been made about cartel behavior on pooling contracts. All contracts (π^i, T^i) with $i < I$ are pooling contracts and cartel behavior *effectively* limits agents to the purchase of only one such policy.

LEMMA 3. For all i, j, $i > j$ implies $c^i(S) \geq c^j(S)$ for each $S \subset C$ and $c^i(S) \subset O^i(S)$, $c^j(S) \in O^j(S)$.

Proof. A proof of this lemma is given by Wilson in Lemma 4 of [9]. It is not repeated here.

Lemma 3 makes the intuitively obvious assertion that higher-risk types always choose policies with no less coverage and premium charge than lower risk types when the two types choose from the same menu of policies.

LEMMA 4. *If S is convex, $O^i[S]$ has at most one element $C^i[S]$.*

Proof. Straightforward from the strict concavity of $V(\cdot)$, assumption 1.

LEMMA 5. *S^* defined in* (D12) *is compact and convex.*

Proof. Straightforward.

LEMMA 6. $A^i(S^*) = \{(\pi^i, T^i)\}$, [illegible]

$$C(S^*) = \sum_{i=1}^{I} (\pi^i, T^i).$$

Proof. Obvious from the definition of S^*.

One more assumption is needed before the main result of this section can be had.

(A4) For each set of contracts

$$\{(\pi, T) \mid (\pi, T) = \lambda(\pi^i, T^i), \text{ some } \lambda \in [0, 1] \text{ and } i \in I^*\} \subset S^*$$

there exist at least two firms supplying those contracts.

This assumption is necessary for the validity of Proposition 2. It prohibits a firm from withdrawing some contract and altering the equilibrium policy set. A detailed discussion follows the proof of Proposition 2.

PROPOSITION 2. *S^* supports a M1 equilibrium in the economy with risk types (p^i), $i = 1,\ldots, I$.*

Proof. By Lemma 6 $A^i(S^*) = ((\pi^i, T^i))_{i \le I}$. Therefore we need only concern ourselves with the expected returns from the I contracts (π^i, T^i) $i \le I$. The expected profit from contract i is

$$\begin{aligned} P[\pi^i, T^i, S^*] &= \sum_{j \ge i} \frac{d_j[\pi^i, T^i, S^*]}{\sum_{j=1}^{I} d_j[\pi^i, T^i, S^*]} (\pi^i - p^j T^i) \\ &= \sum_{j \ge i} \frac{m^j(\pi^i - p^j T^i)}{\sum_{j \ge i} m^j} \\ &= \pi^i - \bar{p}(i)\, T^i = 0. \end{aligned}$$

Now I need only show that there exist no alternative contracts that could profitably be offered to the market. Let S be any subset of C satisfying

$$\pi = qT$$

for some $q > 0$ and all $(\pi, T) \in S$.

Let

$$\hat{\imath} = \min\{i \in I^* \mid \text{there exists } (\pi, T) \in A^i[S^* \cup S] \cap S\ (\pi, T) > (0, 0)\}.$$

Since $C^{\hat{\imath}}(S^*)$ is the optimal contract of a $p^{\hat{\imath}}$-agent selecting from S^* alone we have

$$(-V_T/V_\pi)^{\hat{\imath}}_{C^{\hat{\imath}}[S^*]} \le \bar{p}(\hat{\imath}).^8$$

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[8] Equality holds if $(\pi^{\hat{\imath}}, T^{\hat{\imath}}) > (0, 0)$.

Therefore $q = \pi/T < \bar{p}(\hat{i})$ for all $(\pi, T) \in S$. I claim $s^i = A^i[S^* \cup S] \cap S = \phi$ for all $i > \hat{i}$ and $s^i \supset s^j$ for $i < j$. To see this, note that by hypothesis $s^{\hat{i}+1} = (0, 0)$. This implies that

$$\bar{p}(\hat{i}+1) \geq \pi/T = q \qquad \text{all } (\pi, T) \in S.$$

It is clear that all p^i-agents with $i \leq \hat{i} + 1$ will purchase $C^{i-1}[S^*]$ en route to their total amount of coverage. After purchase of that amount the lowest price at which insurance is available is q for the newly offered policies. By Lemma 3 the additional coverage demanded is nondecreasing in p^i and it follows easily that $S^i \supset S^j$ if $i < j$. This implies $\sum_{s \in S} P[s, S^* \cup S] < 0$ since $q < \bar{p}(\hat{i})$. Since S and q were chosen arbitrarily it has been shown that no firm can offer a profitable set of contracts. Now suppose some firm withdraws some of the policies it is offering and supplies a different set of policies. By (A4) this removal does not alter the set S^* and cannot be profitable by the argument just given. Q.E.D.

It is important to stress that all but one of the contracts purchased will involve pooling of different risk types. These pooling contracts are viable only if the group of firms selling them are sharing information so that no agent can purchase more than one of a given contract.[9] Given that each contract is being supplied by many firms the equilibrium is a Nash equilibrium.

It is interesting to examine the monopolistic market under the assumption that firms never possess information about customer purchases from other firms. In this case it is possible for the market to reach the position where the set of offered contracts is S^* of definition 12.[10] Interestingly, it turns out in this case that S^* does not support a Nash equilibrium. The problem is that given the extreme informational structure, it is not possible for two or more firms to supply a pooling contract. If more than one firm offers a pooling contract, high-risk agents may purchase several of these contracts causing negative profits. However, if for each line of pooling contracts, there exists only one firm supplying those contracts it is possible for that firm to police the number purchased by each agent. Given this market position Proposion 2 shows that there exist no alternative contracts that could be offered and make a profit. But the firms with a monopoly on different pooling contracts are in a unique position. If one of these firms drops a pooling contract the set S^* is altered. The same firm would be able to replace the dropped contract with another giving the same amount of coverage, but a higher premium. As long as the premium were not increased so much that no one would buy the new contract, expected profit would be positive. Although this means that a Nash equilibrium does not exist, there is a rather attractive anticipatory equilibrium. If firms holding a monopoly on the supply of pooling contracts drop them for more profitable alternatives they may find the gain in profit short lived. This is because there now exists incentive for a new firm to offer the dropped pooling contract, which from the consumers point of view dominates the higher premium contract. As firms learn this, it may be reasonable to assume that firms offering pooling contracts will anticipate the market response. The most attractive feature of this equilibrium is that firm anticipation of the markets response to their actions need only be considered by experienced firms already in the market. Proposition 2 shows that no entering firm could upset the anticipatory equilibrium. For the sake of concreteness consider the case with two risk groups. It is clear here that the only firm that could destroy the equilibrium is the unique firm offering the contracts 0α (see Fig. 2). This equilibrium notion seems to be entirely satisfactory in the context of a static market where no disequilibrium trading is allowed to occur. In that case firms should learn that they cannot do better than the equilibrium pooling contracts. In a dynamic model with disequilibrium trading, the anticipatory equilibrium is not so convincing. The short-run profits gained by dropping the pooling contracts may dominate the strategy of remaining with them. This simply points to the fact that the model does not have enough structure to handle such questions. No attempt to answer these questions is made here. There seems to be no compelling reason to assume that firms will not share purchase information when it is profitable. For this reason it seems reasonable to assume that the market with partial information about consumer purchases dominates both alternatives. In the case of pooling contracts it never harms a firm to share purchase information, in an ex post sense, but it will often turn out profitable.

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[9] The statement that firms must share information is a little misleading. All that is required is that firms selling pooling contracts agree to divulge the names of those agents presenting claims for payments. Thus information is shared in an ex post sense. Such an agreement can never decrease profits, but can increase profits.

[10] It is not difficult but extremely tedious to show that if an equilibrium contract set exists, it must be S^*. I leave that exercise for the reader

4. THE EXISTENCE OF AN E1 EQUILIBRIUM

In the previous section each firm was restricted to offering all its policies at a common average price. This assumption includes the Rothschild–Stiglitz assumption that each firm only offers one policy. The very minor generalization of the one-policy assumption was introduced, because in view of Proposition 1 and the discussion preceding it, the idea of firms offering insurance at a given price and restricting coverage seemed to be a natural first step to examine. It is important to determine just how significant that assumption is.

In his treatment of the adverse selection problem Wilson used a more general equilibrium concept than did Rothschild and Stiglitz in the main sections of their paper. Essentially Wilson allowed each firm to offer different contracts with various prices, but assumed that no firm would offer a policy that had negative expected profit. In this section we discover that the same contract set S^* of the previous section is also a Wilson $E1$ equilibrium.

(D13) A contract set C^* is an $E1$ equilibrium if:

(i) $P[(\pi, T), C^*] \geq 0$ for all $(\pi, T) \in C^*$; and

(ii) There exists no S such that

$$P[c., C^* \cup S] \geq 0 \qquad \text{for all } c \in S$$

with $(>)$ for some $c \in S - C^*$.

The first condition requires that all contracts in C^* make nonnegative returns. The second requires that there exist no new set of contracts that if offered would earn nonnegative profits for each policy in the set, with strictly positive profits for some policy not in the old set.

The main result of this section, Proposition 3, establishes that S^* is indeed an $E1$ equilibrium contract set.

The argument of this section turns out to be much more complicated than the last. The intricacies of the problem cannot be adequately expressed with the case of two risk classes. Recall Lemma 3 which states that for any policy set S an optimal policy for any risk type must be greater than or equal to any optimal contract for an agent of lower risk. Any firm attempting to improve its profits when S^* is offered can offer the market an alternative set of policies S, with the restriction that each buyer purchases only one new policy in S. Having done this each buyer is free to recontract in S^*. Lemma 3 still holds for the total amount of coverage purchased, but it is not true that if S^j is the policy purchased by a p^j-agent from S that $S^i \leq S^j$ must hold for $j > i$. Surprisingly $S^i > S^j$ is possible if the price or premium per unit of coverage is lower for S^j. The reader can verify that if $S^i > S^j$ the price of coverage on S^j, $p(S^j)$ must be strictly lower. Consider the two group case. Suppose that S^* is being offered and some firm offers a set of policies S. Recall that any policy s_i in S purchased by a p^i-agent must have an average price $p(s)$ less than $\bar{p}(i)$. Therefore if p^2-agents are the sole buyers of any contract in S that contract is unprofitable. Therefore any contract purchased by p^2-agents must also be purchased by p^1-agents if it is to be profitable. In that case it follows that each contract purchased by p^2-agents has a lower average price than $\bar{p}(1)$. Since p^1-agents must be assumed to be buying all contracts bought by p^2-agents it follows that the expected profit on these policies are negative. This establishes that an $E1$ equilibrium always exists when $I = 2$. Unfortunately this argument fails to generalize in so simple a fashion. Suppose $I = 3$, and that p^3 is close to p^2 with m^3 relatively small compared to m^1 and m^2. Then $\bar{p}(1)$ can be strictly greater than $\bar{p}(1, 3)$, the average probability of the group composed of all p^1 and p^3 agents. Since it is possible that $S^3 < S^2$ can occur we must take account of the possibility that in the new set S, p^1- and p^3-agents pool on the same policy, while p^2-agents buy another with both earning nonnegative returns. Lemma 7 provides the result that rules out this situation. It states that if a lower risk agent purchases a policy with greater premium and coverage than a policy purchased by some higher-risk agent, then the two agents are indifferent between the two policies. In view of (D3) it follows that agents of both types will be purchasing each policy. In the case $I = 3$ just discussed, if $S^2 > S^3 = S^1$ one-half of both risk types 2 and 3 will purchase S^2 and since $p(S^2) < \bar{p}(2)$ must be the case, that contract earns a negative return. Note however that although all three types purchase $S^1 = S^3$ they do so in proportions $(M^1, M^2/2, M^3/2)$ and it is possible that the contract is profitable. Alternatively it is possible that $S^2 < S^1 = S^3$. In this case Lemma 7 shows that p^1- and p^2-agents are indifferent between S^1 and S^2. The demand vector for the contract $S^1 = S^3$ is

$$d[S^1, S^* \cup S] = (M^1/2, M^2/2, M^3).$$

Again profit must be strictly negative on that contract.

Lemma 7. *Suppose $\bar{S}$ is compact and convex with*

$$\pi[T, \bar{S}] = \min\{\pi \mid (\pi, T) \in \bar{S}\}$$

a continuous, piecewise linear, and increasing function of T. Suppose that $S \subset \hat{C}$. Then if $i > j$ and

$$S^i \in A^i[\bar{S} \cup S] \cap S = S(i),$$
$$S^j \in A^j[\bar{S} \cup S] \cap S = S(j),$$

$S^i < S^j$ implies $S^i \in S(j)$ and $S^j \in S(i)$.

Proof. See appendix.

Lemma 8. *Let $\bar{S}$ and S satisfy the hypotheses of Lemma 7. Suppose that $i > k > j$ and that $S' \in S(i) \cap S(j)$, then $S' \in S(k)$.*

Proof. See appendix.

Lemma 9. *Suppose that $\bar{S}$ is compact and convex. Let $S \subset \hat{C}$ and suppose that consumers are restricted to purchasing one policy in S. Then for each $S^i \in S(i)$*

$$d_i[S^i, \bar{S} \cup S] = m^i \cdot \frac{1}{\#S(i)}.$$

Proof. See appendix.

PROPOSITION 3. *S^* with its associated supply behavior by firms is a E1 equilibrium.*

Proof. First note that S^* satisfies the hypotheses of lemma 7. Recall that

$$S(i) = \{c \in A^i(S^* \cup S) \cap S\} \qquad \text{for } S \subset \hat{C}.$$

It follows from Lemma 6 that

$$-(V_T{}^i/V_\pi{}^i)|_{c^i(S^*)} = \bar{p}(i) \qquad \text{for } i \in I^*.$$

This implies that for $c \in S(I)$

$$p(c) \leqslant p^I = \bar{p}(I).$$

Suppose there exists $c \in S(I)$ and $c \notin S(j)$ for all $j \leqslant I$. This implies in view of the inequality directly above, that $P[c, S^* \cup S] < 0$. Assume therefore that $c \in S(j)$ for some $j < I$ for all $c \in S(I)$. I claim that $S(I) \subseteq S(I-1)$. For $c \in S(I)$ let $i(c) = \min\{i \in I^* \mid c \in S(i)\}$. Note that $i(c) \leqslant I-1$ for all c. Therefore by Lemma 8 $c \in S(I-1)$. Since c was chosen arbitrarily in $S(I)$ we have $S(I) \subseteq S(I-1)$.

This means $p(c) < \bar{p}(I-1)$ or $I-1$ would not buy. By Lemma 9

$$d_j[c, S^* \cup S] = m^j \cdot \frac{1}{\#S(j)} \qquad \text{if } c \in S(j).$$

$S(I) \subseteq S(I-1)$ implies $\#S(I) \leqslant \#S(I-1)$. This means that

$$\frac{d_I[c, S^* \cup S]}{d_{I-1}[c, S^* \cup S]} = \frac{m^I}{M^{I-1}} \cdot \frac{\#S(I-1)}{\#S(I)} \geqslant \frac{m^I}{m^{I-1}}.$$

The above inequality implies that if

$$c \notin S(j) \qquad \text{for all } j < I-1,$$

$P[c, S^* \cup S] < 0$ and again the proof is finished. Assume therefore that $c \in S(j)$ for some $j < I-1$ for all $c \in S(I) \subseteq S(I-1)$.

Then Lemma 8 implies $c \in S(I-2)$ and since c is arbitrary in $S(I)$, $S(I) \subseteq S(I-2)$. I claim $S(I-1) \subseteq S(I-2)$. Let $c' \in S(I-1)$. If $S(I-1) = S(I)$ the result is already shown. Therefore assume $c' \in S(I-1) - S(I)$. This implies $c' < \min\{c \in S(I)\}$. Otherwise by Lemma 7 $c' \in S(I)$. This inequality implies there exists $c \in S(I) \cap S(I-2)$ such that $c > c'$. Then Lemma 7 implies $c' \in S(I-2)$. Therefore

$$S(I-1) \subseteq S(I-2).$$

If there exists $c \in S(I)$ such that $c \notin S(j)$ for all $j > I - 2$ Lemma 9 implies

$$P[c, S^* \cup S] < 0.$$

Repeating this argument $I - 3$ times we have either there exists a $c \in S(I)$ such that $P[c, S^* \cup S] < 0$ or $S(i) \subset S(j)$ for all $i, j \in I^*$ such that $i > j$. This last condition implies

$$\text{(a)} \qquad S(I) \subseteq S(i) \qquad \text{all } i \in I^*$$

$$\text{(b)} \qquad \#S(i) < \#S(j) \qquad \text{for } i > j.$$

Thus by Lemma 9

$$P[c, S^* \cup S] < 0$$

for all $c \in S(I)$.

To complete the proof note that if $S(I) = \{(0, 0)\}$ then Lemma 7 implies $S(i) = \{(0, 0)\}$ for all $i \in I^*$. This means p^l-agents do in fact purchase from S if any type does. Q.E.D.

The proof of Proposition 3 establishes that for any set of contracts S, those contracts chosen from S by the highest-risk-type agents must earn negative expected returns. Therefore if firms never offer unprofitable policies S^* has been shown to support a stable equilibrium.

However, the possibility does exist that some firm may be able to offer a set of policies with some contracts earning large enough positive returns to offset the losses on unprofitable contracts. The definition of an $E1$ equilibrium excludes this possibility. Wilson's justification of the exclusion is that firms will realize that if they offer such a set, other firms will be able to offer only those policies in the new set earning positive returns. In that case the original firm will be left with too many unprofitable policies.

Another justification is based on a criticism of the assumption that firms *know* the expected returns to every conceivable policy offer irrespective of the set of existing contracts. More realistically firms would be learning what expected returns would be for various policy offers. In that case I find the argument that firms would always discontinue immediately the supply of unprofitable policies very persuasive. In the context of imperfect information about the expected returns to contract offers the $E1$ equilibrium concept is a very attractive self-confirming expectations equilibrium of the type used by M. Spence for labor markets.

Despite the above remarks it is still interesting to inquire whether or not the contract set S^* is immune to competitive attack by a firm offering a set of policies some of which must subsidize others, but whose aggregate profitability is positive. Proposition 4 establishes that for the case of two risk groups this kind of subsidy offer is not possible.

Proposition 4. *Let $I = 2$. Then S^* defined in* (D12) *is a Nash equilibrium in the general sense that for all alternative sets of policies $S \subset \hat{C}$ $\sum_{s \in S} P[s, S^* \cup S] < 0$.*

Proof. It was established in Proposition 3 that for any alternative offer S there must exist a contract with strictly negative returns. Therefore we need only consider pairs of contracts where one subsidizes the other.

Suppose that S^* is the aggregate supply of contracts and that some firm offers the pair of policies $\{S^1, S^2\}$. If $S^1 > S^2$ Lemma 7 implies that each policy is purchased by agents of both types. The fact that p^1-agents purchase when S^* is available implies that $p(S^i) < \bar{p}(1)$ for $i = 1, 2$. By Lemma 9 expected profit is negative for each policy. Therefore $S^1 < S^2$ is the only possibility for positive profit.

Let $S^+ = \{(S^1, S^2 \mid S^i \in C$ and $M^1[S_1^1 - p^1S_2^1] + M^2[S_1^2 - p^2S_2^2] > 0$, $i = 1, 2$, with $S^1 < S^2$, $p(S^1) < \bar{p}(1)\}$.

Any pair of policies that could destroy the S^* offer must come from S^+. I show that for each pair of policies in S^+ all agents will demand S^1 no one will purchase S^2 and therefore expected profits are negative. Let $C[S^i] \in S^*$ be the optimal contract in S^* purchased by a p^2-agent who purchases S^i from the new offer. I construct a third contract S satisfying

$$V[S^2 + C[S^2], p^2] < V[S + C[S], p^2] < V[S^1 + C[S^1], p^2].$$

Define $S = (\bar{p}(1) S_2^1 + p^2[S_2^2 - S_2^1], S_2^2)$. The contract S offers the same total coverage as S^2. Its premium S_1, is constructed so that if $S_1^1/S_2^1 = \bar{p}(1)$, and S were purchased by all p^2-agents and S^1 by all p^1-agents expected profits from the sum would equal zero. However, since $S_1^1/S_2^1 < \bar{p}(1)$ it follows that

$$M^1[S_1^1 - p^1S_2^1] + M^2[S_1 - p^2S_2] < 0.$$

Since $S_2 = S_2^2$ it follows from the last inequality and the nonnegativity condition on profits in the definition of S^+ that

$$p(S) < p(S^2).$$

This means that $S_1 < S_1^2$ and we have

$$V[S^2 + C[S^2], p^2] < V[S + C[S], p^2].$$

To prove that S^1 is preferred to S I show that the total coverage $S_2 + C_2[S]$ could be obtained cheaper with S^1. First note

$$S_2 + C_2[S] > S_2^1 + C_2[S]$$

with the difference being $S_2 - S_2^1$. The cost of the additional coverage is $p^2(S_2 - S_2^1)$. Note it is easy to show that $S_2 - S_2^1 < L$. But the cost of additional coverage from S^* has an average price bounded above by p^2 and can be purchased at that price up to amount L. Therefore the coverage $S_2^2 - S_2$ can be purchased with S^1 for a premium

$$S_1^1 - p^2[S_2 - S_2^1] + C_1[S] < S_1 + C_1[S] \qquad \text{since } S_1^1 < \bar{p}(1)\, S_2^1.$$

This implies S^1 is preferred to S and therefore S^2. Since (S^1, S^2) was chosen arbitrarily in S^* we have our result. Q.E.D.

The basic property of S^* that prevents a pair of policies with one subsidiezing the other from being profitable is that the cost of the additional coverage obtained on the subsidized policy beyond that offered with the subsidizing policy must be higher than that coverage would cost if purchased from S^*. Actually a stronger result than Proposition 4 can be had. It can be shown that for any finite I there exists no pair of contracts that are profitable when one subsidizes the other. The obvious question to ask is does there exist a set of contracts $\{c^1, \ldots, c^k\}$ $k > 2$ which are profitable when some must subsidize and $I > 2$? I have not been able to give a satisfactory answer to that question. However, I believe the following is true.

Conjecture. For any finite I risk types

$$\sum_{c \in S} P[c, S^* \cup S] < 0$$

for all $S \subset \hat{C}$.

CONCLUSION

Throughout this paper I have termed insurance firms monopolistic competitors. This appellation seems appropriate to me for two related reasons. First, the fact that insurance firms are not price takers, but are able to exercise choice over price by their offer of policies seems a reasonable manifestation of monopolistic competition with product differentiation. Second, the existence of monopolistic competition and the problems caused by nonconvexities referred to in the introduction are very related. My intuition has been that the existence problem in the insurance market has as its basis the fact that consumer choice sets can be nonconvex with the result that firm profit function can be discontinuous. The reader should notice that in this paper not only is the policy set S^* convex, but that the proofs of the propositions and major lemmas exploited this fact heavily. Perhaps more significantly it should be recognized that if firms could not make all or nothing offers so that consumers could mix fractions of any policy offers it would be a trivial matter to prove the conjecture of the last section.

In the present analysis the convexity of the market offer of policies arises not because firms need be so clever that they understand the nature of the market, but as a natural manifestation of the fact that profit maximization requires quantity rationing at given prices. Recall the discussion of Section 2.

In the absence of perfect information market performance is likely to be altered from the smooth operation of a competitive market. In the case of an insurance market with adverse selection this paper has shown that the market mechanism is highly dependent upon the assumptions one makes about the information market participants hold. This paper has argued that monopolistic competition will produce an equilibrium. The troublesome aspect of the equilibrium allocation is that many agents have constructed portfolios of many contracts from different firms. From a purely theoretical point of view there is nothing suspect about this.[11] In the specific example of insurance markets it does seem that firms have been fairly successful in preventing such behavior by consumers. This implies that firms must be cooperating completely in the sharing of purchase information. One approach to understanding why this may have occured is to invoke moral hazard. In fact the introduction of moral hazard does change market behavior significantly.[12] The bottom line of this is that if a good understanding of the workings of financial markets is to be achieved, our models must confront the moral hazard problem.

APPENDIX

LEMMA 7. *Suppose $\bar{S}$ is compact and convex with*

$$\pi[T \mid \bar{S}] = \min\{\pi \mid (\pi, T) \in \bar{S}\}$$

continuous, piecewise linear, and increasing in T. Suppose that $S \subset \hat{C}$. Then if $i > j$ and

$$S^i \in A^i[\bar{S} \cup S] \cap S \quad S(i),$$

$$S^j \in A^j[\bar{S} \cup S] \cap S = S(j),$$

$S^i \lesssim S^j$ implies

$$S^i \in S(j) \qquad \textit{and} \qquad S^j \in S(i).$$

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[11] An example of current interest is the tendency of some credit card applicants to understate the number of credit cards already held. The borrower is then able to increase his line of credit beyond the credit limit of each card. It is conceivable that if lenders could obtain information about total credit held, the borrower would have a lower level of credit on each card or possibly not be able to obtain all cards.

[12] This assertion is based upon results reported on in [6].

Proof. Note that $S^k \in S(k)$ implies that there exists $C(k) \in O^k(\bar{S} \cup S)$ with

$$C(k) = S^k + C^k \text{ for some } C^k \in \bar{S}.$$

By Lemma 3 $C(i) \geq C(j)$ which means $C^i > C^j$ because $S^i < S^j$.
Define $\mu \in [0, 1]$ so that

$$S_2^i + \mu C_2^i = S_2^j + C_2^j. \tag{a}$$

Since $S^j \in S(j)$ it must be true that

$$S_1^i + \mu C_1^i \geq S_1^j + C_1^j. \tag{b}$$

Otherwise agent j could have been made strictly better off by purchasing S^i. If the strict inequality holds in *b* agent i could have purchased S^j and obtained his first $S_2^j + C_2^j$ units of coverage at a strictly lower cost than with S^i. The additional coverage purchased by agent i is

$$C_2(i) - C_2(j) = Y.$$

It can be shown that the cost of the additonal Y units can be no greater with S^j than with S^i. With the policy S^i obtaining total coverage $C_2(j)$ entails by Eq. (a) above, purchasing μC_2^i units of coverage from $\bar{S}$. Define $\lambda \in [0, 1]$ so that

$$\lambda C_2^i = C_2^j \tag{c}$$

This means that with the policy S^j buying total coverage $C_2(j)$ involves purchasing λC_2^i units from $\bar{S}$. From Eq. (a),

$$\mu C_2^i = S_2^j - S_2^i + C_2^j \geq C_2^j = \lambda C_2^i.$$

The difference between the cost of the final Y units via the alternative policies is

$$\pi(\mu C_2^i + Y) - \pi(\mu C_2^i) - (\pi(\lambda C_2^i + Y) - \pi(\lambda C_2^i)) \geq 0 \tag{d}$$

by the assumptions on $\pi(\cdot\cdot)$. This means that with the strict inequality in (b) agent i must strictly prefer S^j to S^i irrespective of the equality or inequality in (d). Since this contradicts the assumption that $S^i \in S$(i) the equality must hold in (b) and it follows that $S^i \in S(j)$. On the other hand if the strict inequality holds in (d) it still follows that S^j must be strictly preferred to S^i by agent i. Since this cannot be, equality holds and we have immediately that

$$S^j \in S(i). \qquad \text{Q.E.D.}$$

LEMMA 8. *Let $\bar{S}$ and S satisfy the hypotheses of Lemma 7. Suppose that*

$i > k > j$ and that $S' \in S(i) \cap S(j)$, then $S' \in S(k)$.

Proof. Let $C^k \in S(k) \neq \phi$ since $C^k = (0, 0)$ is possible. Either $C^k \succ S'$ or $C^k \prec S'$.

The other possibilities can be disposed of easily. In either of the above two cases lemma 7 implies

$$S' \subset S(k). \qquad \text{Q.E.D.}$$

LEMMA 9. *Suppose that $\bar{S}$ is compact and convex and that $S \subset \hat{C}$. Then for each $S^i \in S(i)$*

$$d_i[S^i, \bar{S} \cup S] = M^i \cdot \frac{1}{\#S(i)}.$$

Proof. By definition

$$d_i[S^i, \bar{S} \cup S] = M^i \cdot \frac{\{A_j^i(C^i, \bar{S} \cup S \mid S^i \in A_j^i \text{ for some } C^i \in O^i(\bar{S} \cup S)\}}{\sum_{C^i \in O^i(\bar{S} \cup S)} \#\{A^i(C^i)\}}.$$

For each $C^i \in O^i$ there exists $S^i \in S(i)$ and $C[S^i] \in \bar{S}$ such that

$$C^i = S^i + C[S^i].$$

For each such S^i the fact that $\bar{S}$ is convex implies by Lemma 4 that $C[S^i]$ is unique. This means that C^i is unique for S^i and it follows that the numerator in the demand element above equals $M^i \cdot 1$. By similar reasoning the denominator equals $\#S(i)$. Q.E.D.

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REFERENCES

1. G. AKERLOF, The market for "lemons": Qualitative uncertainty and the market mechanism, *Quart. J. Econ.* **84** (1970).
2. K. ARROW, The organization of economic activity: Issues pertinent to the choice of market versus non-market allocation, *in* "The Analysis and Evaluation of Public Expenditures: The PPB System," pp. 47–64, U.S. Congress, Joint Economic Committee, U.S. Government Printing Office, Washington, D.C., 1969.
3. J. GREEN, The non-existence of informational equilibria, *Rev. Econ. Stud.* (October 1977).
4. S. GROSSMAN, The existence of future markets, noisy rational expectations and informational externalities, *Rev. Econ. Stud.* (October 1977).

5. S. GROSSMAN AND J. STIGLITZ, Information and competitive price systems, *Amer. Econ. Rev.* (May 1976).
6. G. JAYNES, Asymmetric information and the performance of insurance markets, manuscript, Yale University, 1978.
7. D. KREPS, A note on "fulfilled expectations" equilibria, *J. Econ. Theory* **14** (1977), 32–43.
8. M. ROTHSCHILD AND J. STIGLITZ, Equilibrium in competitive insurance markets, *Quart. J. Econ.* **90** (1976), 629–649.
9. C. WILSON, A model of insurance markets with asymmetric information, *J. Econ. Theory* **16** (1977), 167–207.

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Taxing Price Makers*

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INTRODUCTION

It has been early recognized that the existence in the production sector of distortions between marginal rates of transformation and relative prices might invalidate the traditional conclusions of welfare economics (see Lipsey and Lancaster [22] for a defense of this argument). Actually, part of the second best literature is aimed at elucidating the normative implications of the presence in the production sector of distortions which originate in policy constraints [5], in fixed taxes [10, 20] or in some degree of monopoly power [3].

In all of these studies, producers respond to prices only, and their behavior can be interpreted as a "deviant" or non competitive behavior with respect to some price vector and formalized through a correspondence which associates to this vector a set of production plans (as in [12] or [14]). Clearly, this formulation cannot apply to price makers, in particular to genuine monopolists whose supplies depend in general on other variables in addition to prices.

Even though distortions created by monopolies were one of the initial motivations of this literature, the study of price makers in a second best framework remains rather poor. The conventional wisdom is that in an economy with price makers it is possible to restore first best Pareto efficiency with appropriate taxation schemes. This "tradition" seems to rest on partial equilibrium formulations of the problem or on general equilibrium models [4] which do not recognize the full complexity of the problem.

Our work is an attempt at a comprehensive treatment in a general equilibrium framework of the question of decentralization of Pareto optima in economies with price makers where linear taxes are available. The strategic behavior of price makers is formalized through a reaction correspondence

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which in the case of the monopolist has an easy interpretation when he is assumed to know the true demand function. New results stem both from the general equilibrium approach and from the explicit recognition of a fundamental non convexity.

In Section I, the essential discontinuity of the monopolist's reaction function is explained. Then, our basic model and assumptions are discussed. Adopting successively the second best theory point of view and the decentralization point view, Sections II and III develop our theory of optimal taxation of a price maker when, in addition to linear taxes, lump sum transfers are available policy tools. We first show that, in a regular case, a linear excise tax on the price maker (which turns out to be a subsidy in the case of a normal good) enables the Government to restore first best Pareto efficiency. However the non regular case is shown to be non pathological. Indeed an example of a simple economy where a first best Pareto optimum can never be attained is given. The non regular case presents a noteworthy property: the corresponding second best situation is generally highly unstable; a small variation of the environment or a small error in the choice of policy tools induce the price maker to choose a production level which is far from the optimal one. In some sense the optimizing behavior of the Government tends to lead the system, which would have been otherwise stable, to unstable situations ("catastrophic" points). This latter feature is closely related to the fact that the maximization program of the agent that the Government attempts to control is not concave. An analysis similar to this one would be relevant in any control problem involving an agent which has a reaction function discontinuous with respect to his environment (this point has been stressed independently by Mirrlees [16]. Section IV extends the analysis to the case where lump sum transfers are not available. Then, distributional and allocational goals become conflicting. The difficulties created by these types of question are explored under several distributional assumptions, the Diamond-Mirrlees case where profits are 100 % taxed and cases where profits are only partially taxed. Then, even in the regular case, first best Pareto optimality is not in general achievable. The characteristics of the non regular case are extended to this situation. Finally some comments on the informational assumptions underlying our work are offered.

I. PRELIMINARIES

I.1. *A Main Difficulty: Discontinuities in the Monopolist's Reaction Function*

The non concavity of the profit function of a monopolist is now a well known phenomenon. Such a non concavity induces discontinuities in the monopolist's response to changes in the price system and in the parameters of the economy. These discontinuities are the main obstacle to the integration of monopoly theory in a general equilibrium setting of the Arrow-Debreu type; the need for ad hoc assumptions to obtain convex valued supply correspondences has been the major teaching of repeated attempts by Arrow [1], Arrow and Hahn [2], Fitzroy [9], Gabszewicz and Vial [11], Negishi [17], Laffont and Laroque [13], Marschak and Selten [15] to prove the existence of a monopolistic general equilibrium. Recently, Roberts and Sonnenschein [18] have told more explicitly the same story by proving the possibility of the construction of an economy without any monopolistic equilibrium.

Beyond the existence question, this discontinuity in the monopolist's reaction function has also a serious influence on the normative analysis presented here, by requiring special attention to those critical points where marginal changes create non marginal consequences. So, as a preliminary necessary work, we focus our attention on this problem in the remainder of this subsection.

Consider an economy with $L+1$ commodities. Good 0 is supplied by a monopolist. Goods 1,..., L are supplied by competitive firms, and competitive consumers demand all of the goods. Let R_i, $i = 1,\ldots, I$, be the income of consumer i and let $R = (R_1, \ldots, R_I)$. The consumer price vector is denoted $\Pi = (\pi_0, \pi) \in \mathbb{R}_+ \times \mathbb{R}_+^L$ and the producer price vector is denoted $P = (p_0, p) \in \mathbb{R}_+ \times \mathbb{R}_+^L$. Let $T = \Pi - P = (t_0, t) \in \mathbb{R} \times \mathbb{R}^L$ the vector of taxes. Finally let $\xi_0(\Pi, R)$ be the continuous demand function for good 0 and let $C(q, p)$ be the continuous cost function of the monopolist firm.

The profit function of the monopolist is therefore:

$$\begin{aligned}\Psi(p_0, p, T, R) &= p_0\xi_0(P+T, R) - C(\xi_0(P+T, R), p)\\ &= p_0\xi_0(p_0+t_0, p+t, R) - C(\xi_0(p_0+t_0, p+t, R), p)\end{aligned}$$

The function $\Psi(., p, T, R)$ is in general non concave and we have essentially two cases (see Fig. 1).

Case a. *Unique maximizing price.* In Fig. 1a, p_0^* is the unique maximizing price of the monopolist. Moreover, as a function of the environment (p, T, R), p_0^* is a continuous function, since a marginal change of these variables, leading to the dotted curve, induces only a marginal change of the monopolist's optimal price choice.

Case b. *Multiple maximizing prices.* In Fig. 1b, two maximizing prices exist for the monopolist. Suppose p_0^* is chosen. Then, a marginal change of (p, T, R) (leading to the dotted curve) creates a large discontinuity in the behavior of the monopolist for whom a value close to p_0^{**} becomes optimal.

Generally speaking, the maximization of his profit function by the monopolist determines a price correspondence $\mathscr{P}_0$, which associates to every environment (p, T, R) a set of optimal prices. Assuming that demand

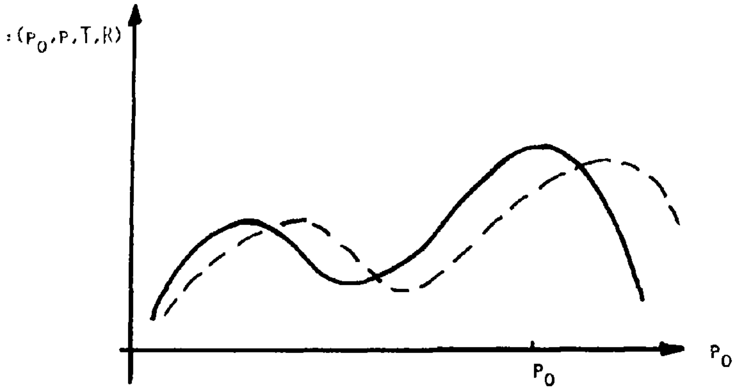

FIGURE 1a

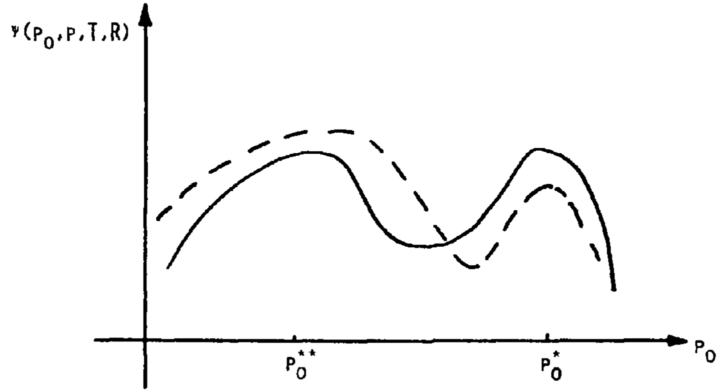

FIGURE 1b

functions are single valued, let Q_0 be the induced quantity correspondence. These correspondences are multivalued and non convex valued at critical points corresponding to case b, and they are single valued at regular points corresponding to case *a*.

Note that the monopoly choice problem can be approached with the quantity instead of the price as a decision variable : If $\pi_0(q, \pi, R)$ is the inverse demand function for the monopolized commodity, the profit function of the monopolist is:

$$\Phi(q, p, T, R) = [\pi_0(q, \pi, R) - t_0]\, q - C(q, p).$$

This alternative formulation does not affect the analysis of the maximisation problem.

Having made explicit the discontinuity of the monopolist's reaction function, we consider in the next sections the more general case of a price maker agent, defined by his price correspondence $\mathscr{P}_0(p, T, R)$.

This formulation may be used in the description of a wide variety of oligopolistic competition. In particular in the case of a Cournot oligopoly $\mathscr{P}_0(p, T, R)$ is the set of agreed prices obtained as Nash equilibria of the associated game.

I.2. *General Features of the Model*

We consider an economy with the following physical characteristics. There are $L + 1$ commodities indexed by $l = 0, \ldots, L$. Commodity 0 is a final commodity. Consumers are indexed by $i = 1, \ldots, I$. Production sets Y_j, $j = 0, \ldots, J$, are associated with private producers. Y_g denotes the production set of the public sector. The general assumptions on preferences and technology are the following.

Assumption 1. Consumer i has a monotone strictly concave utility function $u_i(.)$ defined on a convex consumption set X_i, $i = 1, \ldots, I$.

Assumption 2. The production set Y_j of producer j, $j = 0, \ldots, J$ is convex. In addition the production set Y_g of the public sector defined by $f(y_g) \geqslant 0$ is convex. In addition commodity 0 is produced only by firm 0.

The set of feasible states of this economy given the technological constraints, which we call the set of *physical feasible states*, is defined as follows:

$(x_i)(y_j)(y_g)$ is a physical feasible state iff:

$$x_i \in X_i, i = 1, \ldots, I, y_j \in Y_j, j = 0, \ldots, J, y_g \in Y_g \tag{1.1}$$

$$\sum_{i=1}^{I} x_i \leqslant \sum_{j=0}^{J} y_j + y_g \tag{1.2}$$

In the whole paper we suppose that:

- Firm 0 is a price maker on the market of commodity zero and has a competitive cost minimizing behavior on the other markets. All the other agents have a competitive behavior on all markets.

- Consumers and producers are faced with two different price systems, denoted as in the preceding subsection $\Pi = (\pi_0, \pi)$, $P = (p_0, p)$. ($T = (t_0, t)$ and $R = (R_1, \ldots, R_I)$ denote taxes and incomes).

Assumptions 3 and 4 are regularity assumptions concerning respectively demand and supply functions and the cost function of the price maker. Assumptions 5 and 6 are local regularity assumptions.

Assumption 3. The utility function $u_i(.)$, the indirect utility function $V_i(\Pi, R_i)$ and the demand functions $\xi_{il}(\Pi, R_i)$ of agent i, $i = 1,\ldots, I$, the supply functions $\eta_{jl}(p)$ of the competitive producers $j = 1,\ldots, J$, the production function $f(\cdot)$ of the public sector, are continuously differentiable in the interior of their domains.

Let

$$\xi_l(\Pi, R) = \sum_{i=1}^{I} \xi_{il}(\Pi, R_i), \qquad l = 1,\ldots, L$$

$$\eta_l(p) = \sum_{j=1}^{J} \eta_{jl}(p), \qquad l = 1,\ldots, L$$

Assumption 4. Let q be the production level of commodity 0; the cost function of the price maker $C(q, p)$ and his net demand on markets 1 to L, $x_l(p, q)$ are continuously differentiable in the interior of their domains.

Assumption 5. The (L, L) matrix A defined by

$$a_{lk} = \left(\frac{\partial \eta_l}{\partial p_k} - \frac{\partial x_l}{\partial p_k}\right) \quad \begin{matrix} l = 1,\ldots, L, \\ k = 1,\ldots, L \end{matrix}$$

is of rank $L - 1$,

Assumption 6. The $(L + 1, L + 1)$ matrix S defined by

$$s_{lk} = \sum_{i \in I} \left(\frac{\partial \xi_{il}}{\partial \pi_k} + \xi_{ik} \frac{\partial \xi_{il}}{\partial R_i}\right)$$

which is the matrix of aggregate compensated price effects is of rank L.

Given the institutional constraints we have just mentioned a *feasible state of the model considered in this paper* consists of consumption plans x_i, $i = 1,\ldots, I$, production plans $y_0 = (q, x_0)$, y_j, $j = 1,\ldots, J$, y_g, associated with (Π, P, R) such that:

$$\eta_l(p) + y_{gl} - \xi_l(\Pi, R) - x_{0l} \geqslant 0 \tag{1.3}$$

$$x_{0l} = x_l(p, q) \tag{1.4}$$

$$q - \xi_0(p_0 + t_0, \pi, R) = 0 \tag{1.5}$$

$$p_0 \in \mathscr{P}_0(p, T, R) \tag{1.6}$$

$$f(y_g) \geqslant 0 \tag{1.7}$$

Eventually, in the models we will consider, additional constraints on T, y_g, R will be considered corresponding to different assumptions on the set of policy tools which are available to the Government.

- If there is no other significant constraint than (1.3) to (1.7), the Government can disconnect freely production and consumption prices, decide upon the production plans of the public sector and control the income of each individual through a program of lump sum transfers. This will be the model considered in sections II and III

- In section IV we will assume a more restricted set of policy tools. Lump sum transfers will be supposed impossible in subsection IV.1, i.e. $R_i = 0$, $i = 1, \dots, I$. In subsections IV.2 and IV.3, non labor income will arise from the distribution of the price maker's profit. The assumption of unrestricted commodity taxation will be given up subsection IV.3.

II. TAXING PRICE MAKERS WHEN LUMP SUM TRANSFERS ARE AVAILABLE: THE SECOND BEST THEORY POINT OF VIEW

II.1. *The Maximization Program*

We assume that the Government can realize any distribution of income $(R_1, \dots, R_I)$ through a program of lump sum transfers. This restrictive assumption has been made implicitly in the whole literature on optimal taxation with monopolies. A priori it gives to the Government powerful policy tools for solving distributional problems; in particular the question raised by the pure profit of the monopolist. The Government may also completely disconnect production and consumption prices with linear commodity taxes and control production in the public sector.

Let us first consider a single price maker. Given a social welfare function defined by the weights $\alpha_i > 0$, $i = 1, \dots, I$, the Government's maximization problem can then be written[1]:

$$\text{(I)} \qquad \max \sum_{i \in I} \alpha_i V_i(\Pi, R_i)$$

s.t.

$$\eta_l(p) + y_{gl} - \xi_l(\Pi, R) - x_l(p, q) \geq 0, \; l = 1, \dots, L \tag{2.1}$$

$$q - \xi_0(p_0 + t_0, \pi, R) = 0 \tag{2.2}$$

---

[1] The set of feasible states we are considering is such that all markets are cleared through price adjustments. We may also interpret the program as if the policy tools of the Government were p, T, R and y_g and if he behaved as an auctioneer. We assume throughout the paper that the set defined by equations (2.1) to (2.5) is non empty and that program (I) has solutions. We refer to the literature on existence proofs (see Roberts and Sonnenschein [18]) for sufficient conditions of existence when taxes are zero. Let us note only that the presence of taxes makes the existence question more tractable; for example, when the commodity produced by the monopoly is not a "necessary" good, the economy can be reduced to an economy without monopoly by using a large enough tax on this commodity. Looking at compactness is also outside the scope of this paper.

$$p_0 \in \mathscr{P}_0(p, T, R) \tag{2.3}$$

$$f(y_0) \geqslant 0 \tag{2.4}$$

$$p \geqslant 0 \; \pi \geqslant 0 \tag{2.5}$$

If the producer of good 0 was behaving competitively we know that the Government could then be able to achieve a first best optimum. With production prices it could control private production locally. With consumer prices and lump sum transfers it could control consumption locally. In the set of "physical" feasible allocations the Government could then ensure a displacement in any direction. Any solution of program (I) would then have to be a local optimum of the social welfare function in the set of feasible allocations. Moreover, in the convex economy we defined, such a local optimum would also be a global optimum, i.e., a first best optimum.

The introduction of a price maker defined by the correspondence $\mathscr{P}_0(.)$ raises the question: under which conditions is a solution of program (I) still a first best optimum? Consider such a solution. First in the special case of a monopolist, we show geometrically and hopefully intuitively that if his reaction function is differentiable with respect to the instruments in a neighborhood of the solution, it is possible to control locally the production vector of the monopolist in addition to the behavior of any other agent. Consequently as above, we are at a first best optimum. However, because of a fundamental non concavity in the price maker's profit function there are discontinuities in his responses to changes in the parameters. Therefore, at the solution of program (I), the reaction function of the monopolist may not be differentiable. Then, such a solution is not in general a first best optimum.

II.2. *Local Controllability When the Price Maker Is a Monopolist: A "Geometric" Insight*

We consider in this section a simplified version of the economy presented above with one consumer (u, X), and one public firm (private firms are excluded). The price taker is a monopolist with the profit function given in section.

$$\Phi(q, p, T, R) = [\pi_0(q, \pi, R) - t_0]\, q - C(q, p).$$

The vector $e = (p, T, R)$ defines the environment of the monopolist. We will see that the Government, using the available policy tools, is able to have local control of the economy on the relevant "physical" feasible set around initial situations where the production of the monopolist is regular.

Let us first define precisely what is a regular production plan of the monopolist. Let $B^d Y_0$ be the boundary of the monopolist's production set. The set

Λ_0 of attainable regular production plans of the monopolist is defined as follows:

$$y_0 = (q, x_0) \in \Lambda_0 \Leftrightarrow \exists e = (p, T, R) \text{ such that:}$$

- $Q_0(.)$ is single valued and continuous at (e).
- $q = Q_0(p, T, R)$, $x_0 = x(p, q)$
- the profit function $\Phi(., e')$ is differentiable for any e' in a neighborhood of e.

Λ_0 is the set of production plans which are optimal for the monopolist for some state of the environment and in which small perturbations of the environment induce smooth reactions of the monopolist.

Consider then a state $(\bar{y}_0, \bar{y}, \bar{x})$ of the simplified economy such that:

$$\bar{x} = \xi(\bar{\Pi}, \bar{R}),\ \bar{y}_0 = (\bar{q} = Q_0(\bar{p}, \bar{T}, \bar{R}),\ \bar{x}_0 = x(\bar{p}, \bar{q})),\ \bar{y}_0 \in \Lambda_0,$$
$$\bar{x} = \bar{y}_0 + \bar{y}_g .$$

$\bar{y}_0, \bar{y}_g, \bar{x}$ is a feasible state of the model associated with equations (2.1) to (2.5); it has the additional property that $\bar{y}_0 \in \Lambda_0$.

The following lemma states that in this case, the Government, using adequately the policy tools, can induce the agents to choose any 3-uple $y_0 \in B^d Y_0$, $y_g \in Y_g$, $x \in X$ close to $\bar{y}_0$, $\bar{y}_g$, $\bar{x}$ *and* meeting the constraints $x = y_0 + y_g$. In other words, all agents can be induced to change their demands and supplies in any relevant direction compatible with technological and scarcity constraints.

LEMMA 1. *Let* $\bar{y}_0 = (\bar{q}, \bar{x}_0)$, $\bar{y}_0 \in \Lambda_0$ *associated with* $(\bar{p}, \bar{T}, \bar{R})$; *let* $\bar{x} = \xi(\bar{\Pi}, \bar{R})$, $\bar{y}_g \in Y_g$ *such that* $\bar{x} = \bar{y}_g + \bar{y}_0$. *Under* A1 *to* A4 *for any* $\{y_0 = (q, x_0), y_g, x\} \in \{\mathcal{V}(\bar{y}_0) \cap B^d Y_0\} \times \{\mathcal{V}(\bar{y}_g) \cap Y_g\} \times \{\mathcal{V}(\bar{x}) \cap X\}$ *such that* $x \leq y_0 + y_g$ (*where* $\mathcal{V}(.)$ *are appropriate neighborhoods of* (.)) *then there exist* (p, T, Π, R) *close to* $(\bar{p}, \bar{T}, \bar{\Pi}, \bar{R})$ *such that* $x = \xi(\Pi, R)$, $q = Q_0(p, T, R)$, $x_0 = x_0(p, q)$.

Proof. It is clear that there exist (Π, R) close to $(\bar{\Pi}, \bar{R})$ such that $x = \xi(\Pi, R)$. Since $y_0 \in \mathcal{V}(\bar{y}_0) \cap B^d Y_0$, there exists p close to $\bar{p}$ such that $x_0 = x(p, q)$. Furthermore, considering the first order condition of maximization of profit $(\partial\Phi/\partial q) = 0$

$$t_0 = \frac{\partial \pi_0}{\partial q}(q, \pi, R)\, q + \pi_0(q, \pi, R) - \frac{\partial C}{\partial q}(q, p)$$

where the right hand side defines, for $(\pi, R,)\, p$ close enough to $(\bar{\pi}, \bar{R}, \bar{p})$, the optimal tax t_0-close to $\bar{t}_0$ since the right hand side of the above expression is continuous in all the variables-which induces the monopolist to produce q. Q.E.D.

Let us now consider $\zeta^* = (x^*, y_0^*, y_g^*)$ an optimum of our one consumer economy with monopoly, associated with e^* and such that $x^* = \xi(\Pi^*, R^*)$. $y_0^* \in A_0$.

PROPOSITION. *Let $\zeta^* = (y_0^*, y_g^*, x^*)$ be an optimum of the one consumer economy defined by equations* (2.1) *to* (2.5) *and associated with a regular y_0^*. Then, under $A1$ to $A4$, on the one hand $X(x^*) = \{x \in X, x \succ x^*\}$ and $Y_0 + y_g^*$, and on the other hand $Y_g + y_0^*$ and $X(x^*)$ are locally separated.*

Proof. Let us suppose for example that $Y_g + y_0^*$ and $X(x^*)$ are not locally separated. Then, there exist $x' \in X(x^*)$ such that $x' = y_0^* + y_g'$ for some y_g' close to y_g^*. But the preceding lemma asserts that y_0^*, y_g', x' is a feasible state of our model contradicting the optimality of ζ^*. A similar reasoning applies for $Y_0 + y_g^*$ and $X(x^*)$. Q.E.D.

It follows that, if Y_g and Y_0 are convex, we have in addition that:

$$\pi^* \cdot y_g^* \geq \pi^* \cdot y_g \quad \forall y_g \in Y_g, \; \Pi^* \cdot y_0^* \geq \Pi^* \cdot y_0 \quad \forall y_0 \in Y_0$$

and that ζ^* is a first best optimum.

These propositions are straightforward consequences of lemma 1. A state where the monopolist's production, public production and private consumption can be locally controlled in the sense of lemma 1, can be improved upon in the sense of some social welfare function as soon as it is not a local maximum of the first best problem, (i.e. as soon as it is not a first best optimum because we are in a convex problem). Hence second best optima occuring at regular production plans are first best optima.

However, lemma 1 does not hold when $y_0 \notin A_0$. In this case the Government can only modify the monopolist's production plans in some directions depending on the associated changes in consumption and incomes. Hence the second best allocation where $y_0^* \notin A_0$ will not be (generally) first best.

We will see in the next subsection, that this analysis is general. Two different characterization theorems are obtained according to whether the production plan of the price taker is regular or not. In the first case, first best Pareto optimality is restored. In the second case, optimality conditions are more complicated and shadow prices to be given to public firms are no longer consumption prices.

II.3. *The Regular Case. General Results*

We generalize in this subsection the above results to any type of price maker. We observe that a characterization of the maxima of program (I) can be obtained through standard techniques only when at the optimum the correspondence $\mathscr{P}_0$ is single valued and differentiable. Indeed we have:

THEOREM 1.[2] *Let* (p_0^*, p^*, T^*, R^*) *be an optimal solution of program* (I) *with* $(p_0^*, p^*) \gg 0$. *Under* A1 – A4, *if in a neighborhood of* (p^*, T^*, R^*), *the correspondence* $\mathscr{P}_0$ *reduces to a single valued and continuously differentiable function such that* $(\partial\mathscr{P}_0/\partial t_0)(p^*, T^*, R^*) \neq -1$, *if* A5, A6 *are satisfied at the optimal solution, then the corresponding state is first best Pareto optimal. Moreover* T^* *can be normalized so that only commodity* 0 *is taxed.*

Proof. The Lagrangian $\mathscr{L}$ of program (I) is:

$$\begin{aligned}\mathscr{L} = {} & \sum_{i \in I} \alpha_i V_i(P + T, R_i) + \sum_{l=1}^{L} \rho_l[\eta_l(p) + y_{gl} - \xi_l(p_0 + t_0, p + t, R) \\ & - x_l(p, q)] + \rho_0[-\xi_0(p_0 + t_0, p + t, R) + q] \\ & + \lambda[p_0 - \tilde{\mathscr{P}}_0(p, T, R)] + \mu f(y_g) + \nu p + \gamma\pi\end{aligned}$$

where $\tilde{\mathscr{P}}_0(p, T, R)$ is a continuously differentiable function which coincides with $\mathscr{P}_0$ in a neighborhood of (p^*, T^*, R^*).

Monotonicity of preferences and availability of lump sum transfers imply $f(y_g^*) = 0, \mu^* > 0, \Pi^* \gg 0$. Hence, as $p^* \gg 0$, the Kuhn and Tucker conditions reduce to

$$\sum_{i \in I} \alpha_i \frac{\partial V_i}{\partial p_0} - \sum_{l=0}^{L} \rho_l^* \frac{\partial \xi_l}{\partial p_0} + \lambda^* = 0 \tag{2.6}$$

$$\sum_{i \in I} \alpha_i \frac{\partial V_i}{\partial t_0} - \sum_{l=0}^{L} \rho_l^* \frac{\partial \xi_l}{\partial t_0} - \lambda^* \frac{\partial \tilde{\mathscr{P}}_0}{\partial t_0} = 0 \tag{2.7}$$

$$-\sum_{l=1}^{L} \rho_l^* \frac{\partial x_l}{\partial q} + \rho_0^* = 0 \tag{2.8}$$

$$\sum_{i \in I} \alpha_i \frac{\partial V_i}{\partial p_k} - \sum_{l=0}^{L} \rho_l^* \frac{\partial \xi_l}{\partial p_k} - \sum_{l=1}^{L} \rho_l^* \left(\frac{\partial x_l}{\partial p_k} - \frac{\partial \eta_l}{\partial p_k}\right) - \lambda^* \frac{\partial \tilde{\mathscr{P}}_0}{\partial p_k} = 0 \qquad k = 1, \ldots, L \tag{2.9}$$

$$\sum_{i \in I} \alpha_i \frac{\partial V_i}{\partial t_k} - \sum_{l=0}^{L} \rho_l^* \frac{\partial \xi_l}{\partial t_k} - \lambda^* \frac{\partial \tilde{\mathscr{P}}_0}{\partial t_k} = 0 \qquad k = 1, \ldots, L \tag{2.10}$$

$$\rho_l^* + \mu^* \frac{\partial f}{\partial y_{gl}} = 0 \qquad l = 1, \ldots, L \tag{2.11}$$

$$\alpha_i \frac{\partial V_i}{\partial R_i} - \sum_{l=0}^{L} \rho_l^* \frac{\partial \xi_{il}}{\partial R_i} - \lambda^* \frac{\partial \tilde{\mathscr{P}}_0}{\partial R_i} = 0 \qquad i \in I \tag{2.12}$$

---

[2] Throughout the paper, the regularity conditions of the Kuhn-Tucker theorem are assumed to be valid.

Observing that $\partial V_i/\partial p_0 = \partial V_i/\partial t_0$, $\partial \xi_{il}/\partial p_0 = \partial \xi_{il}/\partial t_0$ and subtracting (2.7) from (2.6) we obtain

$$\lambda^* \left[1 + \frac{\partial \tilde{\mathscr{P}}_0}{\partial t_0}\right] = 0 \tag{2.13}$$

Therefore, if $\partial \mathscr{P}_0/\partial t_0 \neq -1$, we get

$$\lambda^* = 0 \tag{2.14}$$

Observing that $\partial V_i/\partial p_k = \partial V_i/\partial t_k$ and $\partial \xi_{il}/\partial p_k = \partial \xi_{il}/\partial t_k$, we subtract (2.10) from (2.9) and using (2.14) we have:

$$\sum_{l=1}^{L} \rho_l^* \left(\frac{\partial \eta_l}{\partial p_k} - \frac{\partial x_l}{\partial p_k}\right) = 0 \qquad k = 1,\dots, L \tag{2.15}$$

From the homogeneity of degree zero of $\eta_l(.)$ and $x_l(.)$, $l = 1,\dots, L$ and the symmetry of the Jacobians of η_l and x_l we have

$$\sum_{l=1}^{L} p_l^* \frac{\partial \eta_l}{\partial p}(p^*) = 0$$

$$\sum_{l=1}^{L} p_l^* \frac{\partial x_l}{\partial p}(p^*) = 0$$

which implies with assumption $A5$:

$$\rho_l^* = kp_l^* \qquad l = 1,\dots, L \tag{2.16}$$

Now, multiplying equation (2.12) by ξ_{ik}^*, $i \in I$ and adding all these equations to the kth equation of (2.10) we obtain:

$$\sum_{i\in I} \alpha_i \left(\frac{\partial V_i}{\partial t_k} + \xi_{ik}^* \frac{\partial V_i}{\partial R_i}\right) - \sum_{l=0}^{L} \rho_l^* \sum_{i\in I}\left(\frac{\partial \xi_{il}}{\partial t_k} + \xi_{ik}^* \frac{\partial \xi_{il}}{\partial R_i}\right) = 0 \tag{2.17}$$

Since, from Roy's identity

$$\frac{\partial V_i}{\partial t_k} = -\xi_{ik}^* \frac{\partial V_i}{\partial R_i},$$

(2.17) reduces to

$$\sum_{l=0}^{L} \rho_l^* \sum_{i\in I}\left(\frac{\partial \xi_{il}}{\partial t_k} + \xi_{ik}^* \frac{\partial \xi_{il}}{\partial R_i}\right) = 0 \tag{2.18}$$

But, since $\partial\xi_{il}/\partial t_k + \xi^*_{ik}\, \partial\xi_{il}/\partial R_i$ is by the Slutsky equations the derivative of the compensated demand function evaluated at Π^*, we have

$$\sum_{l=0}^{L} \pi_l^* \sum_{i\in I} \left(\frac{\partial\xi_{il}}{\partial t_k} + \xi^*_{ik}\frac{\partial\xi_{il}}{\partial R_i}\right) = 0$$

then, from $A6$,

$$k'\pi_l^* = p_l^* \qquad l = 0, 1,\ldots, L \tag{2.20}$$

which, with (2.16), implies:

$$\pi_l^* = \frac{k}{k'}\, p_l^* \qquad l = 1,\ldots, L \tag{2.21}$$

From (2.16), (2.20), (2.8) we have:

$$\frac{k'}{k}\pi_0^* = \sum_{l=1}^{L} p_l^* \frac{\partial x_l}{\partial q} = \frac{\partial C}{\partial q}(q^*, p^*) \tag{2.22}$$

where $C(q, p)$ is the cost function of the price maker firm.

Therefore, the production plan of the price maker would be competitive with respect to the price system $((k'/k)\,\pi_0^*, p_1^*,\ldots, p_L^*)$.

Also (2.11) implies with $A2$ that the shadow prices of the controlled firm are p_l^*, $l = 1,\ldots, 4$ and that $p^* y_g^* \geqslant p^* y_g$ for any y_g such that $f(y_g) \geqslant 0$.

It follows that the production sector behaves as if it were competitive with respect to the price vector

$$\tilde{P}^* = \left(\frac{k'}{k}\pi_0^*, p_1^*,\ldots, p_L^*\right)$$

Similarly, consumers face the price system $(\pi_0^*, (k/k')\,p_1^*,\ldots, (k/k')\,p_L^*)$ which is proportional to $\tilde{P}^*$.

Hence, the economic state $[(\xi_i(\Pi^*, R_i^*)), \eta_j(p^*), (y_g^*), q^*, (x_l(p^*, q^*))]$ is an equilibrium and therefore (with $A1$) a Pareto optimum (cf. Debreu [6]).

Since there is one degree of freedom in choosing the norms of the production price vector and consumption price vector, we can select k, k' such that:

$$\pi_l^* = p_l^*, \qquad l = 1,\ldots, L$$

The only taxed commodity is then commodity 0 and from (2.22)

$$t_0^* = \frac{\partial C}{\partial q}(q^*, p^*) - p_0^* \tag{2.23}$$

Q.E.D.

The theorem calls for several remarks:

(a) Beyond the use of different techniques, the same ideas underly the proof of theorem 1 and the more intuitive analysis of the preceding subsection: Assumption 6 together with $\partial \mathscr{P}_0/\partial t_0 \neq -1$ assures that through the use of lump sum transfers consumers' demands can be induced to move in any direction. Assumption 5 plays the same role for feasible directions of supply changes for all goods but good 0. Finally the assumption $y_0 \in A_0$ is here replaced by the existence of $\mathscr{P}_0$, single valued and continuously differentiable.

(b) The assumption $\partial \mathscr{P}_0/\partial t_0\,(p^*, T^*, R^*) \neq -1$ means that locally the Government has the power to influence the consumption price π_0 ; if $\partial \mathscr{P}_0/\partial t_0$ $(p^*, T^*, R^*) = -1$ any move of t_0 is counteracted by an opposite move of p_0.

This assumption is sufficient to exclude some types of behavior which would here result in inefficient allocations whatever the economic policy, *even in the regular case*. Let us construct an example of such a behavior. Consider a model in which the price maker desires to achieve an a priori given level $\bar{q}$ of commodity zero, whatever the environment. Suppose that there exists $\pi_0(p, t, R)$ such that for any (p, t, R):

$$\bar{q} = \xi_0(\pi_0(p, t, R), p, t, R)$$

The price making behavior is characterized by:

$$p_0 = \pi_0(p, t, R) - t_0$$

Here $\partial \mathscr{P}_0/\partial t_0 = -1$; such a case is excluded by our assumption (which is clearly not a necessary condition).

Observe finally that in the case of the monopolist mentioned above, this assumption reduces to $\partial \xi_0/\partial \pi_0 \neq 0$. It is always satisfied because of the monopolist's behavior who maximizes profits.

(c) The first part of the analysis rests on the description of feasible states of the economy associated with (2.1)-(2.5). Program (I) does not make any assumption on the reaction function $\mathscr{P}_0$ beyond the assumption that *the demand for commodity* 0 *is satisfied by the price maker*. This is the case if the price maker communicates his reaction function, whatever it may be, to a "referee" and in counterpart is constrained to satisfy the demand he is faced with. It is also the case if the state which finally occurs fulfills the price maker's expectation, i.e., if the reaction function is "true" at the optimum,

$$p_0^* \in \mathscr{P}_0(p^*, T^*, R^*),\ q^* \in Q_0(p^*, T^*, R^*)$$

In general, it is so if the reaction function is derived from the knowledge of the true demand function (see also the concluding comments).

II.4. *More on the Monopolist Case.*

We will now focus the attention on the case just mentioned of a single monopolist who knows the true demand function.

THEOREM 2. *Let us consider the same optimum and the same assumptions as in theorem* 1. *Then, the appropriate* rate of subsidy *on the monopolized commodity which restores Pareto efficiency, equals the inverse of its demand elasticity.*

Proof. The first order condition associated with the monopoly problem is (see Section I)[3] :

$$\xi_0(\Pi^*, R^*) + p_0^* \frac{\partial \xi_0}{\partial p_0}(\Pi^*, R^*) - \frac{\partial C}{\partial q}(q^*, p^*) \frac{\partial \xi_0}{\partial p_0}(\Pi^*, R^*) = 0 \qquad (2.24)$$

Since $\partial \xi_0 / \partial p_0 = 0$ is excluded we have:

$$\left[p_0^* - \frac{\partial C}{\partial q}(q^*, p^*)\right] = -\xi_0^* \left(\frac{\partial \xi_0}{\partial p_0}\right)^{*-1} \qquad (2.25)$$

remembering that the price elasticity of the demand for commodity zero is

$$e_0 = -\frac{p_0}{\xi_0} \frac{\partial \xi_0}{\partial p_0}$$

and using (2.23), we finally obtain: $t_0^* = (e_0^*)^{-1} p_0^*$. Hence the result if we define the rate of tax as (t_0^*/p_0^*). Q.E.D.

If the good is normal the tax is negative; it is a subsidy. If the good is not normal, the tax is positive. However, in that case it might be in the interest of the monopolist to quote a high price and satisfy only part of the demand. To stick to our model he should then be constrained to fulfill all the demand.

COROLLARY 1. *Under the assumptions of theorem* 1, *if taxing the good sold by the price maker is not allowed, appropriate taxes on the other goods restore Pareto efficiency. In the case of the monopolist, these taxes are given by*:

$$t^* = \frac{(e_0^*)^{-1}}{1 - (e_0^*)^{-1}} p^*.$$

---

[3] The reaction function of a monopolist around a point where it is univalent is locally defined by the first order condition (2.24). Moreover, in a neighborhood of such a point, profit is strictly positive; $\Psi = 0$ at a positive level of production is excluded by the differentiability assumption since we have in that case at least two optimal policies, another one corresponding to a zero production level.

Proof. The proof goes through to the last paragraph as in Theorem 1, where a different normalization of prices is chosen, namely $\pi_0 = p_0$

From (2.20), (2.16)

$$t_l^* = k'' p_l^* \qquad l = 1, \dots, L$$

with

$$k'' = (k - k')/k'.$$

From (2.20), (2.16), (2.8) we have:

$$(k'' + 1) \sum_{l=1}^{L} p_l^* \frac{\partial x_l}{\partial q} = p_0^*.$$

Using (2.25) we obtain:

$$(k'' + 1)\left[p_0^* + \xi_0^* \left(\frac{\partial \xi_0}{\partial p_0}\right)^{*-1}\right] = p_0^*$$

or

$$k'' = \frac{-\xi_0^* \left(\frac{\partial \xi_0}{\partial p_0}\right)^{*-1}}{p_0^* + \xi_0^* \left(\frac{\partial \xi_0}{\partial p_0}\right)^{*-1}} = \frac{(e_0^*)^{-1}}{1 - (e_0^*)^{-1}} \qquad \text{Q.E.D.}^{4}$$

Theorems 1, 2 generalize immediately to the case of several price makers, each one choosing the prices of his own subset of monopolized goods. However, corollary 1 clearly does not generalize.

In the case of a price maker producing for intermediary as well as final demand, it is left to the reader to verify that if the monopolized commodity is sold at consumer prices to competitive firms, the same results as above obtain, where ξ_0 is now the demand for good 0 by consumers and competitive firms. However, if on the contrary it is sold at producer prices, Pareto efficiency cannot be usually restored by a tax-system.

Coming back to the case of the monopolist producing for final demand, it is clear that if there is a single good (such as leisure) which cannot be taxed, the same results as in Theorems 1, 2 still obtain since we have one degree of freedom in the normalization of prices. If at the same time, a good, say good 1, cannot be taxed $t_1 = 0$ and the monopolized good cannot be subsidized $t_0 \geq 0$, Pareto optimality cannot be restored in general.

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[4] Since $\partial C/\partial q > 0$, $p_0 + \xi_0(\partial \xi_0/\partial p_0)^{-1} \geq 0$ from (2.24) hence $1 - 1/e_0 \geq 0$, i.e. at the optimal production level, the absolute value of the elasticity of demand is always larger or equal to one. Therefore $k'' > 0$ (< 0) when the good is normal (not normal), paralleling the comment made on theorem 2.

We already discussed the fact that our formulation of the price making behavior was consistent with the existence of several firms deciding jointly on the price to be quoted. However, Theorem 1 assumes that there is only a single price maker producing efficiently: the reader will notice that this assumption is used for obtaining formula (2.22). When there are several price makers two cases have to be distinguished:

— In the first case, the supply of the group of price makers is efficient (relatively to the total production set of this group): This is the case for example when the price maker sector consists of identical firms—i.e. firms with identical production sets—forming an oligopoly "à la Cournot" and when only the symmetrical Nash equilibria—i.e. those corresponding to equal shares of the total production for all firms—are considered (see Ruffin [19]). In this case, Theorem 1 remains true and Pareto optimality can be restored; the set of price makers can be identified with a single fictitious price maker whose production set is the sum of the individual production sets and the argument of Theorem 1 applies.

— In the second case, the supply of the group is not efficient; since policy tools do not discriminate among oligopolists, first best Pareto optimality cannot in general be restored. In addition, the Kuhn-Tucker multipliers of the associated program which are the shadow prices to be given to public firms for the production of commodities 1,..., L, are neither proportional to production prices nor proportional to consumer prices.

II.5. *What Goes Wrong in the Non Regular Case*

Consider first the case of a monopolist. Let $\zeta^* = (p_0^*, p^*, T^*, R^*)$ be a maximum of program (I) where $\mathscr{P}_0$ is not single valued. This means that at $e^* = (p^*, T^*, R^*)$, the maximization of profit by the monopolist yields several maxima among which p_0^* (say 2 and let us call them $p_0^{1*} = p_0^*$ and p_0^{2*}). We also suppose that in a neighborhood of e^*, there are around p_0^{*1}, p_0^{2*}, two continuously differentiable branches denoted $\mathscr{P}_0^1$ and $\mathscr{P}_0^2$ on which the necessary conditions for maxima of the monopolist's profit are satisfied.

ζ^* is necessarily an optimum of the following program:

$$\text{(II)} \qquad \max \sum_{i \in I} \alpha_i V_i(\Pi, R_i)$$

$$\eta_l(p) + y_{gl} - \xi_l(\Pi, R) - x_l(p, q) \geqslant 0 \qquad l = 1, \dots, L \tag{2.26}$$

$$q - \xi_0(p_0^1 + t_0, \Pi, R) = 0 \tag{2.27}$$

$$p_0^1 = \mathscr{P}_0^1(p, T, R) \tag{2.28}$$

$$p_0^2 = \mathscr{P}_0^2(p, T, R) \tag{2.29}$$

$$\Psi(p_0^1, p, T, R) \geqslant \Psi(p_0^2, p, T, R) \tag{2.30}$$

$$f(y_g) \geqslant 0 \tag{2.31}$$

$$p \geqslant 0, \pi \geqslant 0 \tag{2.32}$$

Let (ρ_l), λ_0, λ_1, λ_2, τ, μ be the multipliers associated with (2.26) to (2.31) respectively.

All the functions written in this program are now supposed to be differentiable, and the standard Kuhn and Tucker approach can be used.

Comparing with subsection II.3, one sees that the presence of the (generally) non zero multiplier τ associated with the constraint (2.30) prevents λ_1, the multiplier associated with the constraint (2.28) to be zero. This expresses here the fact already noticed that the production of the monopolist is not locally controllable. Consequently none of the results mentioned in subsection II.3 obtain. In particular, it must be emphasized that the shadow price to be given to *public firms* are neither *the production prices* (hence production efficiency no longer necessarily holds) nor the *consumption prices*.

In view of Guesnerie [12], where deviant price takers—whose supply functions depend only on production prices—are studied, the fact that production efficiency does not hold here is not surprising. Monopolists when they are not fully controllable appear as generalized deviant agents. Production prices have two functions, allocation of production and control of deviants. Consequently it may be desirable in order to perform better the second function, to have production prices different from the shadow prices given to public firms.

However, in Guesnerie [12], it remains desirable, even in genuine second best situations, to have the shadow prices given to public firms equal to the *consumption prices*. Intuitively, this difference can be seen as follows. Here, the "imperfect" controllability of the price maker can be improved by eventually introducing distortions within the consumers-public production sector, since, through the demand function the objective function of the price-maker depends on incomes and consumption prices. On the contrary, in Guesnerie [12] nothing can be gained from such distortions since the objective function of the price-taker depends only on the production prices which can be controlled independently of the instruments used to control the consumers-public production sector.

Let us finally notice that the reasoning presented here from program (II) applies to any price maker which has a differentiable objective function.

III. TAXING PRICE MAKERS WHEN LUMP SUM TRANSFERS ARE AVAILABLE: THE DECENTRALIZATION POINT OF VIEW

III.1. *Decentralization of First Best Optima in an Economy with Monopoly: The Problem*

A superficial view of the preceding results could lead one to believe that in a given economy, the presence of monopolies does not prevent the attainement of almost all first best optima. Such an assertion could be defended on the following grounds: the situations where the monopoly's optimal decisions are multivalued—the fact that decisions are single valued and continuous is the main requirement for proving theorem 1, differentiability being only a technical matter—are in some sense negligible; for most demand functions ξ_0, case b of section I occurs only for values of (p, T, R) which are of Lebesgue measure zero. Hence, theorem 1 would apply almost always. Such a conclusion which is more or less implicitly supported in the literature is not correct. The first part of the argument is right but it does not imply the second part.

In order to clarify this important point, we shall take a complementary point of view, the decentralization point of view. Considering in a given economy a given first best Pareto optimal state, we ask: is it possible, with only linear taxes, to decentralize this Pareto optimal state when commodity 0 (which is a final commodity) is produced by a monopoly?

Let ζ be a first best Pareto optimal state of the economy defined in Section II with a strictly positive production of commodity zero, and let $(x_i(\zeta))$, $y_g(\zeta)$, $(y_j(\zeta))$, $y_0(\zeta)$ be respectively the consumption vectors of consumer i, $i = 1, \dots, I$, and the production vectors of the Government sector, the competitive sector and the firm producing commodity 0. Let us assume that this Pareto optimum can be decentralized with a unique price system $P(\zeta) = (p_0(\zeta), p(\zeta))$, and consequently with a unique vector of incomes $R_1(\zeta), \dots, R_I(\zeta)$, such that $x_i(\zeta) = \xi_i(P(\zeta), R_i(\zeta))$, $y_g(\zeta) = \eta_g(p(\zeta))$, $y_j(\zeta) = \eta_j(p(\zeta))$, $y_0(\zeta) = \eta_0(P(\zeta))$.[5]

If all firms, except the firm producing commodity zero, are competitive, is it possible to decentralize this Pareto optimum through an appropriate taxation of the monopolized commodity?

Let us consider the profit function defined in Section I:

$$\Phi(q, p, t, t_0, R) = [\pi_0(q, \pi, R) - t_0]\, q - C(q, p)$$

Since Pareto optimality requires the proportionality of consumer and producer prices π_l, p_l for $l = 1, \dots, L$, it is enough to consider the taxation programs which tax only commodity 0. Clearly, a necessary and sufficient condition for ζ to be decentralized, despite the presence of the monopoly is that there exists t_0^* such that $\Phi(q, p(\zeta), 0, t_0^*, R(\zeta))$ reaches a positive maximum at $q = q_0(\zeta)$. If we call $Q_0(\zeta, t_0)$ the profit maximizing production levels of the monopoly when the tax is t_0 and with demand and price conditions associated with ζ, ζ is decentralizable when $q_0(\zeta)$ belongs to the range of the correspondence $Q_0(\zeta, t_0)$.[6]

The possible non decentralization of $q_0(\zeta)$ seems intuitively to be linked to the non concavity, with the possibility of jumping from one level of production to another level when t_0 varies, never choosing intermediary values (see III.2). From this point of view, the possibility of being unable to decentralize a given Pareto optimum does not appear any longer to be pathological (see III.3).

---

[5] As η_j denotes the competitive supply function of competitive firm j, η_g, η_0 respectively denote the competitive supply functions associated with Y_g and Y_0, the production sets of the public sector and of the monopoly.

[6] Obviously $q_0(\zeta) \in Q_0(\zeta, t_0)$ is impossible for $t_0 \leq -p_0(\zeta)$.

III.2. *Sufficient Conditions for Decentralization*

In the next theorem, we give conditions on the characteristics of the given Pareto optimum which make it decentralizable.

THEOREM 3. *Let ζ be a Pareto optimum for which the production $q^* = q_0(\zeta)$ of good 0 is positive and such that:*

a) *The profit function $\Phi(q, \zeta, t_0)$ exists, is strictly concave and differentiable with respect to $q \in]0, \infty[$*[7],

b) $(\partial C/\partial q)(q^*, p(\zeta))\, q^* - C(q^*, p(\zeta)) - (\partial \pi_0/\partial q)(q^*, \pi(\zeta), R(\zeta))\, q^{*2} \geq 0$. *Then ζ is decentralizable.*

When there are several decentralization prices, this result must hold for one production correspondence constructed from one decentralization price.

Proof. Given a) a necessary and sufficient condition for q^* to be chosen by the monopolist is:

$$\Phi(q^*, \zeta, t_0) \geq 0$$

$$\frac{\partial \Phi}{\partial q}(q^*, \zeta, t_0) = 0$$

But $(\partial \Phi/\partial q)(q^*, \zeta, t_0) = (\partial \pi_0/\partial q)(q^*, \pi, R)\, q^* + \pi_0(q^*, \pi, R) - t_0 - (\partial C/\partial q)(q^*, p) = 0$ where π, p, R are $\pi(\zeta), p(\zeta), R(\zeta)$.

Let us choose:

$$t_0^* = \frac{\partial \pi_0}{\partial q}(q^*, \pi, R)\, q^* + \pi_0(q^*, \pi, R) - \frac{\partial C}{\partial q}(q^*, p)$$

This t_0^* is admissible if moreover $\Phi(q^*, \zeta, t_0^*) \geq 0$, i.e. $[\pi_0(q^*, \pi, R) - t_0^*]\, q^* - C(q^*, p) \geq 0$ or

$$\frac{\partial C}{\partial q}(q^*, p)\, q^* - C(q^*, p) - \frac{\partial \pi_0}{\partial q}(q^*, \pi, R)\, q^{*2} \geq 0^{8} \qquad \text{Q.E.D.}$$

---

[7] The profit function is written here as a function of q, t_0 and ζ which summarizes the other characteristics of the given Pareto optimum. Its existence relies on the invertibility of $\pi_0(\cdot)$. Observe also that a profit function strictly concave in q for a value of t_0 is strictly concave for any t_0.

[8] The same result obtains more generally if $\partial Q_0/\partial t_0 \neq 0$ since

$$q - \bar{q} = \frac{\partial Q_0}{\partial p}(p - \bar{p}) + \frac{\partial Q_0}{\partial t}(t - \bar{t}) + \frac{\partial Q_0}{\partial R}(R - \bar{R}) + \frac{\partial Q_0}{\partial t_0}(t_0 - \bar{t}_0).$$

Remark. The condition b) is particularly simple if the production set of the monopolist is convex. Then,

$$\frac{\partial C}{\partial q}(q^*, p)\, q^* - C(q^*, p) \geq 0,$$

so that a *sufficient* condition for decentralization is (besides a):

$$\frac{\partial \pi_0}{\partial q}(q^*, \pi, R) \leq 0$$

i.e. that the monopolized commodity be a *normal* good at the given Pareto optimum.

III.3. *Examples of Non Decentralizable Pareto Optima*

(1) *An example of a unique Pareto optimum which is not decentralizable.* Let us consider a positive number q^* and a function $\beta_0(q)$ (which is to be interpreted as the inverse demand function) with the following property:
For

$$q_1 = 3q^*, \ \beta_0(q_1) = \tfrac{1}{2}\beta_0(q^*)$$
$$q_2 = \tfrac{1}{10}q^*, \ \beta_0(q_2) = 8\beta_0(q^*)$$

Observe that:

$$[\tfrac{1}{2}\beta_0(q^*) - t_0]\, 3q^* - 3q^* > [\beta_0(q^*) - t_0]\, q^* - q^*$$

is equivalent to:

$$\beta_0(q^*) > 4t_0 + 4$$

$$[8\beta_0(q^*) - t_0]\frac{q^*}{10} - \frac{q^*}{10} > [\beta_0(q^*) - t_0]\, q^* - q^*$$

is equivalent to:

$$\beta_0(q^*) < 4.5t_0 + 4.5$$

In view of these inequalities, whatever t_0, a monopoly faced with such an inverse demand function $\beta_0(q)$ and having a linear cost function, will prefer either q_1 or q_2 (or both) to q^*. Consequently, he will never choose q^*.

Now, let us consider the following simple economy: there are two commodities labeled 0 and 1; 0 is a consumption good, 1 is labor. There is one firm producing commodity 0 from commodity 1, with constant returns to scale (the transformation curve is $y_0 + y_1 = 0$).

There is a single consumer with preferences such that the only optimum of the economy corresponds to the production plan $y_0^* = 1$, $y_1^* = -1$; the demand for commodity 0 when the consumer is faced with the price system

[.5, 1] is 3 and the demand for commodity 0 when the consumer is faced with the price system [8, 1] is $\frac{1}{10}$.

(In Fig. 2, a family of indifference curves meeting these conditions is constructed).

In such an economy, if the firm is a monopoly, there exists no tax scheme which enables the decentralization of the unique optimum, since from the discussion above the producer will never want to produce the plan $\begin{bmatrix}1\\-1\end{bmatrix}$.

The indifference curves are constructed so that:

- The slopes of the right hand side of the indifference curves passing through A, B, C are identical and take a value between $-\frac{1}{10}$ and 0;

- the left hand sides of the indifference curves passing through B, A, C haves lopes which are respectively between $-\frac{1}{10}$ and -1, -1 and -2, -2 and $-\infty$.

These indifference curves can be appropriately completed to represent a continuous quasi-concave function. Also, a differentiable quasi-concave utility function could easily be constructed to give A, B, C as tangency points.

(2) *Example where no Pareto optimum is decentralizable.* Consider an economy with I consumers, two commodities (commodity 0 and commodity 1) and one firm. All consumers have the same utility function:

$$\begin{aligned} u(x_0, x_1) &= x_0 x_1 \text{ if } x_0 \geq \frac{x_1}{2} \\ &= x_0 \quad \text{if } x_0 < \frac{x_1}{2} \end{aligned}$$

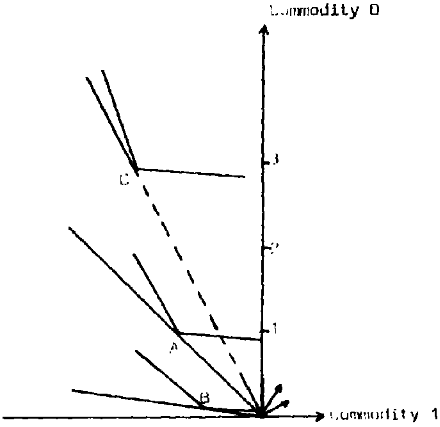

FIGURE 2

The endowment of the economy in commodity 1 is 1 and the technology is such that $y_0 + y_1 = 0$.

Let $(\pi, 1)$ be the normalized consumption price vector. Good 0 is sold by a monopolist. (π identifies here with π_0 of general notation.)

In this economy, all Pareto optima are associated with a production of commodity 0 at the level $\frac{1}{2}$. The decentralization price vector is $(1, 1)$ so that total associated income is always 1.

Let $(R_1, \ldots, R_l)$ be the vector of income in the economy. The demand functions for commodity 0 are easily derived as:

$$\begin{aligned} \xi_{0i}(\pi, 1) &= \frac{R_i}{2\pi} \quad \text{if } \pi \leqslant 2 \\ &= \frac{R_i}{2 + \pi} \quad \text{if } \pi \geqslant 2, \; i = 1, \ldots, l \end{aligned}$$

At any Pareto optimum, $\sum_{i=1}^{l} R_i = R = 1$ and the profit function of the monopolist can then be written as:

$$\begin{aligned} \Psi &= \tfrac{1}{2} - (t_0 + 1)\, q \text{ if } \pi < 2 \text{ or } q > \tfrac{1}{4} \\ &= 1 - (t_0 + 3)\, q \text{ if } \pi \geqslant 2 \text{ or } q \leqslant \tfrac{1}{4} \end{aligned}$$

From the profit function pictured in Fig. 3, it is clear that there exists no tax system such that the monopoly agrees to produce $q = \frac{1}{2}$. Hence, the result.

Note that, with a lower bound ϵ on the production set of the firm, there would exist in this economy a monopolistic equilibrium at $q^* = \epsilon$ for any tax $t_0 > -3$.

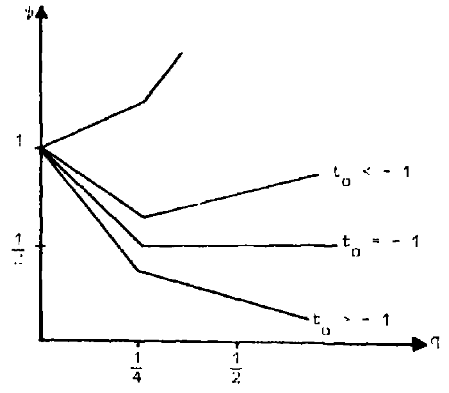

FIGURE 3

Let us comment somewhat on this example. If there is no subsidy or if the subsidy is too small, the monopolist wants to produce a minimal amount to exploit the fact that commodity zero is a necessary good. The Center then tries to have the monopolist produce more by increasing the subsidy (here without success) and at some level of subsidy the monopolist would like to produce an infinite amount.

In designing the optimal tax there is clearly a tendency to stop the increase of subsidy at a point where further increase would involve a jump in behavior. The monopolist does not produce enough; one attempts to have him produce more and stop the level of subsidy at a point at which he is ready to jump to an excessive production.

Hence, in this case, the optimal taxation always puts the monopolist at a "catastrophic" level of production (the catastrophy is not due to special values of exogenous variables, but is the result of the optimal choice of the control variables). The study of the generality of this result would certainly deserve further work.

(3) *First conclusions in the case where lump sum transfers are available.* (i) In the "good" regular case, first best Pareto optimality can be restored with linear taxes. In the "bad" case (not regular), maximization of a social welfare function leads to a genuine second best. Furthermore, the production of the price maker in such a state is highly unstable with respect to small changes of the environment or of the control variables. No simple results concerning optimal taxation and optimal public production can be exhibited.

(ii) When the profit function of the monopolist is not concave, the "bad" case cannot be viewed as pathological. It will be often the case that by varying the weights of the social welfare function, bad case and good case will successively appear i.e. from the decentralization point of view that a subset of first best Pareto optima will not be reachable.

These conclusions can be illustrated through the following diagram in a 2 consumer, 3 good economy (Fig. 4)

· In Fig. 4, we represent in the space of utilities on the one hand the attainable levels of utilities in the physical model, the upper frontier of which is the solid curve *OO'* (corresponding then to first best Pareto optima); on the other hand, the attainable levels of utilities when linear taxes can be used in an economy with a price maker, the upper frontier of which is the dotted curve *ABCDEFO'*.

The solid curve and the dotted curve coincide on *CD* and *FO'*: the corresponding first best optima are decentralizable in the model with a price maker. The solid curve and the dotted curve differ on *ABC*, *DEF*. Social welfare functions leading to such optima in the utility space give second best situations.

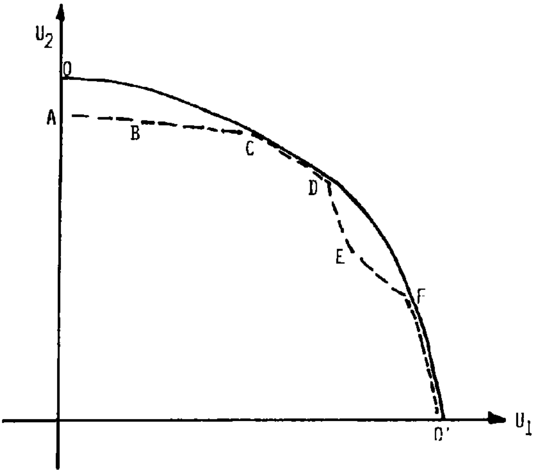

FIGURE 4

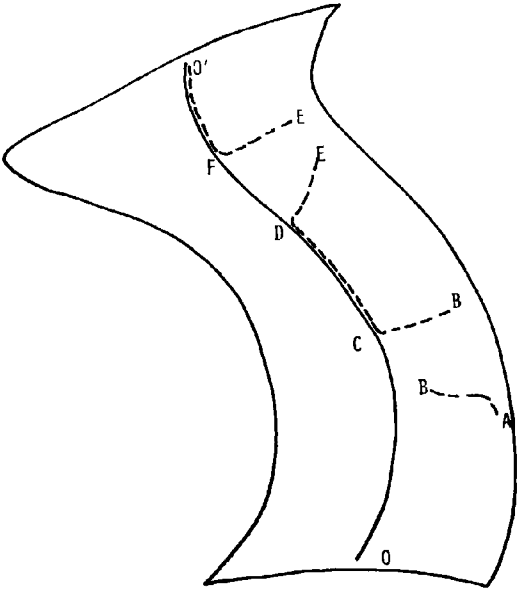

FIGURE 5

• Figure 5 visualizes on the production frontier of the monopolist the path of the monopolist's optimal production plans associated with the optima *ABCDEFO'* of Fig. 4.

The solid line OO' associated with OO' in Fig. 4 represents production plans associated with first best allocations: under standard assumptions it is a continuous line.

On the contrary the dotted line $ABCDEFO'$ corresponding to the monopolist's optimal production plans in the model with linear taxes is discontinuous: on the figure it jumps at B and E which correspond either to discontinuities in the control variables or in the responses of the monopolist.

This dotted line coïncides with OO' on CD and FO' where the monopoly response is stable, while it is unstable on ABC, DEF.

Let us finally notice that the topological properties of the line $ABCDEFO'$ suggested in Fig. 5 have to be considered as illustrative. A more precise study of discontinuity points would be needed.

IV. TAXING PRICE MAKERS: THE INTERACTION OF ALLOCATIONAL AND DISTRIBUTIONAL ISSUES

The developments of the preceding sections rested on the assumption that lump sum transfers were feasible. This assumption, which allowed us to put aside distributional issues and to concentrate on the corrective role of taxation in the allocation of resources, is now given up.

IV.1. *A "Diamond-Mirrlees" Economy*

We first consider an economy "à la Diamond-Mirrlees" [8] in which there is a price maker producing for final demand.

The feasible states of the economy are defined by the following system of inequalities:

$$\eta_l(p) + y_{0l} - \xi_l(\Pi, \mathbf{0}) - x_l(p, q) \geqslant 0 \qquad l = 1,\dots, L \tag{4.1}$$

$$q - \xi_0(\Pi, \mathbf{0}) \geqslant 0 \tag{4.2}$$

$$p_0 \in \mathscr{P}_0(p, T, \mathbf{0}) \tag{4.3}$$

$$f(y_0) \geqslant 0 \tag{4.4}$$

$$p \geqslant 0, \pi \geqslant 0 \tag{4.5}$$

where $\mathbf{0}$ is the vector $(0,\dots, 0)$ of non labor incomes.

A system of indirect linear taxes enables the Government to disconnect the production and consumption price systems. Profits of competitive firms and of the price maker are 100% taxed so that consumers have no other income than their labor income, i.e., $R_i = 0$, $i = 1,\dots, I$. This latter hypothesis rules out part of the distributional problems created by the existence of a monopoly which are related with the distribution of its profits to shareholders. Therefore, the model of this section must be considered a step in the process of understanding the more realistic but more complicated model of subsections IV.2, IV.3 in which profits are distributed.

Let us consider the two following maximization programs:

$$\max \sum_i \alpha_i V_i(\Pi, 0) \tag{P}$$

under (4.1) to (4.5)

$$\max \sum_i \alpha_i V_i(\Pi, 0) \tag{P'}$$

under (4.1), (4.4), (4.5) and

$$\xi_0(\Pi, \mathbf{0}) \leq \eta_0(P) \tag{4.6}$$

where $\eta_0(P)$ is the competitive supply of commodity 0 when the production price system is P, i.e., the supply of a competitive firm which would have the same production possibilities as the price maker.

(P) is the program to consider in an economy "à la Diamond-Mirrlees" when commodity 0 is produced by a price maker and (P′) is the analogous program when the supply of commodity 0 is competitive.

A partial characterization of the maxima of program (P) is given by:

THEOREM 5. *Let* (P^*, T^*, y_0^*) *be an optimal solution of program* (P) *with* $P^* \gg 0$. *Under* $A1$ *to* $A4$, *and under regularity conditions on demand (of the Diamond-Mirrlees type)*[9], *if in a neighborhood of that optimal solution the correspondence* $\mathscr{P}_0$ *reduces to a single valued and continuously differentiable function such that* $(\partial \mathscr{P}_0/\partial t_0)(p^*, T^*, R^*) \neq -1$, *and if* $A5$ *is satisfied at the optimum, then*:

(a) *The shadow prices associated with the optimal production plan of the public firm coincide with the production prices.*

(b) *The optimum of program* (P) *satisfies the Kuhn and Tucker conditions of program* (P′).

Proof. It is easy to see that the necessary conditions associated with program (P) are formally identical to conditions (2.6) to (2.11) of Section II. $\mu^* > 0$ is here assured by the Diamond-Mirrlees demand conditions. It follows that the dual variable λ^* associated with constraint (4.3) equals zero. The argument leading to formula (2.16) can be repeated. Hence:

$$\rho_l^* = kp_l^*, \qquad l = 1, \ldots, L$$

---

[9] Such a condition requires that there is one consumption prices move which would be agreed upon by all consumers: it is fulfilled if for example each consumer has a positive net demand for one good, and it assures here that efficiency in the production sector alone is desirable. See Weymark [27].

From (2.8), we have:

$$p_0^* = k\tilde{p}_0$$

where

$$\tilde{p}_0 = \sum_{l=1}^{L} p_l^* \frac{\partial x_l}{\partial q}$$

is the marginal cost of producing commodity zero.

Let $(p_0', \ldots, p_L')$ be a vector of dual variables associated with an optimal solution of program (P'). It is immediately seen that the optimal solution of program (P) satisfies the Kuhn and Tucker conditions of (P') by choosing $p_l' = p_l^*$, $l = 0, \ldots, L$, as defined above. Q.E.D.

Theorem 5 states that at an optimum where the response function of the price maker is differentiable, production must be efficient (a) and the optimum is a local maximum "à la Diamond-Mirrlees" (b).

The optimal tax on commodity 0, $t_0^* = \pi_0^* - p_0^*$, equals now the sum of a "corrective" tax $\tilde{p}_0 - p_0^*$ and a distribution tax $(\pi_0^* - \tilde{p}_0)$ "à la Diamond-Mirrlees" on which nothing general and independent of the normalization rule can be said.

If the differentiability assumption on $\mathscr{P}_0$ does not hold at the optimum of the program, the necessary conditions must be derived along the lines investigated in subsection II.5. The reader can easily see that, generally in this case, production efficiency is no longer desirable, the response of the price maker is unstable with respect to the value of policy tools and the tax formula are no more of the same form as the tax formula "à la Diamond-Mirrlees" obtained in theorem 5.

There are no grounds for asserting that an optimum of program (P) in theorem 5 is a *global* optimum of program (P'). It was only proved that it satisfies the Kuhn and Tucker conditions of program (P'). Contrary to what happened in section II, necessary conditions are not sufficient for obtaining a global optimum.

Given a *second best* Pareto optimum solution of program (P') one may ask, as in section III, if it can be decentralized when firm 0 is a monopoly. A sufficient condition for decentralization similar to the one derived in theorem 3, is easily obtained. However, the similarity with sections II, III should make clearly non pathological the fact that some second best Pareto optima corresponding to the Diamond-Mirrlees assumptions are not decentralizable when commodity zero is produced by a monopoly.

The reader will find useful to illustrate these conclusions with diagrams similar to the ones presented in section III.

IV.2. *Distribution of Profits with a Complete System of Taxes*

The next step is to give up the assumption that profits of the price maker are taxed 100%. We consider now the case where the profit of the monopoly, B, is distributed to consumers according to fixed shares $\theta_i(\sum_{i=1}^{I} \theta_i = 1)$ and where consumption prices and production prices can still be disconnected with a system of linear indirect taxes.

A feasible state of the economy is then defined by a vector (Π, P, R, B, q, y_o) which satisfies

$$\eta_l(p) + y_{ol} - \xi_l(\Pi, R) - x_l(p, q) = 0 \qquad l = 1,\dots, L \tag{4.7}$$

$$\xi_0(\Pi, R) - q = 0 \tag{4.8}$$

$$p_0 \in \mathscr{P}_0(p, T, R) \tag{4.9}$$

$$f(y_o) \geqslant 0 \tag{4.10}$$

$$B - p_0\xi_0(\Pi, R) + C(q, p) = 0 \tag{4.11}$$

$$R_i = \theta_i B \qquad i = 1,\dots, I \tag{4.12}$$

$$B > 0, p_0 \geqslant 0, p \geqslant 0 \tag{4.13}$$

Consider a feasible state of (4.7) to (4.13); the price quoted by the monopoly satisfies (see subsection II.4)

$$1 = \frac{1}{e_0} + \frac{\partial C/\partial q}{p_0} \tag{4.14}$$

where e_0 is the price elasticity of the demand for commodity zero and the profit of the monopoly is $B = p_0\xi_0(\Pi, R) - C(q, p)$. To the same feasible state we can associate a different normalization of prices, for example, multiply p by λ. From (4.14) we know that p_0 is then multiplied also by λ. The associated profit of the firm is now $B' = \lambda B$ which can be made as desired by choosing an appropriate λ.

Consequently, the constraint (4.11) can be ignored, B is now a decision variable and (4.12) is equivalent to having positive lump sum transfers in fixed proportions. In the differentiable case, the first order conditions of the above maximization problem are closely similar to those obtained by Diamond [7] in his model with a poll tax. In particular, production efficiency holds by the monotonicity of preferences and a tax formula of the many person Ramsey type is obtained.

However, this first approach to the distributional questions raised by the distribution of the monopolist's profits is quite unsatisfactory; it leads to a crucial role being played by the highly unrealistic assumption of an independent choice of normalization for consumer and producer prices.

The case $B = 0$ has been excluded in the above analysis because it is a non differentiable case: if we consider a non zero production level with such a profit, we know that there exists another maximum of the profit function for the zero production level.

This brief analysis shows that if one wants to capture the purely redistributive problems raised by the existence of monopolies, the complete disconnection of production and consumption prices must be given up.

IV.3. *Distribution of Profits without a Complete Disconnection of Consumption and Production Prices*

In the above subsection although profits were not explicitly taxable, they could be controlled through an appropriate normalization of production prices. To rule out this phenomenon, we introduce a new rigidity by assuming that commodity one is not taxed, i.e., $t_1 = 0$.

(a) *Profits are taxable.* Let ϵ be the rate of profit taxation, $\epsilon \in [0, 1]$. The feasible states of the model are now defined by

$$\eta_l(p) + y_{gl} - \xi_l(\Pi, R) - x_l(p, q) \geq 0 \qquad l = 1,\ldots, L \tag{4.15}$$

$$q - \xi_0(\Pi, R) = 0 \tag{4.16}$$

$$p_0 \in \mathscr{P}_0(p, T, R) \tag{4.17}$$

$$f(y_g) \geq 0 \tag{4.18}$$

$$B = \Psi(p_0, p, T, R) \tag{4.19}$$

$$R_i = \theta_i \epsilon B, \qquad i = 1,\ldots, I \tag{4.20}$$

$$t_1 = 0 \tag{4.21}$$

$$p \geq 0, \pi \geq 0, B \geq 0 \tag{4.22}$$

Let us consider a second best Pareto optimum for which $\mathscr{P}_0$ is differentiable, $\epsilon \in]0, 1[$ and let β, ν_i, $i = 1,\ldots, I$ be the Kuhn and Tucker multipliers associated with (4.19), (4.20).

The Kuhn and Tucker conditions associated with B, ϵ, are

$$\beta - \sum_i \nu_i \theta_i \epsilon = 0 \tag{4.23}$$

$$-\sum_i \nu_i \theta_i B = 0 \tag{4.24}$$

which imply $\beta = 0$.

The other Kuhn and Tucker conditions are then similar to those obtained in the previous section. As in Section II one obtains $\lambda = 0$, i.e., the only effective constraints in addition to (4.15), (4.16), (4.18), (4.22) are (4.20), (4.21).

To sum up we have:

Theorem 6. *Let $\zeta^* = (P^*, T^*, y_g^*)$ be a second best Pareto optimum solution under the constraints* (4.15) *to* (4.22). *Under A1-A4 and the differen-*

tiability and regularity conditions analogous to those of Theorem 5, and if in addition $\epsilon^* \in]0, 1[$, ζ^* *satisfies the Kuhn and Tucker conditions associated with the constraints* (4.15), (4.16), (4.18), (4.20), (4.21), (4.22).

In particular the necessary Kuhn and Tucker conditions are the same as those obtained in a program where the Government can fix independently the production level of the monopolist and the level of the monopolist's profit while still being constrained by the cost minimization behavior of the monopolist. Because of (4.21) production efficiency is not generally desirable.

When the differentiability assumption is no longer satisfied, the above results do not hold anymore. In addition, we obtain then the usual "instability" of the monopolist's behavior.

(b) *Monopoly profits are not taxable.* In this case, the feasible states are defined by (4.15) to (4.22) with $\epsilon = 1$ in (4.20).

Even in the differentiable case, one does not get $\beta = 0$ and $\lambda = 0$. The two objectives of the Government achieved in (a), namely the "control" of the production level of the monopoly and of the level of profits of the monopoly are no longer jointly attainable; they become conflicting objectives. In the non differentiable case we obtain in addition the instability mentioned above.

CONCLUDING COMMENTS

The theory of optimal taxation of a price maker developed in this paper has made a number of very strong informational assumptions concerning the Government and the economic agents that we can now discuss.

Our direction is to assume that the Government and the price makers know the *true* demand functions. This assumption of perfect knowledge of demand conditions has been found unrealistic in the literature because of the complexities of the general equilibrium computations which are implicit in it. However this well taken criticism has led through the use of perceived demand function to an unsound reasoning in taxation theory.

Let $\hat{\xi}_0(\Pi, R)$ be the demand function perceived by the price maker.[10] This function which must be true at the equilibrium is therefore indexed by $e^* = (\Pi^*, R^*)$ (equilibrium values which are assumed to be unique), and is denoted $\hat{\xi}_0^{e^*}(\Pi, R)$. Let us consider the economy of Section IV without taxes and let us assume that there exists a unique equilibrium e^*. Then let us consider a perceived demand function $\hat{\xi}_0^{e^*}(\Pi, R)$. In order to define the optimal taxation problem, we must not only assume that the Center knows this *perceived* demand function, but also the way this function changes with the different equilibria associated with different tax schemes. If this perceived function was fixed during a long period, one could argue that by observing the behavior of the monopolist, the Government could discover this function at least locally (even then one would probably have to assume that the production possibilities of the price maker are known to the Center; otherwise there would be an identification problem). However, if this function is substantially different from the true one it is obvious that there will be some learning over time; then, it is clearly impossible to assume that the Government also knows the way the price maker learns. The informational issue is clearly at the heart of the matter and a theory of optimal taxation with perceived demand functions must face it.

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[10] Negishi [17] was the first to introduce in a different context the concept of perceived demand function with the only requirement that the function be correct at the equilibrium.

Another possibility, as a first step in the study of optimal taxation with price makers, is to assume that the perceived demand functions are close enough to the true ones to be stable. Then, one must face another difficulty, the non concavity of profit functions; it was essentially the topic of this paper which can be to some extent interpreted with such perceived demand functions instead of true ones.

The second main assumption of our approach has been to consider price makers with a Nash behavior with respect to the whole environment except the price of their own output. It is a strong informational hypothesis and to assume a more strategic behavior would have been unwise in a first step. It is clear that more realistic assumptions can be made, in particular in the case of oligopolists, and this should be the object of further work. Note however, that it is easy to see that if the monopolist takes into account the effect of his price change over the distribution of income through the change in his own profit, the results would be unchanged. The reason is that by the first order condition, the derivative of his profits with respect to his price is zero.

Finally, it is important to observe that we did not touch the question of strategic behavior of the monopolist with respect to the Government. The monopolist knowing that he will be taxed may attempt to distort his behavior in order to conceal his true characteristics and obtain a more favorable tax.

REFERENCES

1. K. ARROW, The firm in general equilibrium theory, *in* "The Corporate Economy: Growth, Competition and Innovative Potential" (R. Marris and A. Wood, Eds.), Macmillan & Co., 1971.
2. K. ARROW AND F. HAHN, "General Competitive Analysis," Holden-Day, San Francisco, 1971.
3. A. BERGSON, Optimal pricing for a public enterprise, *Quart. J. Econ.* **136** (1972), 518–542.
4. P. BOHM, On the theory of second best, *Rev. Econ. Stud.* **14** (1967), 301–314.
5. M. BOITEUX, Sur la gestion des monopoles publics astreints à l'équilibre budgétaire, *Econometrica* **24** (1956), 22–40.

6. G. Debreu, "Theory of Value," Wiley, New York, 1959.
7. P. Diamond, A Ramsey many person tax rule, *J. Public Econ.* **4** (1975), 335–342.
8. P. Diamond and J. Mirrlees, Optimal taxation and public production, *Amer. Econ. Rev.* **61** (1971), 8–27, 261–278.
9. F. Fitzroy, Monopolistic equilibrium, non convexity and inverse demand, *J. Econ. Theory* **7** (1974), 1–16.
10. H. A. Green, The social optimum in the presence of monopoly and taxation, *Rev. Econ. Stud.* **29** (1962).
11. J. Gabzewicz and J. P. Vial, Oligopoly "à la Cournot" in general equilibrium analysis, *J. Econ. Theory* **4** (1972), 381–400.
12. R. S. Guesnerie, Public production and taxation in a simple second best model, *J. Econ. Theory* **10** (1975), 127–156.
13. J. J. Laffont and G. Laroque, Existence d'un équilibre général de concurrence imparfaite: une introduction, *Econometrica* **44** (1976), 183–194.
14. D. McFadden, A simple remark on the second best Pareto optimality of market equilibria, *J. Econ. Theory* **1** (1969), 26–38.
15. J. Marschak and R. Selten, "General Equilibrium with Price Setting Firms," Springer-Verlag, Berlin/Heidelberg/New York, 1974.
16. J. Mirrlees, "The Theory of Optimal Taxation," Nuffield College, September 1977.
17. T. Negishi, Monopolistic competition and general equilibrium, *Rev. Econ. Stud.* **28** (1961), 196–201.
18. J. Roberts and H. Sonnenschein, On the foundations of the theory of monopolistic competition, *Econometrica* **45** (1977), 101–113.
19. A. Ruffin, Cournot oligopoly and competitive behaviour, *Rev. Econ. Stud.* **38** (1971), 493–502.
20. A. Sandmo and J. Drèze, Discount rate for public investments in closed and open economies, *Econometrica* **38** (1971), 395–412.
21. J. Weymark, On Pareto impairing price changes, *J. Econ. Theory*, **19** (1978), 294–320.
22. R. Lipsey and R. Lancaster, The general theory of second best, *Rev. Econ. Stud.* **24** (1956), 11–32.

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The Core of a Monetary Economy without Trust*

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The concept of trust is formalized by the "sequential core." The absence of trust induces a sequential structure on an economy, even if markets are complete at the first day. The use of money (securities) can replace trust.

There is no room for money (as a store of value) in an Arrow-Debreu economy. It is often thought that this has something to do with the presence of a complete set of markets. In this paper I shall argue that it has virtually nothing to do with the completeness of markets. A store of value is only needed when there is a sequence of budget constraints,[1] and a sequence of budget constraints is no less necessary when there are complete markets than when there are incomplete markets. One rationale for a sequence of budget constraints is the absence of trust. This provides a rationale for the use of money (as a store of value) and may also cause us to revise some of our ideas about sequence economies. This paper is, briefly, about trust, money and sequences of budget constraints. In Sections 1 and 2 the problem and various concepts are informally introduced. Section 3 contains a formal description of an exchange economy with many, individually insignificant agents and uncertainty in the form of several states of nature. Sections 4 through 6 are devoted to defining three important concepts and Section 7 contains three theorems which form the substance of the paper. The proofs are relegated to the Appendix. In Section 8 some connections with the literature are indicated.

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[1] The importance of a sequence of budget constraints has been studied most thoroughly by Hahn [8]. See also Starrett [19]. Both these writers show that the central difference between an Arrow-Debreu economy and a sequence economy is that in the former each agent is confronted by a single budget constraint.

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1. THE PROBLEM OF TRUST

It is well known that one of the crucial assumptions of the Arrow-Debreu model of Walrasian equilibrium is that there is a complete set of markets at the first moment when trading begins. This means that all transactions for the duration of the economy can be arranged at one time. Commodities are distinguished by the date and location at which they are to be delivered.[2] For example, goods with the same physical characteristics which are sold for delivery on different dates are treated as different commodities and are traded on separate markets. One can also define contingent commodities, that is, contracts for delivery of goods contingent on the occurrence of an event.[3] Here too goods are distinguished by the event in which they are to be delivered. By defining commodities in this way and correspondingly expanding the set of markets, it is possible to introduce space, time and uncertainty as different dimensions of the commodity space. There is no *formal* distinction between an Arrow-Debreu model in which time, space and uncertainty appear among the *differentiae* of the commodities and one in which they do not. For this reason, people have been led to say that the Arrow-Debreu world is timeless and certain. In a sense they are right. But even if one accepts that the introduction of time and uncertainty does not change the *form* of the Arrow-Debreu model it is still clear that it changes the interpretation. In particular, the assumption that there exists a complete set of markets at the first date becomes, from a descriptive point of view, untenable. This fact is so well known that it need not detain us here.

There is another aspect of the interpretation of the model which has received less attention and which hides an implicit assumption quite as important as the assumption of complete markets. In the Arrow-Debreu model the description of the trading process runs in terms of the exchange of commodities. In an economy in which there is no uncertainty and all goods are delivered at the same place on the same date, it is possible to imagine economic agents actually handing goods back and forth. When commodities by definition have different delivery dates etc. agents are really writing contracts at the first date, the terms of which are to be fulfilled later. To retain the language of exchange, imagine that at the first date agents actually exchange pieces of paper bearing the words "I promise to give the bearer one unit of X if Y" where X denotes some physical good and Y denotes a place, time and event. If one thinks of the pieces of paper as "commodities", there is nothing inconsistent in saying that all "commodities" are exchanged at the first date. This much is trivial and makes no difference to the formal theory.

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[2] I have in mind here the discussion in Debreu, 1959, pp. 28–32.

[3] Debreu, 1959, pp. 98–102.

In terms of interpretation it is quite important because it is only permissible to identify commodities with pieces of paper if it is assumed that the contracts will be carried out. The Arrow-Debreu model concerns itself with trading at the first date only, because it implicitly assumes that a promise to deliver a commodity is as good as the thing itself. For want of a better expression I shall describe this state of affairs by saying that in an Arrow-Debreu model agents *trust* each other (to keep their promises). In an equilibrium characterized by trust there is no reason to distinguish between contracts, agreements, etc. and their execution. The fact that contracts are fulfilled through time and contingent on uncertain events is irrelevant when there is complete trust, but not otherwise. So, like the assumption of complete markets, the assumption that agents trust one another is necessary if the Arrow-Debreu treatment of time and uncertainty is to make sense.

This little observation is not very interesting in itself but it suggests an interesting question. What happens in the absence of trust? To fix ideas consider the following simple example of an exchange economy. There are two agents, a and b. There are two dates and one state of nature (i.e. no uncertainty). At each date there is one commodity available, a consumption good. Each agent's preferences are described by a Cobb-Douglas utility function with equal weights. Suppose that agent a's endowment is represented by the vector $(1 - \epsilon, \epsilon)$, where ϵ is a small, positive number, and agent b's endowment is represented by $(\epsilon, 1 - \epsilon)$.[4] To reach any allocation on the contract curve of this economy agent a must give agent b a positive amount of the consumption good at the first date and agent b must give agent a a positive amount of the consumption good at the second date.[5] Suppose the two agents agree, at the first date, on an allocation lying on the contract curve. Then at the second date, agent b, having already consumed part of a's endowment, will have an incentive to break the agreement he made at the first date, i.e. he will be better off if he breaks his promise and keeps his entire endowment of the consumption good at the second date.[6]

In the absence of trust the fact that commodities are delivered sequentially becomes all-important. The problem is simply this: If I exchange a unit of consumption today for a unit of consumption tomorrow, I am giving up an actual unit of consumption goods and receiving the mere promise of one in return. There is no reason for me to expect an agent who is only interested in maximizing utility to keep his promise. For this reason I may be unwilling to use forward markets even when these markets exist and are costless. The real problem though is to give a formal account of how agents behave in the absence of trust and this is more difficult than making critical remarks about the conventional theory. There may be several solutions to the problem and at this stage one can do no more than investigate one of them.

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[4] That is, agent a has $(1 - \epsilon)$ units of the consumption good at the first date and ϵ at the second. Agent b has the opposite distribution.

[5] Efficiency requires that the final allocation should have the form (z, z), $(1 - z, 1 - z)$, i.e. each agent has a constant level of consumption. Since each agent must be at least as well off as he would be if he consumed his endowment it follows that $2\log z \geqslant \log \epsilon + \log(1 - \epsilon)$ and $2\log(1 - z) \geqslant \log \epsilon + \log(1 - \epsilon)$. For sufficiently small ϵ this implies that $z > \epsilon$ and $z < 1 - \epsilon$ as claimed.

[6] Then instead of having consumption $(1 - z, 1 - z)$ he would have $(1 - z, 1)$.

The rationale for studying the Walrasian equilibrium is found in the theory of the core, more precisely, in the classical theorems on the equivalence between the core and the set of Walras allocations of a non-atomic economy. The definition of the core itself embodies an assumption of trust. The strategy adopted here is to characterize the absence of trust in terms of a concept like the core, and then to find a corresponding equilibrium concept. Consider once again the simple example mentioned above. Since there are only two agents in the economy the core coincides with the contract curve. In this very simple case the core is the set of attainable allocations such that (a) neither agent can make himself better off by remaining self-sufficient and (b) the two agents cannot both be made better off by any feasible redistribution of their joint endowment. In other words, each is at least as well off as he would be consuming only his initial endowment and the allocation is Pareto efficient. In general terms, the core consists of attainable allocations which cannot be improved upon by any coalition. In the earlier discussion of this example, it was pointed out that agent *b* could improve on his core allocation by agreeing to the core allocation at the first date and then, breaking the agreement, remain self-sufficient at the second date. The classical definition of the core ignores certain opportunities for improving coalitions to form because it ignores the essentially sequential nature of consumption and the absence of trust. This suggests a revision of the classical concept of the core. We can define the *sequential core* to be the set of attainable allocations that cannot be improved on by any coalition either in the classical sense or in the new sense of accepting an allocation to start with but improving on it at a later date. Objections can be raised to this concept, as they can to the classical core, and some of these will be discussed later, but it will be argued below that this notion of the sequential core is, in a precise sense, at least as reasonable as the classical core and represents the natural extension of that concept to deal with the absence of trust. These ideas are not easy to grasp so it may be helpful to describe some of them more formally.

Consider a simple exchange economy $\mathscr{E}$ (more general cases are dealt with in the sequel) consisting of a finite number m of consumers. There is no uncertainty and time is divided into two intervals or *dates* indexed $t = 0, 1$. At each date there is a finite number l of *commodities* available for delivery so the commodity space is $\mathbb{R}^{2l}$. The i-th consumer is characterized by a *consumption set* $\mathbb{R}^{2l}_{+}$, an *endowment* $e_i \in \mathbb{R}^{2l}_{+}$ and a *preference relation* $\succ_i$ on $\mathbb{R}^{2l}_{+}$. An *allocation* for $\mathscr{E}$ is an m-tuple (x_i) of vectors $x_i \in \mathbb{R}^{2l}_{+}$ and an allocation is *attainable* if $\sum x_i = \sum e_i$.[7] A *coalition* is a non-empty subset S of the set of all agents $\{1,..., m\}$. A coalition S can *improve on an allocation* (x_i^*) if there exists an attainable allocation (x_i), say, such that

(a) $x_i \succ_i x_i^*$ for every i in S and

(b) $\sum_{i \in S} x_i = \sum_{i \in S} e_i$.

Part (b) of the definition is crucial: it requires an improving coalition to be self-sufficient *at every date*. The *classical core* is defined as the set of attainable allocations which cannot be improved on by any coalition. It is denoted by $C(\mathscr{E})$.

In defining the sequential core it is necessary to allow for the formation of coalitions which do not satisfy condition (b), because in the absence of trust a coalition may decide to form at the second date, having already traded with the rest of the economy at date 1. The following notation is helpful: if z is a vector in $\mathbb{R}^{2l}$ we write $z = (z(0), z(1))$ where $z(t) \in \mathbb{R}^l$ for $t = 0, 1$. A second period allocation is an m-tuple $(x_i(1))$ where $x_i(1) \in \mathbb{R}_+^l$ for each $i = 1,..., m$. A second period allocation is attainable if $\sum x_i(1) = \sum e_i(1)$. The coalition S can *improve on the allocation* (x_i^*) *at date* 1 if there exists an attainable second period allocation $(x_i(1))$ such that

(c) $[x_i^*(0), x_i(1)] \succ_i x_i^*$ for every i in S, and

(d) $\sum_{i \in S} x_i(1) = \sum_{i \in S} e_i(1)$.

Condition (d) says that if coalition S does not form until date 1 then it is only required to be self-sufficient at date 1. The *sequential core* of $\mathscr{E}$ is defined to be the set of attainable allocations which cannot be improved on by any coalition either in the classical sense (conditions (a) and (b)) or at date 1 (conditions (c) and (d)). It is denoted by $C^s(\mathscr{E})$. An allocation can be said to be *trustworthy* only if it belongs to the sequential core. If the allocation does not belong to the sequential core then even if all agents agree to it at the outset some coalition has an incentive to break the agreement later on. So the sequential core formalizes the notion of trust-worthiness and gives a precise, though tentative, picture of an economy without trust.

The sequential core is, in a precise sense, the natural extension of the classical core to deal with the absence of trust. The definition of the sequential core is just the application of the classical definition of the core to the situations in which agents find themselves at successive dates. In fact, once consumption at date 0 has occurred, it is possible to define a "subeconomy" at date 1 which is formally an exchange economy in its own right. Suppose the allocation (x_i^*) is chosen at date 0 and the i-th agent consumes $x_i^*(0)$. At the beginning of date 1 he finds himself with a consumption set $\mathbb{R}_+^l$ (instead of $\mathbb{R}_+^{2l}$), an endowment $e_i(1)$ (instead of e_i) and a preference ordering on $\mathbb{R}_+^l$ induced by $\succ_i$ and $x_i^*(0)$ according to the following definition:

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[7] $\mathbb{R}^{2l}$ denotes $2l$-dimensional, real, Euclidean space. $\mathbb{R}_+^{2l}$ denotes the non-negative orthant of $\mathbb{R}^{2l}$. A preference relation $\succ_i$ is a transitive, irreflexive binary relation. In expressions like $\sum x_i$, where the range of summation is not shown, it is understood that the summation is from 1 to m.

$$x \textit{ is preferred to } y \textit{ if and only if } (x_i^*(0), x) \succ_i (x_i^*(0), y) \textit{ for any } x, y \in \mathbb{R}_+^l.$$

Now if the classical core is the appropriate solution concept for the original economy $\mathscr{E}$ then it is also the appropriate solution concept for the subeconomy defined at date 1. For the only difference between the two economies is the dimensionality of the commodity space—which obviously should not influence the choice of a solution concept—and the fact that at the beginning of date 1 there exist agreements reached at date 0. But these agreements do not influence agents' behavior at date 1; in fact, this is precisely what is meant by the absence of trust. If one accepts the classical core as the appropriate solution concept for an exchange economy then in the absence of trust an allocation should belong both to the (classical) core of the original economy $\mathscr{E}$ and to the (classical) core of the "subeconomy" at date 1. This is exactly what the sequential core requires. In this sense the sequential core must be held to be at least as reasonable, as a solution concept, as the classical core.[8]

Of course, there remain objections to the sequential core as a solution concept. In the first place, most objections to the classical core would apply equally well to the sequential core. For example, the core can be interpreted rather literally as implying that if a non-core allocation is proposed then an improving coalition actually forms to "block" it. In this case, the definition of an improving coalition seems unreasonable. The fact that there exists an allocation which is attainable for the coalition, and makes its members better off, does not guarantee that they will achieve that allocation. Indeed, some of the members of the coalition may be worse off at a core allocation. For this reason, potential members of an improving coalition may prefer the *status quo*. This criticism applies *a fortiori* to the sequential core. In the absence of trust an improving coalition ought to take into account the fact that a sub-coalition may be improving at a later date.[9]

[8] This idea of applying the classical core as a solution concept to each "subeconomy" is like the game-theoretic notion of a "perfect solution." A solution is perfect if it is a solution of every subgame.

[9] A referee has suggested the following definition of an improving coalition. A coalition is improving in his sense if it is improving in my sense and no subcoalition is improving

In defence of the core it can be argued that it should not be taken literally as a description of the allocation process. Instead, it should be treated as a *criterion* of stability. But this simply begs the question of how a core allocation is chosen. A better argument would be that the core is an interesting concept because it characterizes the Walrasian equilibrium in an atomless economy. The competitive model is a useful vehicle for studying many problems, not because of its realism, but because of its simplicity and attractive normative properties. In other words, it serves as a "benchmark". By extension, the core may be a useful starting point for the study of trust, despite its flaws as a descriptive model. The problems mentioned should be the subject of future research. In the present paper they will be overlooked.

An immediate problem which cannot be avoided is the possibility that the sequential core may be empty. This is certainly the case in the two-person example mentioned earlier. The coalition consisting of agent b alone can improve at date 1 on any allocation in the classical core. Since the sequential core is always contained in the classical core, it must be empty in this example. If the sequential core is so easily shown to be empty it may be doubted whether it is a useful solution concept. This view is unduly pessimistic, however. If trust disappeared overnight and if it were not replaced by another institution, one would not be surprised if the allocation process broke down. In practice, however, institutions do develop to meet such needs and it can be shown that the institution of money can act as a substitute for trust. Precisely, if money is introduced in an appropriate way, then the sequential core (appropriately defined) is non-empty. This is where the twin concepts of trust and the sequential core become fruitful in producing theorems.

2. An Economy with Money

The exchange economy $\mathscr{E}$ described above can be turned into a monetary economy by giving each agent an endowment of money at the beginning of the first date and assuming that each has to pay a tax of an equal amount of money at the end of the second date. The money introduced here is fiat money, pieces of paper issued by the Government and having no intrinsic worth. The artificial device of imposing taxes to soak up the money at the

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at a later date. It is not clear from his suggestion whether the subcoalition is to be improving in my sense or in his sense. If the latter, then the subcoalition must not contain an improving sub-subcoalition and so on. Since the referee's suggestion was intended to increase the size of the sequential core, which is sometimes empty, I presume he wanted the subcoalition to be improving in my sense. In that case, there are substantially fewer coalitions. It seems rather arbitrary to stop after considering one subcoalition, however. Moreover, it is not clear whether interesting theorems follow from this definition.

end of date 1 is often used in models of monetary equilibrium.[10] It ensures that money has a positive terminal value. Consider the two person example of the last section. The two agents are eventually required by the tax constraint to return to the Government the amount of money they started with, but in the interim they can exchange it among themselves. An allocation must specify not only the consumption of each agent at each date but also his money holdings at the end of the first date. Thus, in the example, if agent *b* gives part of his money endowment to agent *a* at the first date he must somehow retrieve it at the second date. The sequential core as previously defined was empty because agent *a*, having given some of his endowment to agent *b*, had no way of persuading agent *b* to keep his promise. In particular, he could not retaliate if agent *b* broke their agreement. But if, in exchange for consumption at date 0, agent *b* gives agent *a* some money, this provides agent *b* with an incentive to co-operate with agent *a* at date 1, i.e. to deliver the consumption goods as promised. Otherwise he will not receive the money he needs to pay his taxes. It is not obvious what will happen at date 1; agent *a* may use his newfound power to extort more than the agreed amount of consumption from agent *b* at date 1. This is a complex bargaining problem. But there is a fairly natural way of extending the definition of the sequential core to a monetary economy. An improving coalition is allowed to form only if its current money endowment is sufficient to cover the second period taxes of all its members. The introduction of money thus reduces the number of coalitions that can form and hence enlarges the sequential core. The sequential core in this example is now non-empty. These ideas are easily formalized.

A *monetary economy* $\mathscr{E}^m$ is defined just like $\mathscr{E}$ except that each agent $i = 1, \dots, m$ is given a *positive* endowment of money $\bar{m}_i$. At the end of date 0 the i-th agent holds a non-negative amount m_i of money and at the end of date 1 he holds at least $\bar{m}_i$. An attainable allocation is now an m-tuple $\{(x_i, m_i)\}$ such that for every $i = 1, \dots, m$, $x_i \in \mathbb{R}^{2l}_+$ and $m_i \geq 0$ and such that $\sum (x_i, m_i) = \sum (e_i, \bar{m}_i)$. At date 0 every agent holds the exact amount of money he needs to pay his taxes so the improving coalitions are the same as before. At date 1 a coalition S can improve on an allocation (x_i^*, m_i^*) if and only if there exists an allocation $(x_i(1))$ at date 1 such that (c) and (d) are satisfied, that is, $(x_i^*(0), x_i(1)) \succ_i (x_i^*(0), x_i^*(1))$ for all $i \in S$ and $\sum_{i \in S} x_i(1) = \sum_{i \in S} e_i(1)$, and in addition,

$$\text{(e)} \quad \sum_{i \in S} m_i^* \geq \sum_{i \in S} \bar{m}_i .$$

The *sequential core of the monetary economy* $\mathscr{E}^m$ is the set of attainable allocations which cannot be improved on by any coalition, either in the classical sense at date 0 (conditions (a) and (b)) or at date 1 (conditions (c)-(e)) as defined above. An interesting property emerges immediately. Recall that a *Walras allocation* for the exchange economy $\mathscr{E}$ is an attainable allocation (x_i^*) such that for some price vector p and every $i = 1,\ldots, m$, x_i^* is maximal with respect to $\succ_i$ in the budget set $\{x \in \mathbb{R}_+^{2l} \mid p \cdot x \leq p \cdot e_i\}$.

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[10] Cf. Hahn [8], Starr [18], Heller [10], and Kurz [12].

PROPOSITION. *If (x_i^*) is a Walras allocation for $\mathscr{E}$ then for some m-tuple (m_i^*), the allocation (x_i^*, m^*) belongs to the sequential core of $\mathscr{E}^m$.*

Proof. Let p be the equilibrium price vector corresponding to the allocation (x_i^*). Then define

$$m_i^* = \bar{m}_i - p(0) \cdot (x_i^*(0) - e_i(0)) \text{ for } i = 1,\ldots, m.$$

Since (x_i^*) is attainable, $\sum m_i^* = \sum \bar{m}_i$. Furthermore, since $\bar{m}_i > 0$, by choosing p small enough we can ensure that $m_i^* > 0$. Hence (x_i^*, m_i^*) is attainable. Since (x_i^*) is attainable it belongs to the classical core of $\mathscr{E}$ so no coalition is improving at date 0. At date 1, suppose S is an improving coalition. Then there exists an allocation $(x_i(1))$ at date 1 such that $(x_i^*(0), x_i(1)) \succ_i (x_i^*(0), x_i^*(1))$ for each $i \in S$. Then for each $i \in S$, $p \cdot (x_i^*(0), x_i(1)) > p \cdot (x_i^*(0), x_i^*(1))$ or $p(1) \cdot x_i(1) > p(1) \cdot x_i^*(1)$. Summing over S we have

$$p(1) \cdot \sum_{i \in S} x_i(1) > p(1) \cdot \sum_{i \in S} x_i^*(1).$$

Condition (d) requires that $\sum_{i \in S} x_i(1) = \sum_{i \in S} e_i(1)$ so

$$p(1) \cdot \sum_{i \in S} e_i(1) > p(1) \cdot \sum_{i \in S} x_i^*(1).$$

By definition,

$$\begin{aligned} m_i^* - \bar{m}_i &= -p(0) \cdot (x_i^*(0) - e_i(0)) \\ &= p(1) \cdot (x_i^*(1) - e_i(1)) \end{aligned}$$

and therefore

$$\sum_{i \in S} (m_i^* - \bar{m}_i) = p(1) \cdot \sum_{i \in S} (x_i^*(1) - e_i(1)) < 0$$

contradicting condition (e). Thus, (x_i^*, m_i^*) belongs to the sequential core as required. ∎

The significance of this result is obvious: the allocation (x_i^*), which was untrustworthy in the non-monetary economy, is trustworthy in the monetary economy for an appropriate distribution of money, in this case (m_i^*), at the end of date 0. A number of points should be noted.

First, it is not the invention of paper money which restores trustworthiness. The Walras allocations are trustworthy in the monetary economy only because there is, in the background, a government which can enforce, evidently at no cost, the payment of taxes. Thus, we have introduced not just a new commodity (money) but a new social institution. In reality this institution would probably be costly, in which case there is a real loss of utility when trust disappears and has to be replaced.

Second, the role of money may seem rather artificial since it can be argued that the government, which enforces the payment of taxes, could just as easily have enforced the original commodity contracts, i.e. intervened directly in the market to make sure that agents kept their promises, without introducing "the social contrivance of money". But money has a number of advantges in the general case. First of all, there may be many goods, many dates and many states of nature. Since there are correspondingly many transactions the use of money (or securities) reduces the number of obligations the government has to enforce. Second, by using money the government avoids the need to gather information about the transactions carried out in the economy. It merely sets taxes equal to endowments of money, thus economizing on information costs. In short, the advantage of money is that it allows transactions to be made trustworthy in a *decentralized* way.

A third point is that the model is based on the extreme assumption that agents *never* renege on tax obligations. If the penalties are sufficiently gruesome this may seem reasonable but it would be better to tell a less fanciful story. Of course, the assumption that agents *never* keep their promises unless it is in their interest to do so is also extreme. These assumptions are adopted here for lack of better ones, but they obviously require further, critical study.

To sum up, the monetary arrangement described in the proposition works because it is a social institution, i.e. it is backed up by the power of government to enforce taxes. This may be costly but it does economize on the cost of enforcing transactions because money decentralizes enforcement.

The proof of the proposition implicitly makes use of a notion of equilibrium. Before turning to the formal analysis of the relationship between the core and this equilibrium concept it may be helpful to compare it briefly with another definition of equilibrium. Consider a sequence economy, that is, an economy with the same set of characteristics $(e_i, \succ_i, \mathbb{R}^{2l}_+)$ as the exchange economy $\mathscr{E}$ but with an *incomplete* set of markets at date 0. Formally, it is assumed that at each date $t = 0, 1$ there is a linear subspace of $\mathbb{R}^{2l}$, denoted by $Z(t)$, representing the commodities which can be traded at that date. At date t, an agent chooses a trade vector $z(t) \in Z(t)$ and the result of this *trade plan* $z = (z(0), z(1))$, as it is called, is a consumption vector $x = e_i + z(0) + z(1)$. At each date there is a price system $\hat{p}(t) \in Z(t)$ at which trades are carried out. The budget set of the i-th agent given the price system $\hat{p} = (\hat{p}(0), \hat{p}(1))$ is defined to be the set of trade plans z such that

(i) $z(t) \in Z(t)$

(ii) $\hat{p}(t) \cdot z(t) \leq 0$

(iii) $e_i + z(0) + z(1) \geq 0$

The important condition from the point of view of this paper is (ii). It is the trades *arranged* at date t which are subject to constraint. A *sequence equilibrium* is then a price system p^* together with an m-tuple (z_i^*) of trade plans such that z_i^* maximizes preferences over the budget set and $\sum z_i^* = 0$.

If money is introduced by giving each agent an endowment $\bar{m}_i$ and demanding it back at the end of date 1, the budget set is changed to the set of plans (z, m) such that (i) and (iii) hold and

$$\text{(ii)}' \quad \hat{p}(0) \cdot z(0) + m - \bar{m}_i \leq 0$$

$$\hat{p}(1) \cdot z(1) + \bar{m}_i - m \leq 0$$

$$\text{(iv)} \quad m \geq 0.$$

This kind of formulation has been used in several papers dealing with sequence economies which are regarded as being extensions of the Arrow-Debreu model. The crucial feature of these models is the definition of the budget constraints. In the first place, if there were a single budget constraint of the form $\sum \hat{p}(t) \cdot z(t) \leq 0$ then, as long as there existed a market for each commodity at some date, the model would be indistinguishable from an Arrow-Debreu model in spite of the incompleteness of markets. And yet there is nothing in the structure of the model that requires separate budget constraints for each date. In the second place, without separate budget constraints there is certainly no need, in an equilibrium theory, for a store of value. Now it would not be surprising if the absence of trust provided a rationale for a sequence of budget constraints. The budget constraint in the Walrasian equilibrium is justified by reference to the core of an atomless economy. Similarly one might expect that the sequential core corresponds, in an atomless economy, to a concept of equilibrium in which there is a sequence of budget constraints. This turns out to be true, but the constraints are not like the ones defined by (ii) or (ii)′ above.

In the proof of the proposition, the holdings of money at the end of date 0, i.e. m_i^*, satisfied

$$p(1) \cdot (x_i^*(1) - e_i(1)) + \bar{m}_i - m_i^* = 0.^{11}$$

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[11] There is a slight abuse of notation in using 'p' as a generic symbol for prices at date t in both the "sequence equilibrium" and the "sequential equilibrium." In the former, $\hat{p}(t) \in Z(t)$ represents the prices of goods traded at date t. In the latter $p(t) \in \mathbb{R}^l$ represents the prices of commodities available for delivery at date t, irrespective of the date at which the trade was arranged.

Since (x_i^*) was by hypothesis a Walras allocation, x_i^* satisfied the equation

$$p \cdot x_i^* = p \cdot e_i$$

and in fact was maximal in the budget set $\{x \in \mathbb{R}_+^{2l} \mid p \cdot x \leqslant p \cdot e_i\}$. Furthermore, it is easy to see that $x_i^*(1)$ is maximal in the second period budget set $\{x \in \mathbb{R}_+{}^l \mid p(1) \cdot x + \bar{m}_i \leqslant p(1) \cdot e_i(1) + m_i^*\}$.[12] The allocation (x_i^*, m_i^*), together with the price system p, constitutes a *sequential equilibrium* for the monetary economy.

It is also possible to define a sequential equilibrium without money. It is an attainable allocation (x_i^*) together with a price system p such that

(a) x_i^* is maximal in the budget set $\{x \in \mathbb{R}_+^{2l} \mid p \cdot x \leqslant p \cdot e_i\}$

(b) $x_i^*(1)$ is maximal in the budget set $\{x \in \mathbb{R}_+{}^l \mid p(1) \cdot x \leqslant p(1) \cdot e_i(1)\}$.

Without the intervention of money a sequential equilibrium will not in general exist, *even under the usual convexity assumptions*.[13] With money, existence is generally guaranteed under traditional assumptions.

In the usual definition of a sequence equilibrium, the transactions arranged at a given date must satisfy the budget constraint. In the sequential equilibrium the allocation must be *both* a Walras allocation for the original economy (condition *a*) *and* a Walras allocation for the subeconomy at date 1 (condition b). Combining (a) and (b) shows that actual deliveries at each date, i.e. $x_i(t) - e_i(t)$, must satisfy a budget constraint. This is quite intuitive in view of the lack of trust. It is not clear what the rationale for the other constraint is.

The rest of this paper is devoted to developing a formal analysis of a monetary exchange economy. In general, it can be shown that if a consumption allocation belongs to the classical core of a (non-monetary) economy, then there is an allocation in the corresponding monetary economy which belongs to the sequential core and assigns the same consumption to each agent. Corresponding to the notion of the sequential core is a definition of sequential equilibrium. The relation between the two is rather messy in a finite economy but in an atomless economy the sequential core is equal to the set of equilibrium allocations, appropriately defined. Thus in large economies the absence of trust implies that agents must satisfy a sequence of budget constraints. However, the form of the constraints differs from the one frequently assumed. Finally, these results can be generalized to give an analogue of a classical theorem of welfare economics. Every Pareto efficient allocation corresponds to an equilibrium of a monetary economy with an appropriate distribution of taxes and hence belongs to the sequential core of that economy. In this case, taxes do not in general equal initial endowments so there are implicit, lump-sum transfers. Conversely, the sequential core of a monetary economy (in which taxes differ from endowments) corresponds to the set of equilibrium allocations (where equilibrium is appropriately defined to take account of the net tax burden).

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[12] $x^*(1)$ is maximal in the sense that x^* is maximal for $\succ_i$ over the set of vectors $(x^*(0), x)$ where x belongs to the second period budget set.

[13] The reason is fairly obvious. If (x_i^*) is a Walras allocation and p is the equilibrium vector then one does not, in general, expect $p(t) \cdot (x_i(t) - e_i(t)) = 0$ to obtain. However, this is what the sequential equilibrium requires.

The concepts and theorems sketched above accomplish three things. They formalize the notion of trust, provide a rationale for a sequence of budget constraints and illustrate the role of money in replacing trust. Indeed they provide a rationale for money.

3. THE SEQUENTIAL ECONOMY

3.1 *Commodities and Markets*

The commodities in an exchange economy are ordered in a natural way by time and uncertainty. This ordering can be represented by the *set of date-event pairs*, that is, an ordered pair $(U, >)$ where U is a finite set and $>$ is an ordering on U.

Time is assumed to consist of a finite number of elementary time intervals called *dates* and indexed by $t = 0, 1,..., T$. There is a finite set S of *states of nature* representing the stochastic nature of the environment. The *observable events* at t are represented by a partition $\mathscr{S}_t$ of S. The sequence of partitions $\mathscr{S}_0, \mathscr{S}_1,..., \mathscr{S}_T$ is assumed to be monotone nondecreasing in fineness and we assume $\mathscr{S}_0 = \{S\}$. A *date-event pair* is an ordered pair (t, E) where $t = 0, 1,..., T$ and $E \in \mathscr{S}_t$. The set of such pairs is denoted by U. If $u, v \in U$, $u = (t, E)$ and $v = (t', E')$ then $u < v$ if and only if $t < t'$ and $E' \subset E$. Then $(U, >)$ is the required structure. A more detailed discussion is contained in Radner [17]. In the rest of the paper it is only necessary to refer to the set $(U, >)$ of date-event pairs and in particular the symbols t, S, T will be used in different senses from those introduced in this section.

The pair $(0, S)$ is denoted by 0. If $u \in U$ and $u \neq 0$ then $0 < u$. If $u, v \in U$ then $u \geqslant v$ means $u > v$ or $u = v$. For any $u \in U$ if $u \neq 0$ there exists a unique element $u^- \in U$, called the *predecessor* of u, such that if $u = (t, E')$ then $u^- = (t - 1, E)$ and $E' \subset E$.

Commodities are distinguished by the date-event pair at which they are to be delivered. At each date-event pair $u \in U$ there is a finite number l^u of commodities to be delivered. The total number of commodities available for delivery in the economy is denoted by $l := \sum_{u \in U} l^u$.[14] The commodity space is identified with $\mathbb{R}^l$.

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[14] The notation $:=$ indicates that the expression on the left hand side is defined by the expression on the right.

Typically, if $x \in \mathbb{R}^l$ then x^u denotes the sub-vector corresponding to the date-event pair $u \in U$.

In order to give a complete description of the market structure it is necessary to specify which commodities can be traded at each date-event pair. Various assumptions are made throughout the paper, depending on the type of equilibrium under consideration.

3.2 *Securities*

There is a finite set of securities in the economy. A security is just a piece of paper, issued by the government, which can be used for paying taxes. Securities can be traded at each date-event pair and thus represent a source of credit in the economy. In general there may be any number of securities but there is one case that is of special interest, namely, the one in which there is a security for each terminal date-event pair. If $u \in U$ then u is a *terminal date-event pair* if there exists no $v \in U$ such that $v > u$. Define $m := \#\bar{U}$ and $n := m \cdot \#U$ where $\bar{U}$ denotes the set of terminal date-event pairs.

Every agent a is given an *endowment of securities* $\sigma(a)$ at the date-event pair 0, where $\sigma(a) \subset \mathbb{R}_+^m$. The agent also has a *tax liability* $\tau(a) \in \mathbb{R}_+^m$. The vector $\tau(a)$ is an array $(\tau_u(a))$ of non-negative numbers, one for each $u \in \bar{U}$, representing the number of securities of each type he must pay the government. However, he is only liable to pay the tax in u-securities at the date-event pair $u \in \bar{U}$. In other words, if $u \in \bar{U}$ the agent's liability to the government is represented by an array $t^u(a) \in \mathbb{R}^m$ where $t_v^u(a) = \tau_v(a)$ if $u = v$ and $= 0$ otherwise, for all $v \in \bar{U}$. Generally, for any $u \in U$ the actual tax liability of agent a is represented by $t^u(a) \in \mathbb{R}^m$ where the array $(t_v^u(a))$ is defined by

$$\begin{cases} t_v^u(a) = \tau_v(a) & \text{if } v \in \bar{U} \text{ and } u < v \text{ or } u = v; \\ t_v^u(a) = 0 & \text{for all } v \in \bar{U} \text{ otherwise.} \end{cases}$$

A *security plan* is a specification of the amount of each security an agent wants to hold at the end of each date-event pair. It can be represented by a vector $s \in \mathbb{R}_+^n$. At the beginning of any date-event pair an agent's holdings of securities are simply those carried forward from the predecessor, or in the case of 0 the initial endowment. The possible security plans for an agent a are those which allow him to meet his tax obligations. Define the *security set* of agent a by

$$S_a := \{s \in \mathbb{R}_+^n \mid s^u \geqslant t^u(a) \text{ for all } u \in \bar{U}\}.$$

Notation. If a adopts the security plan s then at any date-event pair $u \in U$, his initial holdings are s^{u-} where u^- is the predecessor of u and, by convention, $s^{u-} = \sigma(\mathrm{a})$ if $u = 0$. It is convenient to introduce a shift operator T_a on $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$(T_a s)^u = s^{u-}$$

for all $s \in \mathbb{R}^n$, where it is understood that $s^u = \sigma(a)$ if $u = 0$. Then for an $s \in \mathbb{R}^n$, $T_a s$ is the array of initial holdings at each date-event pair corresponding to the plan s, and $(s - T_a s)$ is the corresponding array of net trades.

3.3. *Preference Relations*

The possible consumption plans for agent a are represented by his *consumption set* X_a. To simplify matters it will be assumed that $X_a := \mathbb{R}_+^l$. The tastes and preferences of agent a are represented by a *preference relation* $\succ_a$ on $\mathbb{R}_+^l$, that is, $\succ_a$ is an open subset of $\mathbb{R}_+^l \times \mathbb{R}_+^l$ representing an irreflexive and transitive binary relation on $\mathbb{R}_+^l$. Let $\mathscr{P}$ denote the set of monotonic preference relations on $\mathbb{R}_+^l$.[15] $\mathscr{P}$ is endowed with the topology of closed convergence and becomes a metrizable space (see Hildenbrand [11, 1.1, Theorem 1, p. 96]).

3.4 *Sequential Economies*

The set of agents in an economy is represented by a measure space $(A, \mathfrak{A}, \nu)$ where A is a metric space, $\mathfrak{A}$ the Borel σ-field and ν a probability measure on $(A, \mathfrak{A})$.

The characteristics of an agent $a \in A$ are represented by

$$\mathscr{E}^s(a) := (\succ_a, e(a), \sigma(a), \tau(a))$$

where $\succ_a$ is his preference relation, $e(a)$ his commodity endowment, $\sigma(a)$ his security endowment and $\tau(a)$ his tax obligations. Formally, a *sequential economy* is a triple $(\mathscr{E}^s, U, \succ)$ where $(U, \succ)$ is a set of date-event pairs and

$$\mathscr{E}^s : (A, \mathfrak{A}, \nu) \to \mathscr{P} \times \mathbb{R}_+^l \times \mathbb{R}_+^m \times \mathbb{R}_+^m$$

is a measurable function such that $\int e \, d\nu$ is finite and $\int \sigma \, d\nu - \int \tau \, d\nu$ is also finite. (The functions e, σ and τ are defined precisely at the end of the section.)

A *commodity* (resp. *security*) *allocation* for a sequential economy $(\mathscr{E}^s, U, \succ)$ is a measurable function f (resp. g) on A to $\mathbb{R}_+^l$ (resp. $\mathbb{R}_+^n$) such that, a.e. in A, $f(a)$ (resp. $g(a)$) belongs to the consumption set (resp. security set) of agent a. A commodity (resp. security) allocation for $(\mathscr{E}^s, U, \succ)$ is *attainable* if $\int f \, d\nu = \int e \, d\nu$ (resp. $\int g^u \, d\nu = \int \sigma \, d\nu$ for all $u \in U$). An *allocation* for $(\mathscr{E}^s, U, \succ)$ is a pair (f, g) where f (resp. g) is a commodity (resp. security) allocation for $(\mathscr{E}^s, U, \succ)$. The allocation (f, g) is attainable if f and g are both attainable. Since the set of date-event pairs is fixed in what follows, the sequential economy $(\mathscr{E}^s, U, \succ)$ will be referred to as $\mathscr{E}^s$.

Notation. Let $\mathbf{e}$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ represent respectively the projections of $\mathscr{P} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$ into the second, third and fourth factor spaces. Then $e := \mathbf{e} \circ \mathscr{E}^s$, $\sigma := \boldsymbol{\sigma} \circ \mathscr{E}^s$ and $\tau := \boldsymbol{\tau} \circ \mathscr{E}^s$.

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[15] That is, $x \succ_a y$ implies $x \neq y$ and $x \succ_a y \succ_a z$ implies $x \succ_a z$, where $x, y \in \mathbb{R}_+^l$.

4. SUBECONOMIES

For every $u \in U$ define the set F^u by

$$F^u := \{v \in U \mid v = u \text{ or } v > u\}.$$

If $v \notin F^u$ then either v corresponds to a date earlier than u's or it corresponds to an event which is impossible at u. Thus, F^u represents the present and possible future of the economy, once u has been reached. If $v \notin F^u$ then decisions concerning v cannot be changed once u is reached; hence only date-event pairs in F^u are relevant to decision-making at u. Let $\mathscr{F} := \{F^u : u \in U\}$.

Suppose that $\mathscr{E}^s$ is a sequential economy, (f, g) an attainable allocation for $\mathscr{E}^s$, F an element of $\mathscr{F}$ and u the first element in F (i.e. $v \in F$ implies $v \geq u$). We define the *F-subeconomy relative to* $(\mathscr{E}^s, f, g)$ as follows. For each $a \in A$ put

$$\mathscr{E}^F(a) := (X_a^F, \succsim_a^F, e^F(a), \sigma^F(a), \tau^F(a))$$

where

$$X_a^F := \{x \in \mathbb{R}_+^l \mid x^v = f^v(a), v \notin F\}$$

$$\succsim_a^F := \succsim_a \cap (X_a^F \times X_a^F)$$

$e^F(a)$ is defined by

$$\begin{cases} e^{Fv}(a) := f^v(a) \text{ if } v \notin F \\ \qquad\quad := e^v(a) \text{ otherwise} \end{cases}$$

$$\sigma^F(a) := g^{u^-}(a)$$

$$\tau^F(a) := t^u(a).$$

The definition means the following. The consumption set X_a^F is defined to include only those consumption plans which agree with $f(a)$ outside F. In other words, consumption can differ from $f(a)$ only at date-event pairs in F. The new preference ordering is simply the restriction of the original one to the new consumption set X_a^F. The new endowment $e^F(a)$ is defined to equal the original endowment on F and $f(a)$ outside F. This ensures that $e^F(a) \in X_a^F$. Finally, the initial endowment of securities is equal to the value of $g(a)$ at the predecessor of u, i.e. the amount that $g(a)$ implies a will hold at the beginning of u, and $\tau^F(a)$ is set equal to the tax obligations of a at u. The definition may seem slightly artificial but it is necessary (for the sake of the proofs) to define the consumption set and security set to be subsets of the corresponding sets in the original economy. This is why a certain amount of awkwardness is unavoidable.

Clearly $\mathcal{E}^F$ is a measurable function on $(A, \mathcal{A}, \nu)$ to $\mathcal{P}^F \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$, where $\mathcal{P}^F$ is the same as $\mathcal{P}$ except for having a smaller-dimensional commodity space. Then $(\mathcal{E}^F, F, \leq)$ is a sequential economy according to the definition given in Section 5. Strictly speaking we ought to write $(\mathcal{E}^F, F, \leq, f, g)$ but when there is no risk of confusion we refer to the F-subeconomy relative to $(\mathcal{E}^s, f, g)$ as $\mathcal{E}^F$.

The security set for agent a must also be redefined as part of the definition of the subeconomy. The *security set for agent a relative to g* is denoted by (S_a^F, g) or, if there is no ambiguity, simply by S_a^F. It is defined by

$$S_a^F := \{s \in \mathbb{R}_+^n \mid s^v = g^v(a), v \notin F, s^v \geq t^v(a), v \in F \cap \bar{U}\}.$$

The interpretation of the concept of the F-subeconomy is obvious. It is a model of the economic activity in the date-event pairs in F, *taking as given agents' actions in* $U \backslash F$ since these cannot be changed once u is reached.

Notation. For each $F \in \mathcal{F}$ define the shift operator T_a^F from $\mathbb{R}^n$ to $\mathbb{R}^n$ (relative to g) by putting

$$\begin{aligned}(T_a^F s)^v &= g^v(a) \text{ if } v \notin F\\ & g^{u-}(a) \text{ if } v = u \text{ is first in } F\\ &= s^{v-} \text{ otherwise.}\end{aligned}$$

Two points should be noted about this definition. First, $T_a^F s$ agrees with $g(a)$ outside F and has the usual definition inside F, bearing in mind that $\sigma^F(a) = g^{u-}(a)$. Second, no "shifting" occurs outside F so the initial and final holdings at date-event pairs outside F are equal.

5. SEQUENTIAL CORE

An allocation (f, g) for $\mathcal{E}^s$ can be *improved upon* if there is a coalition $E \in \mathcal{A}$ and an allocation (f^*, g^*) such that

(i) $f^*(a) \succ_a f(a)$ a.e. in E

(ii) $\nu(E) > 0$, $\int_E f^* \, d\nu = \int_E e \, d\nu$ and $\int_E \sigma \, d\nu \geqslant \int_E \tau \, d\nu$.

The crucial difference between the usual definition of "improving" and this one is that here a coalition can only form *if it has a sufficiently large endowment of securities to meet its tax obligations.*

The *core* of a sequential economy $\mathcal{E}^s$ is defined to be the set of attainable allocations for $\mathcal{E}^s$ which cannot be improved upon. It is denoted by $C(\mathcal{E}^s)$. This concept is of no interest in itself but allows us to define the sequential core rather neatly. The *sequential core* of $\mathcal{E}^s$ is defined to be the set of attainable allocations (f, g) for $\mathcal{E}^s$ such that for every $F \in \mathcal{F}$, $(f, g) \in C(\mathcal{E}^F)$, where $(\mathcal{E}^F, F, \succ)$ is defined relative to $(\mathcal{E}^s, f, g)$, of course. The sequential core is denoted by $C^s(\mathcal{E}^s)$.

An allocation in the sequential core is completely trustworthy since it cannot be improved upon at any subsequent date-event pair. Indeed, the concept of the sequential core can be taken to characterize trustworthiness in a sequential economy because an attainable allocation which is not in the sequential core can be improved upon at a date-event pair $u \geqslant 0$; and this means *either* the allocation does not belong to the core $(u = 0)$ *or* there is an incentive for some coalition to agree to the allocation at 0 and then break their agreement at $u > 0$. Hence we define an attainable allocation (f, g) for $\mathcal{E}^s$ to be *trustworthy* if and only if $(f, g) \in C^s(\mathcal{E}^s)$.

6. SEQUENTIAL EQUILIBRIUM

A *sequential equilibrium* for $\mathcal{E}^s$ is an attainable allocation (f, g) and a price vector $(p, q) \in \mathbb{R}^l \times \mathbb{R}^n$ such that for every $F \in \mathcal{F}$

a.e. in A, $(f(a), g(a))$ *is maximal for* $\succ_a^F$ *in the budget set*

$$\{(x, s) \in X_a^F \times S_a^F \mid p \cdot x + q \cdot s \leqslant p \cdot e^F(a) + q \cdot T_a^F s\}.$$

Notice that the price vector (p, q) is independent of F. An *equilibrium allocation* is an attainable allocation (f, g) such that for some price vector (p, q), (f, g, p, q) is a sequential equilibrium. $W^s(\mathcal{E}^s)$ denotes the set of equilibrium allocations for $\mathcal{E}^s$.

Remark 1. Implicit in the definition of a sequential equilibrium is an assumption about the set of tradable commodities at each date-event pair; specifically, that at any date-event pair $u \in U$ agents can exchange commodities for delivery at *any* date-event pair $v \in F^u$. By way of contrast, in the definition of the Walras equilibrium it was implicitly assumed that there existed a unitary market at 0, that is, *all* trading was carried out at the initial date-event pair.

The definition of a sequential equilibrium is obviously motivated by the definition of the sequential core, bearing in mind the classical theorem on the equivalence of the core and the Walras allocations of a large economy. From the point of view of equilibrium theory, however, the definition seems a bit odd since most of the markets assumed to exist are superfluous. But the sequential equilibrium is a special case of a quite natural equilibrium concept, the *equilibrium of plans, prices and price expectations* of Radner, 1972. Assume that at each date-event pair only *spot trading* is allowed.[16] Then at any date-event pair $u \in U$, for any agent $a \in A$, the budget constraint is

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[16] This assumption does not appear to be essential. It is made here just to fix ideas and simplify the notation. If all trades are made on spot markets there is no need to distinguish between transactions and deliveries as far as notation is concerned.

$$p^u \cdot x^u + q^u \cdot s^u = p^u \cdot e^u(a) + q^u \cdot s^u .$$

Now if (f, g, p, q) is a sequential equilibrium then at each $u \in U$, for every $a \in A$,

$$p \cdot f(a) + q \cdot g(a) = p \cdot e^F(a) + q \cdot T_a{}^F g(a),$$

that is,

$$\sum_{t \geq u} p^t \cdot f^t(a) + q^t \cdot g^t(a) = \sum_{t \geq u} p^t \cdot e^t(a) + q^t \cdot g^{t^-}(a).$$

Since this equation holds for all $u \in U$ it clearly implies that

$$p^u \cdot f^u(a) + q^u \cdot g^u(a) = p^u \cdot e^u(a) + q^u \cdot g^{u^-}(a)$$

for every $u \in U$. Hence a sequential equilibrium is formally a special case of the Radner equilibrium (but the converse is obviously false).[17] The difference in interpretation hinges on the assumptions made about the existence of forward markets. In the sequential equilibrium interpretation we assume at each date-event pair $u \in U$ there exist forward markets for commodities deliverable at any $r \geq u$. In the Radner equilibrium interpretation only spot markets exist. For analytical purposes, the sequential equilibrium is more convenient but the Radner equilibrium is the more economically interesting concept. For this reason theorems stated in terms of the sequential equilibrium may be interpreted in terms of the Radner equilibrium.

7. THREE THEOREMS

The following two theorems describe the relationship between the sequential core and the set of equilibrium allocations. They mimic the classical theorems on the core and set Walras allocations of a exchange economy (see Hildenbrand [11, 2.1, Proposition 1, p. 131 and Theorem 1, p. 133]). They are stated and proved for the case in which the number of securities is $m = \#\overline{U}$ but this assumption is inessential; the theorems remain true if there were fewer or none at all, though in that case the sequential core might be empty.

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[17] The model presented in Radner, 1972, does not have securities of the sort used here. If the securities were removed the sequential equilibrium would be formally a special case of the Radner equilibrium but it would probably not exist. The treatment of securities is entirely in the spirit of Radner's model, however, particularly the way in which he handles stock markets, so there seems no harm in using the phrase "Radner equilibrium" in this slightly extended sense.

Theorem 1. *If $\mathscr{E}^s$ is a sequential economy then $W^s(\mathscr{E}^s) \subset C^s(\mathscr{E}^s)$.*

Theorem 2. *If $\mathscr{E}^s : (A, \mathscr{A}, \nu) \to \mathscr{P} \times \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^n$ is an atomless sequential economy such that $\int e \, d\nu \gg 0$ and $\int \sigma \, d\nu \gg 0$ then $C^s(\mathscr{E}^s) \subset W^s(\mathscr{E}^s)$.*

The proofs are gathered in the Appendix.

Theorem 2 is the more interesting result. It says that in the case of an atomless economy, (f, g) is trustworthy only if for some price vector (p, q) and any $F \in \mathscr{F}$ the budget constraint

$$p \cdot f(a) + q \cdot g(a) \quad p \cdot e^F(a) + q \cdot T_a{}^F g(a)$$

is satisfied. Thus, even if markets are complete at every date-event pair, the absence of trust induces a sequential structure on the economy. In a finite economy there are two opposing forces at work. A man who has promised to deliver a commodity would rather renege but on the other hand there may be "gains from trade" with his creditor. The gains from trade may be more important than the forward contract in which case he will keep his promise, even if there is no corresponding money transfer. In an atomless economy there are no gains from trade with an individual so every forward contract must be covered by money. This is just the "no surplus" condition at work.

Before stating the final theorem some new notation is needed. Let $\mathscr{E}$ denote an *exchange economy*, that is, a measurable function from $(A, \mathscr{A}, \nu)$ to $\mathscr{P} \times \mathbb{R}_+{}^l$ such that $\int e \, d\nu$ is finite. A sequential economy could equally well be denoted $\mathscr{E}^s$ or $(\mathscr{E}, \sigma, \tau)$. An exchange economy is said to be *convex* if for any $a \in A$, if $\{a\}$ is an atom (i.e. $\nu(\{a\}) > 0$) then $\succ_a$ is convex.

Theorem 3. *Let $\mathscr{E} : (A, \mathscr{A}, \nu) \to \mathscr{P} \times \mathbb{R}_+{}^l$ be a convex exchange economy and f an efficient consumption allocation for $\mathscr{E}$. Suppose $\int e \, d\nu \gg 0$. Then there exist functions σ, τ and an attainable security allocation g for $\mathscr{E}^s := (\mathscr{E}, \sigma, \tau)$ such that* (i) *$\mathscr{E}^s$ is a sequential economy* (ii) $(f, g) \in C^s(\mathscr{E}^s)$.

It is essential for the validity of this theorem that there be at least $m = \#\overline{U}$ securities.

This theorem is proved by showing that for some triple (g, σ, τ), (f, g) is an equilibrium allocation for $(\mathscr{E}^s, \sigma, \tau)$ and then invoking Theorem 1. f can be any efficient allocation. This is reminiscent of the classical theorem that with suitable transfers of income any efficient allocation is a Walras allocation for a convex economy. In this case, the "income transfers" are represented by the net taxes $\tau(a) - \sigma(a)$. We are most interested in the case in which $\tau(a) - \sigma(a) = 0$, a.e. in A. In this case, since the actual values of $\sigma(a)$ are unimportant, the government requires no information. The process of generating trustworthiness can be completely decentralized. If f is a Walras allocation then there is no loss of generality in supposing $\sigma(a) =$

$\tau(a)$, a.e. in A. Let $W(\mathscr{E})$ (resp. $C(\mathscr{E})$) denote the set of Walras allocations (resp. the classical core) of an exchange economy $\mathscr{E}$. Then we have the following simple result.

COROLLARY TO THEOREM 3. *Let $\mathscr{E}$ and f be as in Theorem 3. Then if $f \in W(\mathscr{E})$ we can, without loss of generality, assume that $\sigma = \tau$. The same is true if $f \in C(\mathscr{E})$ and $\mathscr{E}$ is an atomless exchange economy.*

8. APPLICATIONS

There is a great variety of models of sequence economies in the literature. A small and possibly unrepresentative sample might include models of pairwise trade ([13, 14, 15, 16]), models of equilibrium with incomplete market structures ([17], further studied in [5, 6, 9]), models of equilibrium with transaction costs ([7, 8, 10, 12, 18, 19]), and models of temporary equilibrium ([2, 3, 4] and many others). What all these have in common is a "sequence of budget constraints". The presence of these constraints is usually taken as axiomatic but it is not always clear whether their imposition is justified. As was pointed out in the introductory remarks to this paper, one of the purposes of analysing economies without trust is to develop a theory which may justify the presence of a sequence of budget constraints. It appears now that the form of the constraints which arise as "equilibrium conditions" from the theory of the sequential core differs from that found in models of equilibrium in sequence economies. The interesting question of whether the theory developed in preceding sections provides a rationale for these models, or more precisely for the preferred definition of equilibrium, must be tentatively answered in the negative, at least in some cases. In other cases the correspondence is more satisfactory but even where no rationale has been provided an important lesson may be learned—that some theories are in serious need of revision. There are considerable problems in applying the precise results developed in earlier sections to most of these models, for the simple reason that they are structurally quite different from the sequential economy described in Section 3. Nonetheless, the general concepts described in this paper can be used to gain insights into the meaning and validity of the models. In this section, three models, those of Ostroy-Starr [16], Hahn [7, 8], and Radner [17] are studied in this way. The remarks that follow are not intended to do more than loosely indicate the connections.

Ostroy-Starr

Ostroy and Starr study the problem of reaching a Walras allocation by means of pairwise trades. Unlike the sequential economy the Ostroy-Starr model has only one time period (in my sense) but within this single time period agents trade sequentially with each other. It is a model with one Hicksian *week* divided into several trading *days*. There is a finite number of commodities and agents, the latter indexed $i = 1, 2, \ldots, m$. During each day of the week each agent meets another agent, except possibly for an odd-man-out who does not meet anyone. The agents trade commodities which they carry forward from day to day. There are given an equilibrium price vector p and an m-tuple (z_i) of Walrasian excess demands, one for each agent i. Each agent i has an endowment vector e_i. The problem is how to get each agent from e_i to $(e_i + z_i)$ by means of pairwise exchanges. This problem is rather different from the problem of trust studied earlier. In this case it is not a question of agents failing to keep agreements once some trades have been carried out; instead the problem is that agents must decide how much to trade before they have met all other agents. The problem is essentially one of "speculation".

There is assumed to be a finite number of trading periods indexed $t = 1, \ldots, T$. The order of meetings is represented by a T-tuple of permutations $\{\pi^t\}$ on the set of agents $\{1, \ldots, m\}$, satisfying

(a) $\pi^t \circ \pi^t = id$ for each $t = 1, \ldots, T$;

(b) $\forall i, j = 1, \ldots, m, i \neq j, \exists t = 1, \ldots, T, \pi^t(i) = j$;

(c) $\exists$ at most one i such that $i = \pi^t(i)$, for each $t = 1, \ldots, m$.

The interpretation is that agent i meets agent j on date t if and only if $\pi^t(i) = j$. Let e_i^t denote the bundle of commodities held by agent i at the beginning of day t. Ostroy-Starr are interested in the informational requirements of pairwise exchange. To formalize the idea of "informational requirements" they introduce the concept of a *trading rule*, a function mapping past and present excess demands and the pairing sequence $\{\pi^t\}$ onto today's actual (pairwise) trades. A *centralized* trading rule is one which uses all of this information. Ostroy-Starr show that if T is the minimum number of periods required for every pair to meet at least once then there exists a centralized, pairwise trading rule such that

(i) $e_i^1 = e_i$ and $e_i^T = e_i + z_i$ for all $i = 1, \ldots, m$;

(ii) $p \cdot e_i^t = p \cdot e_i^{t+1}$ for each $i = 1, \ldots, m$ and $t = 1, \ldots, T$;

(iii) $e_i^t \geqslant 0$ for all $i = 1, \ldots, m$ and $t = 1, \ldots, T$.

The question that interests us is why should (ii) be satisfied? Without (ii) the problem is trivial. If agents trust one another there is no reason to impose (ii) but in the absence of trust it is easy to see why it must be satisfied.

The Walras allocation obviously belongs to the (classical) core when agents have the endowments (e_i). What is needed is some condition that will also ensure it belongs to the (classical) core when the endowments are (e_i^t), for each $t = 1, \ldots, T$. Clearly (ii) is just such a condition. To see this, notice that if $((x_i), p)$ is a Walrasian equilibrium, where $x_i := (e_i + z_i)$, when agents hold endowments (e_i), then it is a Walrasian equilibrium for any other distribution of endowments, (e'_i) say, such that each agent's wealth is unchanged, i.e. $p \cdot e_i \quad p \cdot e'_i$. Hence, for every $t \quad 1,\ldots, T$ if (ii) is satisfied then $(e_i + z_i)$ is a Walras allocation relative to the endowments (e_i^t). Since a Walras allocation belongs to the core this means that the sequence of trades is, in an obvious sense, trustworthy.

To show the converse, that trustworthiness implies condition (ii), one would have to assume that the economy is atomless, but it is not difficult to see how standard arguments could be deployed to this end. Ostroy-Starr also discuss cases in which a *decentralized* rule will allow agents to reach a Walras allocation by means of pairwise exchanges satisfying conditions similar to (i)-(iii) above. One of these involves the use of money. These cases could be analysed in a similar way.

There is a difficulty with the Ostroy-Starr model. Although trade is carried out pairwise, agents somehow come to know the equilibrium price vector before trade has even begun. A similar problem arises from the interpretation of (ii) as reflecting lack of trust. Lack of trust was previously identified with the feeling that (other) agents would break their promises if it was to their advantage. But here agents trade only once with any particular partner. It is not clear what could or should count as a broken agreement.

Both aspects of the model require a certain amount of sympathy to make sense of them. It must be assumed that the "auctioneer" is able to call out correct, equilibrium prices from his knowledge of aggregate excess demands, but unable to co-ordinate individual, pairwise exchanges. Similarly, in applying a concept similar to the sequential core it is assumed, on any day t, that any coalition can decide, without meeting, to seek a different set of excess demands (different, that is, from those corresponding to the original Walrasian equilibrium) as long as it remains self-sufficient. And this ignores the problem of reaching the desired allocation by means of pairwise exchanges. For example, some members of the coalition may not meet, according to the original order of meeting represented by $\{\pi^t\}$, after the date on which the coalition "forms". It seems to be asking a lot to appeal to the "auctioneer" to provide the necessary information, bearing in mind the fact that the model is essentially about an economy in which agents have very limited ability to communicate. These problems are inherent in the model's dichotomy between the choice of the final allocation (Walrasian equilibrium) and the trading process by which it is reached. Subject to this qualification the absence of trust does seem to provide some sort of rationale for condition (ii).

Hahn [7, 8], *Radner* [17]

When we come to models of equilibrium with transaction costs it is more difficult to relate the present theory to the literature. The main reason is that, by construction, an allocation in the sequential core is Pareto efficient, whereas equilibrium allocations are rarely efficient in sequence economies. An exception is the model of Starrett [19]. He gives an example of an equilibrium with transaction costs which is not merely Pareto inefficient but inefficient among allocations which are attainable through the transaction technology, i.e. constrained inefficient. He then shows that the inefficiency results from the presence of budget constraints at each date and that if each agent has access to costless credit facilities (so that there is effectively only one budget constraint) then equilibrium is always constrained efficient. Clearly, the same will be true if agents have sufficiently large holdings of money which they must return to the government at the last date. In that case, if spot markets are costless, the Starrett model is a special case of the model of this paper. Only spot markets are used as long as agents have sufficient credit facilities or money balances, so there is no need to worry about the transaction costs of using forward markets. What the Starrett model does not explain, of course, is why the budget constraints should be there in the first place. The absence of trust explains this feature of the model, as we have seen, but it should be noticed that there is a difference between money and credit here. If the government enforces the payment of taxes but not the payment of debts, only money will guarantee trustworthiness.

The Starrett model has an attractive simplicity. The general case is not so pretty. In general, the set of attainable allocations is limited by the transaction technology. The model described in this paper has an attainable set unrestricted by any constraints other than the size of the total resource endowment but even if we introduced costs of coalition formation there is no reason to think this would yield a comparable attainable set. "Market trading" and "coalition formation" are two quite different methods of allocating resources and there is, in general, no reason why the respective sets of attainable allocations should coincide. For this reason I believe it would be misguided to attack the problem by looking for a definition of the sequential core that will coincide with the set of equilibrium allocations of the Hahn model, for instance.

Nonetheless, I think there are insights to be gleaned from the present model, in which we ignore the costs of getting information, forming coalitions and so on, which can be applied to descriptive models with incomplete markets [17] or transactions costs [8]. The most important insight comes from the recognition that trustworthiness (in the present model) requires that the budget constraint at any date-event pair u applies to *net trades* and covers all date-event pairs in F^u. In the two papers referred to above it is assumed that the budget constraint at a date-event pair applies to the gross trades arranged at that date-event pair.

The distinction between what I have called *net* and *gross* trades can be made clearer by looking at a simple example of an exchange economy of the

Radner-type. There are two dates $t = 0, 1$ and no uncertainty. Each agent $i = 1, \ldots, m$ chooses a trade plan $z_i = (z_i(0), z_i(1))$ subject to the constraints

(i) $z_i(t) \in Z(t)$

(ii) $e_i + \sum_t z_i(t) \geq 0$

(iii) $\hat{p}(t) \cdot z_i(t) \leq 0$

where e_i is the i-th agent's endowment, $Z(t)$ is the subspace of commodities which can be traded at date t and $\hat{p}(t)$ is the vector of prices for commodities traded at date t. The planned consumption corresponding to the trade plan z is simply $[e_i + \sum_t z_i(t)] = x_i$. The *net trades* here are represented by the vector $(x_i - e_i)$. The *gross trades* at date t are denoted by $z_i(t)$. Writing $x_i = (x_i(0), x_i(1))$ and $e_i = (e_i(0), e_i(1))$ the net trades at date t are $(x_i(t) - e_i(t))$ and the budget constraints suggested by the sequential core have the form

(iii)′ $\sum_t p(t) \cdot (x_i(t) - e_i(t)) \leq 0,\ p(1) \cdot (x_i(1) - e_i(1)) \leq 0$

where $p := (p(0), p(1)) \in \mathbb{R}^l$. The vectors $(p(t))$ must be distinguished from $(\hat{p}(t))$; $p(t)$ is a vector of prices of commodities delivered at date t whereas $\hat{p}(t)$ is a vector of prices of commodities traded at date t. An equilibrium subject to these new constraints is *inessential* (in the sense of Hahn [8]) by definition. The constraints (iii)′ imply

(iii)″ $p(t) \cdot (x_i(t) - e_i(t)) \leq 0 \ (t = 0, 1)$.

If constraints of the form (iii)″ are accepted as part of the *definition* of equilibrium in place of (iii)′ then there is a natural extension of the equilibrium concept to include essential sequence equilibria. Suppose there is a complete set of spot markets at each date and some futures markets exist at date 0. Let q denote a vector of futures prices and y_i the i-th agent's vector of trades in futures markets. Then the i-th agent's budget constraints can be written

(iii)‴ $p(0) \cdot (x_i(0) - e_i(0)) \leqslant 0$

$p(1) \cdot (x_i(1) - e_i(1)) + (p^1 - q) \cdot y_i \leqslant 0$

where p^1 is the vector of date 1 prices of commodities which could be traded on a futures market at date 0. Obviously, p^1 is a "subvector" of $p(1)$. In this case we can identify $(p(0), q)$ with $\hat{p}(0)$ and $p(1)$ with $\hat{p}(1)$. A comparison of (iii) and (iii)‴ gives the flavour of the distinction, alluded to above, between constraints on gross and net trades.

It is harder to relate the present theory to Hahn's model of equilibrium with transaction costs because he introduces different buying and selling prices for the same commodities. But Hahn's constraints are very much in the Radner spirit and subject to similar remarks.

In other words, if an agent buys wheat on Monday for delivery on Thursday and then sells the same wheat on Tuesday, these two transactions cannot be used to offset one another. This seems odd on two counts. First, it assumes the presence of trust to the extent that forward sales today can be set against spot purchases today. Second, it does not allow apparently trustworthy arrangements such as the following: on Monday an agent purchases a quantity of wheat for delivery Thursday without making any corresponding sale; on Tuesday he sells an appropriate quantity of rye for delivery on Thursday without making a corresponding purchase. In this way the traditional assumption seems to go too far and not far enough in trying to ensure trustworthiness. And a side glance at reality (e.g. commodity futures markets) shows that the constraint is not strictly accurate empirically either.

It could be argued that every forward purchase automatically entails a commitment to sell money forward (this applies only to the Hahn model since Radner's does not contain money) and that in this sense the constraint is both natural and implied by the absence of trust. But such an argument is fallacious because it ignores the distinction between selling money forward and delivering money at a later date. For example, if I make a forward purchase I do not necessarily have to sell money forward to pay for it; I may simply agree to pay cash on delivery. In the Hahn model this could be done costlessly in one of two ways: either by carrying cash balances forward from one period to the next or by selling another commodity on a spot market (if there existed a costless one) on the date the first commodity was to be delivered. But the budget constraint rules out either of these two procedures. Suppose that an agent wishes to purchase wheat on Monday for delivery on Thursday. He cannot pay for it in cash on Thursday using cash balances carried forward costlessly from Monday; he must make a sale, e.g. sell Thursday money forward, even though this is costly and the first procedure is evidently trustworthy.

These remarks cannot be conclusive, of course, in the absence of formal results, but they ought to give pause for thought especially since, as Starrett [19] shows, the form of the constraints is crucial to the inefficiency results of Hahn [8].

One implication of the constraints suggested in this paper is that bond markets (forward markets in money) would never be needed. This is because the constraints do not allow the transfer of wealth between periods except by means of securities. But a more sophisticated model would allow for the enforcement of contracts by other means. If it were costly to enforce forward trades in money this would provide a rationale for [8, Theorem 6], which shows that if forward money markets are costly but spot markets are not, *then equilibrium (in his sense) is inefficient if any forward market is active. The theorem does not appear to depend on the precise form of the budget constraints.*

APPENDIX

This appendix contains proofs of the theorems stated in Section 9. The techniques used to prove Theorems 1 and 2 are extensions of the standard methods used in the analysis of the classical core (cf. Hildenbrand [*11*, 2.1]). The only mathematical novelties arise from the treatment of security prices. As far as possible I have tried to follow Hildenbrand [11] in notation and have in several places referred the reader to arguments or definitions in his book rather than reproduce them here. Theorem 3 is a different kind of result but again uses an extension of standard arguments.

THEOREM 1. *Let $\mathscr{E}^s$ be a sequential exchange economy. Then $W^s(\mathscr{E}^s) \subset C^s(\mathscr{E}^s)$.*

Proof. Let $(f, g) \in W^s(\mathscr{E}^s)$ and $(p, q) \in \mathbb{R}^l \times \mathbb{R}^n$ be the associated price vector. For every $F \in \mathscr{F}$ define the F-subeconomies $(\mathscr{E}^F)$ in the usual way. If $(f, g) \notin C^s(\mathscr{E}^s)$ then for some $F \in \mathscr{F}$, $(f, g) \notin C(\mathscr{E}^F)$. There exists a coalition $E \in \mathscr{A}$ with $\nu(E) > 0$ and a consumption allocation h such that

$$h(a) \succ_a f(a) \text{ for all } a \in E, \int_E h\,d\nu = \int_E e^F\,d\nu$$

and if u is the minimal element of F then

$$\int_E (t^u(a) - g^{u-}(a))\,d\nu \leq 0$$

Then for every $a \in E$, $p \cdot h(a) > p \cdot f(a)$ and since $\int_E h\,d\nu = \int_E e^F\,d\nu$, $p \cdot \int_E f\,d\nu < p \cdot \int_E e^F\,d\nu$. By definition, however, $p \cdot (f(a) - e^F(a)) + q \cdot (g(a) - T_u{}^F g(a)) = 0$ so

$$q \cdot \int_E (g(a) - T_u{}^F g(a))\,d\nu > 0,$$

contradicting the hypothesis that E is self-sufficient in securities. To see this, note that for any $s \in S_u{}^F$, $a \in A$,

$$q \cdot (g(a) - T_a{}^F g(a)) \leqslant q \cdot (s - T_a{}^F s)$$

because otherwise the pair $(f(a), g(a))$ would not be maximal. In particular, if s is defined by $s \in S_a^F$, $s^v = t^u$, for all, $v \in F$ then

$$q^u \cdot (t^u(a) - g^{u-}(a)) \geq q \cdot (g(a) - T_a^F g(a))$$

whence

$$0 \geq \int_E q^u \cdot (t^u(a) - g^{u-}(a))\, d\nu \geq \int_E q \cdot (g(a) - T_a^F g(a))\, d\nu. \quad \blacksquare$$

An attainable allocation (f, g) for $\mathscr{E}^s$ is called *degenerate* if there exists a price vector $q \in \mathbb{R}^n$, $q > 0$ such that for every $F \in \mathscr{F}$

$$\text{a.e. in } A,\ q \cdot g(a) = q \cdot T_a^F g(a) \text{ and } q \cdot s \geq q \cdot T_a^F g(a) \text{ for any } s \in S_a^F.$$

Otherwise an allocation is called *non-degenerate*.

Let Z^F denote the F-subspace of $\mathbb{R}^l \times \mathbb{R}^n$: Z^F is the linear subspace of $\mathbb{R}^l \times \mathbb{R}^n$ carrying vectors $z \in \mathbb{R}^l \times \mathbb{R}^n$ such that if $z = (x, s)$ then $x^u = 0$ for any $u \notin F$ and $s^u = 0$ for any $u \notin F$. If $\mathbb{R}^l \times \mathbb{R}^n$ is thought of as the vector space carrying the *actions*[18] of agents in $\mathscr{E}$, then Z^F can be thought of as the vector space carrying the actions of the agents in the sub-economy $\mathscr{E}^F$.

THEOREM 2. *If* $\mathscr{E}^s : (A, \mathscr{A}, \nu) \to \mathscr{P} \times \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^n$ *is an atomless sequential economy such that* $\int e\, d\nu \gg 0$ *and* $\int \sigma\, d\nu - \int \tau\, d\nu \gg 0$ *then* $C^s(\mathscr{E}^s) \subset W^s(\mathscr{E}^s)$.

Proof. Suppose that $(f, g) \in C^s(\mathscr{E}^s)$ is a non-degenerate allocation. For every agent $a \in A$ and every component $F \in \mathscr{F}$ define

$$\succ_a^F(f) := \{x \in X_a^F \mid x \succ_a^F f(a)\}$$

and

$$\chi^F(a) := \succ_a^F(f) - \{e^F(a)\}$$

For every agent $a \in A$ and component $F \in \mathscr{F}$ define

$$\zeta^F(a) := S_a^F - \{T_a^F g(a)\}$$

where T_a^F is the shift operator defined in Section 6 and S_a^F is the security set of agent a in the F-subeconomy. For every $a \in A$, put $\psi^F(a) := (\chi^F(a) \times \zeta^F(a)) \cup \{0\}$ for each $F \in \mathscr{F}$. Notice that if z is a vector belonging to $\psi^F(a)$ and $z = (x, s) \in \mathbb{R}^l \times \mathbb{R}^n$, then $x^u = 0$ and $s^u = 0$ for all $u \notin F$, that is, $\psi^F(a)$ is contained in the linear subspace Z^F of $\mathbb{R}^l \times \mathbb{R}^n$.[19]

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[18] Actions here may be identified with excess demands for commodities and securities.

[19] This follows from the definition of a subeconomy. Cf. Section 4.

$$\text{Let } \Omega := \{z \in \mathbb{R}^l \times \mathbb{R}^n \mid z \ll 0\} \text{ and let}$$

$$C := co \bigcup_{F \in \mathscr{F}} \left\{ \int \psi^F \, d\nu \right\}^{20}$$

We need to show that $C \cap \Omega = \emptyset$. Suppose not. Then there exists an array $\{\lambda^F : F \in \mathscr{F}\}$ of non-negative numbers such that $\sum \lambda^F = 1$ and a vector $z \in \Omega$ such that

$$z \in \sum_{F \in \mathscr{F}} \lambda^F \int \psi^F \, d\nu.$$

There exists a set $G \in \mathscr{F}$ such that $F \in \mathscr{F}$ and $F \subset G$, $F \neq G$, imply $\lambda^F = 0$. Let G be a maximal set of this kind, so that $\lambda^G \neq 0$. Let $(\mathscr{F} \mid G)$ denote the subfamily of $\mathscr{F}$ defined by

$$(\mathscr{F} \mid G) := \{F \in \mathscr{F} \mid F \subset G\}.$$

The sequential structure of $(u^{\cdot})$ ensures that if $F \in (\mathscr{F} \mid G)$ then either $F \cap G = \emptyset$ or $G \subset F$. Let proj_G denote the natural projection from $\mathbb{R}^l \times \mathbb{R}^n$ into Z^G. Then

$$\begin{aligned}
\mathrm{proj}_G \left[\mathrm{co} \bigcup_{F \in (\mathscr{F} \mid G)} \int \psi^F \, d\nu \right] &= \mathrm{co} \left[\mathrm{proj}_G \bigcup_{F \in (\mathscr{F} \mid G)} \int \psi^F \, d\nu \right] \\
&= \mathrm{co} \left[\bigcup_{F \in (\mathscr{F} \mid G)} \mathrm{proj}_G \int \psi^F \, d\nu \right] \\
&= \mathrm{co} \left[\bigcup_{F \in (\mathscr{F} \mid G)} \int \mathrm{proj}_G \, \psi^F \, d\nu \right] \\
&\subset \mathrm{co} \int \psi^G \, d\nu = \int \psi^G \, d\nu
\end{aligned}$$

The first three equations follow from the definitions, the inclusion relationship follows from the construction, since for any $F \in (\mathscr{F} \mid G)$

$$\mathrm{proj}_G \, \psi^F(a) \subset \psi^G(a)$$

for every $a \in A$, and the last equation is a theorem since $(A, \mathscr{A}, \nu)$ is an atomless economy (Hildenbrand, 1974, D.II.4, Theorem 3, p. 62) and by construction for any $F \in (\mathscr{F} \mid G)$ $\mathrm{proj}_G \, \psi^F(a) \subset \psi^G(a)$ for every $a \in A$. Then $\mathrm{proj}_G(z) \in \int \psi^G \, d\nu$. Let h be a measurable selection from ψ^G such that $\int h \, d\nu = \mathrm{proj}_G(z)$ and let $E := \{a \in A \mid h(a) \neq 0\}$. Then

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[20] For any subset Y of a Euclidean space the notation *co Y* denotes the *convex hull* of Y.

$\nu(E) > 0$ and $\int_E g^{u-}\,d\nu \geq \int_E t^u\,d\nu$ where u is the initial element of G.[21] Corresponding to h is a measurable function $h^* : A \to \mathbb{R}^l$ such that a.e. in E, $h^*(a) \in \succ_a^G(f) - \{e^G(a)\}$ and $\int_E h^*\,d\nu \leq 0$. Putting

$$f^*(a) = h^*(a) + e^G(a) - \int_E h^*\,d\nu/\nu(E)$$

we have an attainable allocation (f^*, g) such that $f^*(a) \succ_a^G f(a)$ for $a \in E$, contradicting the hypothesis that $(f, g) \in C(\mathscr{E}^G)$.

Hence, $co[\bigcup_F \int \psi^F\,d\nu] \cap \Omega = \emptyset$. Also, $0 \in \int \psi^F\,d\nu$ for all $F \in \mathscr{F}$ so $co[\bigcup_F \int \psi^F\,d\nu] \neq \emptyset$. By the Minkowski theorem there exists a hyperplane separating the two sets. Precisely there exists a price vector $(p, q) \in \mathbb{R}_+^l \times \mathbb{R}_+^n$ such that $(p, q) \neq 0$ and

$$(1) \qquad 0 \leq \inf\{(p, q) \cdot C\}$$

where $C := \mathrm{co}[\bigcup_F \int \psi^F\,d\nu]$.

The graph of the correspondence ψ^F is measurable for every $F \in \mathscr{F}$. Clearly it is sufficient to show that the correspondences χ^F and ζ^F have measurable graphs. The first follows from an argument in Hildenbrand [11, 2.1, Theorem 1, bottom of p. 134]. The graph of ζ^F is the set

$$\{(a, s) \in A \times \mathbb{R}^n \mid s \in S_a^F - \{T_a^F g(a)\}\}.$$

Since T_a^F is continuous for fixed $a \in A$ and g is measurable $T_a^F g(a)$ defines a measurable function on A to $\mathbb{R}^n$. By Hildenbrand [11, D.II.3, Proposition 1, p. 59] it is sufficient to show that S_a^F, regarded as a correspondence from A to $\mathbb{R}^n$ has a measurable graph. Now $S_a^F = \varphi \circ h(a)$ where h is a function from A to $\mathbb{R}^n \times \mathbb{R}^m$ defined by $h(a) = (g(a), t(a))$ for every $a \in A$, and φ is the

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[21] Let $s(a)$ denote the security holdings of agent $a \in E$ which correspond to the aggregate vector z. By hypothesis, $s(a) \in S_a^G$. Since $z \in \Omega$ it follows that

$$\int_E [s^u - s^{u-}]\,d\nu \leq 0, \text{ if } u \in G,$$

and

$$\int_E (s^{\bar{u}} - r^{\bar{u}})\,d\nu \geq 0, \text{ if } \bar{u} \in \bar{U} \cap F.$$

Also, if u is first in G then

$$\int_E s^{u-} d\nu = \int_E g^u\,d\nu.$$

From these relationships it easily follows that

$$\int_E g^{u-} \geq \int_E t^u.$$

correspondence from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ defined by $\varphi(s, t) = \{y \in \mathbb{R}_+^n \mid y^v \geq s^v$ if $v \notin F$ and $y^u \geq t^u(a)$ if $u \in \bar{U}\}$. h is measurable and φ has a closed (and hence measurable) graph so $\varphi \circ h$ has a measurable graph [11, D.II. 3, Proposition 1, p. 59].

From (1) it obviously follows that

$$(2) \qquad 0 \leq \inf(p, q) \cdot \int \psi^F \, d\nu \text{ for each } F \in \mathscr{F}.$$

Since ψ^F has a measurable graph it follows [11, D.II.4, Proposition 6, p. 63] that

$$\inf(p, q) \cdot \int \psi^F \, d\nu = \int \inf(p, q) \cdot \psi^F \, d\nu$$

so from (2): $0 \leq \int \inf(p, q) \cdot \psi^F \, d\nu$. By definition $0 \in \psi^F(a)$ for every $a \in A$ so $0 \leq \inf(p, q) \cdot \psi^F(a)$, a.e. in A. Thus

(3) a.e. in A, $(p, q) \cdot (e^F(a), T_a^F g(a)) \leq (p, q) \cdot (x, s)$ for any pair (x, s) such that $x \succ_a^F f(a)$, $s \in S_a^F$.

From (3) it follows that a.e. in A, $p \cdot e^F(a) + q \cdot T_a^F g(a) \leq p \cdot f(a) + q \cdot g(a)$. Now suppose that the inequality is strict for a set of agents with positive measure. Then $p \cdot \int e^F \, d\nu + q \cdot \int T_a^F g \, d\nu < p \cdot \int f \, d\nu + q \cdot \int g \, d\nu$, contradicting the hypothesis that (f, g) is attainable. Then a.e. in A, $p \cdot e^F(a) + q \cdot T_a^F g(a) = p \cdot f(a) + q \cdot g(a)$.

Suppose that $p = 0$ and $q > 0$. Then $q \cdot g(a) = q \cdot T_a^F g(a)$ for every $F \in \mathscr{F}$ and $q \cdot s = q \cdot T_a^F g(a)$ for all $s \in S_a^F$ for every $F \in \mathscr{F}$, a.e. in A. But this means (f, g) is degenerate, a contradiction, so $p > 0$. Since $\int e \, d\nu > 0$, $p > 0$ implies that $\nu\{a \in A \mid p \cdot e(a) > 0\} > 0$. By a well-known argument (see [11, 2.1, Theorem 1, p. 135]) for any agent a such that $p \cdot e(a) > 0$, property (3) implies that $(f(a), g(a))$ is maximal for $\succ_a$ in the budget set

$$\{(x, s) \in \mathbb{R}_+^l \times S_a \mid p \cdot x + q \cdot s \leq p \cdot e(a) + q \cdot T_a s\}$$

and this, together with monotony of preferences, implies that $p \gg 0$. Then for any $F \in \mathscr{F}$, (3) implies that $(f(a), g(a))$ is maximal in the budget set

$$\{(x, s) \in X_a^F \times S_a^F \mid p \cdot x + q \cdot s \leq p \cdot e^F(a) + q \cdot T_a^F s\}.$$

If for some $F \in \mathscr{F}$ and some pair (x, s) in the budget set $x \succ_a^F f(a)$ then clearly $q \cdot (s - T_a^F s) \geq q \cdot (g(a) - T_a^F g(a))$ since $p \cdot (f(a) - e^F(a)) + q$ $(g(a) - T_a^F g(a)) = \inf(p, q) \cdot \psi^F(a) = 0$. This implies that $p \cdot x \leq p \cdot f(a)$. Since $x \succ_a^F f(a)$ it must be the case that $p \cdot x = p \cdot f(a) = \inf p \cdot X_a^F$. Then $p > 0$ implies $x = f(a)$, a contradiction.

This completes the proof that $(f, g) \in W^s(\mathscr{E}^s)$ under the assumption that (f, g) is non-degenerate. Suppose now that (f, g) is degenerate and let $\bar{q} \in \mathbb{R}^n$, $\bar{q} > 0$ be the price vector referred to in the definition. For some $\bar{u} \in \bar{U}$ and $u \in U$, $\bar{q}_{\bar{u}}^u > 0$ and there is no loss of generality in supposing that $u < \bar{u}$ or $u = \bar{u}$. Indeed, if we define $H^{\bar{u}} = \{u \in U \mid u < \bar{u} \text{ or } u = \bar{u}\}$ for each $\bar{u} \in \bar{U}$, then if $u \notin H^{\bar{u}}$ there is no tax constraint for $\bar{u}$-securities in the definition of S_a^F, where $F = F^u$. Let u be a maximal date-event pair not in $H^{\bar{u}}$ such that $q_{\bar{u}}^u > 0$. Define $s \in S_a^F$ by $s_{\bar{v}}^v = g_{\bar{v}}^v(a)$ if $\bar{v} \neq \bar{u}$ or if $\bar{v} = \bar{u}$ and $v < u$ and by $s_{\bar{v}}^v = 0$ otherwise. Let $E = \{a \in A \mid g_{\bar{u}}^u(a) > 0\}$. Then it is clear that $s \in S_a^F$ where $F = F^u$ and $a \in E$ and that $\bar{q} \cdot s < \bar{q} \cdot g(a)$, a contradiction. So suppose instead that $u \in H^{\bar{u}}$, indeed let u be a maximal $u \in H^{\bar{u}}$ such that $\bar{q}_{\bar{u}}^u > 0$. If $u \neq \bar{u}$ then by the previous argument it is possible to find a set of agents $E \in \mathscr{U}$ with positive measure such that for each $a \in E$ there is a point $s \in S_a^F$, where $F = F^u$, such that $\bar{q} \cdot s < \bar{q} \cdot g(a)$ (if $\bar{u}$-securities are a free good at $\bar{u}$ then the tax constraint is irrelevant to the problem "minimize $\bar{q} \cdot s$ subject to $s \in S_a^F$"). But if $q_{\bar{u}}^{\bar{u}} > 0$ then for *any* $u \in H^{\bar{u}}$, $\bar{q}_{\bar{u}}^u > 0$. For if $u \in H^{\bar{u}}$ and $q_{\bar{u}}^u = 0$, define $s \in S_a^F$, where $F = F^u$, by $s_{\bar{v}}^v = g_{\bar{v}}^v(a)$ for any $\bar{v} \neq \bar{u}$ or if $\bar{v} = \bar{u}$, for any $v < u$, by $s_{\bar{u}}^v = g_{\bar{u}}^v(a) + \epsilon$ for $v = u$ or $u < v < \bar{u}$ and $s_{\bar{u}}^{\bar{u}} = g_{\bar{u}}^{\bar{u}}(a)$, for any $a \in A$. Obviously $\bar{q} \cdot s < \bar{q} \cdot g(a)$ for all $a \in A$.

The preceding argument proves that for $\bar{u} \in \bar{U}$ *either* $\bar{q}_{\bar{u}}^u > 0$ for all $u \in H^{\bar{u}}$ *or* $\bar{q}_{\bar{u}}^u = 0$ for all $u \in H^{\bar{u}}$. Let $\bar{q}$ have a maximal set of positive prices, that is, there is no other separating price vector q with a strictly larger set of securities $\bar{u} \in \bar{U}$ such that $q_{\bar{u}}^u > 0$ for all $u \in H^{\bar{u}}$. Let $U^* = \{\bar{u} \in \bar{U} \mid \bar{q}_{\bar{u}}^u = 0, u \in H^{\bar{u}}\}$. Let L be the linear subspace of $\mathbb{R}^l \times \mathbb{R}^n$ defined by the condition $(p, q) \in L$ if and only if $(p, q) \in \mathbb{R}^l \times \mathbb{R}^n$ and $q_{\bar{u}}^u = 0$ if $\bar{u} \notin U^*$ for any $u \in U$. Now apply the argument used in the non-degenerate case[22] but apply the separating hyperplane theorem to the linear subspace L. Then there exists a price vector $(p, q) \in \mathbb{R}^l \times \mathbb{R}^n$ such that $(p, q) > 0$ and if $\bar{u} \notin \bar{U}^*$ then $q_{\bar{u}}^u = 0$ for all $u \in U$. Suppose $p = 0$. Then for any $F \in \mathscr{F}$ and a.e. in A,

$$q \cdot s \geq q \cdot g(a) = q \cdot T_a^F g(a)$$

but then it is easy to see that $\bar{q}$ is not maximal since $\bar{q} + q$ will have a larger set of securities $\bar{u} \in \bar{U}$ such that $(\bar{q}_{\bar{u}}^u + q_{\bar{u}}^u) > 0$ for $u \in H^{\bar{u}}$. So $p > 0$. Furthermore, it is not difficult to show that

$$0 \leq \inf\{(p, q + \bar{q}) \cdot C\} \tag{1)'$$

where $C = \mathrm{co}[\bigcup_F \int \psi^F \, d\nu]$. The rest of the proof proceeds as before with $(p, q + \bar{q})$ in place of (p, q). This completes the proof of the theorem. ▮

THEOREM 3. *Let $\mathscr{E} : (A, \mathscr{U}, \nu) \to \mathscr{P} \times \mathbb{R}^l$ be a convex exchange economy such that $\int e \, d\nu \gg 0$ and let f be an efficient consumption allocation for $\mathscr{E}$.*

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[22] i.e. show that $C \cap \Omega = \varnothing$ where $C := \mathrm{proj}_L \, [\mathrm{co} \bigcup_F \int \psi^F \, d\nu]$.

There exist integrable functions $\sigma, \tau: (A, \mathcal{U}, \nu) \to \mathbb{R}_+^m$ *and a security allocation* $g: (A, \mathcal{U}, \nu) \to \mathbb{R}_+^n$ *such that* (i) $\mathcal{E}^s := (\mathcal{E}, \sigma, \tau)$ *is a sequential economy and* (ii) $(f, g) \in C^s(\mathcal{E}^s)$.

Proof. In the statement of the theorem it is understood that there is a set of date-event pairs $(U, \leq)$ associated with $\mathcal{E}$, that $n := m \cdot \# U$ and that there are $m := \# \overline{U}$ securities.

Let τ be a ν-integrable function on A to $\mathbb{R}_+^m$ such that for all $a \in A$, $\tau(a) = (1)$. That is, each agent has an obligation to pay one unit of security u in the (terminal) date-event pair $u \in \overline{U}$. Define a set S in $\mathbb{R}^n$ by putting

$$S := \{s \in \mathbb{R}_+^n \mid s_u^u \geq 1 \text{ for all } u \in \overline{U}\}$$

and define the shift operator T on $\mathbb{R}^n$ to $\mathbb{R}^n$ by putting

$$(Ts)^u = \begin{cases} s^{u-} & \text{if } u \neq 0 \\ (1) & \text{if } u = 0 \end{cases}$$

for every $s \in \mathbb{R}^n$. Now $(S - TS) \cap \Omega^n = \varnothing$ where $\Omega^n := \{x \ll 0\}$, because the definition of S requires $s_u^u \geq 1$ if $u \in \overline{U}$ and the definition of T requires $(Ts)^0 = (1)$. Also, S is obviously convex.

Next define

$$\succ_a(f) := \{x \in X_a \mid x \succ_a f(a)\}$$

and

$$\chi(a) := \succ_a(f) - \{e(a)\}.$$

Since the economy is convex, $\int \chi \, d\nu$ is convex. Further, f is efficient, preferences are monotonic, so

$$\left(\int \chi \, d\nu\right) \cap \Omega^l = \varnothing$$

where $\Omega^l := \{x \in \mathbb{R}^l \mid x \ll 0\}$. In summary, $(\int_\chi d\nu) \times (S - TS)$ is convex and disjoint from $\Omega := \{x \in \mathbb{R}^l \times \mathbb{R}^n \mid x \ll 0\}$. Applying the Minkowski theorem, there exists a price system $(p, q) \in \mathbb{R}^l \times \mathbb{R}^n$ and $(p, q) \neq 0$ such that

$$0 \leq \inf (p, q) \cdot \left\{\left(\int \chi \, d\nu\right) \times (S - TS)\right\}. \tag{1}$$

Suppose that $p = 0$. This corresponds to the "degenerate" case in the proof of the previous theorem. By construction $\mathbb{R}_+^n \subset (S - TS)$ and this proves that $q > 0$, for if $q_{\bar{u}}^u < 0$ the vector with $\epsilon > 0$ in the corresponding place and 0 elsewhere belongs to $(S - TS)$ and for sufficiently large ϵ its value would be negative, contradicting (1). Similarly, if $u \notin H^{\bar u}$ then $q_{\bar u}^{u} = 0$ otherwise the security plan s defined by

$$s_{\bar v}^{v} = \begin{cases} 0 \text{ if } v \neq u \text{ and } \bar v = \bar u \\ 1 \text{ otherwise.} \end{cases}$$

has a negative value, i.e. $q \cdot s < 0$, contradicting (1). By an argument used in the previous theorem's proof one can show that for any $\bar u \in \overline{U}$ *either* $q_{\bar u}^{u} > 0$ for all $u \in H^{\bar u}$ *or* $q_{\bar u}^{u} = 0$ for all $u \in H^{\bar u}$. Let $\bar q$ be a price vector with a maximal set of positive prices, that is, there is no other separating price vector q with a strictly larger set of securities $\bar u \in \overline{U}$ such that $q_{\bar u}^{u} > 0$ for all $u \in H^{\bar u}$. Let $U^* := \{\bar u \in \overline{U} \mid \bar q_{\bar u}^{u} = 0, u \in H^{\bar u}\}$. Let L be the linear subspace of $\mathbb{R}^l \times \mathbb{R}^n$ defined by the condition $(p, q) \in L$ if and only if $q_{\bar u}^{u} = 0$ for any $u \in U$ if $\bar u \notin U^*$. By the Minkowski theorem there exists a price system $(p, q) \in L$, $(p, q) \neq 0$, such that (1) is satisfied. Then $(p, q + \bar q)$ also satisfies (1), that is,

$$(1)' \qquad 0 \leqslant \inf\{(p, q + \bar q) \cdot C\}$$

where $C := (\int \chi \, d\nu) \times (S - TS)$. Since $\bar q$ is "maximal" it follows that $p \neq 0$. Since $0 \in (S - TS)$, (1)′ implies that

$$0 \leqslant \inf \left\{ p \cdot \int \chi \, d\nu \right\}$$

so by a well known argument $p \gg 0$. In what follows I shall write (p, q) in place of $(p, q + \bar q)$ and assume $p \gg 0$, $q > 0$ and assume that q is "maximal" in the sense defined above. There is no loss of generality in this.

Consider a security allocation g such that $\int g_{\bar u}^{u} \, d\nu = 1$ for all $u \in U$, $\bar u \in \overline{U}$. Then $\int (g - Tg) \, d\nu = 0$. Suppose that σ is a security endowment and g, σ are chosen so that, for every $F \in \mathscr{F}$,

$$p \cdot f(a) + q \cdot g(a) = p \cdot e^F(a) + q \cdot T_a^F g(a)$$

for every $a \in A$, where the operator T_a^F is defined in terms of g and σ in the usual way. Now define $\psi^F(a)$ by putting

$$\psi^F(a) := \chi^F(a) \times \zeta^F(a) \quad \text{where} \quad \chi^F(a) := \succ_a^F(f) - \{e^F(a)\}$$

$$\succ_a^F(f) := \{x \in X_a^F \mid x \succ_a^F f(a)\} \qquad \zeta^F(a) := S_a^F - \{T_a^F g(a)\}$$

for each $F \in \mathscr{F}$ and $a \in A$. If $F = U$ write $\psi(a)$ for $\psi^F(a)$. From the definition of g and σ it is clear that

$$0 \leqslant \inf \left\{ (p, q) \cdot \int \psi \, d\nu \right\}$$

Also, for each $F \in \mathscr{F}$, $\psi^F(a)$ is contained in the closure of $\psi(a)$ so it easily follows that

$$0 \leq \inf \left\{ (p, q) \cdot \int \psi^F \, d\nu \right\} \tag{2}$$

for every $F \in \mathscr{F}$. The rest of the proof follows the lines of the proof of Theorem 2. First it is shown that ψ^F has a measurable graph. Then it follows from a result of Hildenbrand [11, D.II.4, Proposition 6, p. 63] that

$$\inf (p, q) \cdot \int \psi^F \, d\nu = \int \inf (p, q) \cdot \psi^F \, d\nu$$

so from (2) : $0 \leq \int \inf (p, q) \cdot \psi^F \, d\nu$. By definition of g and σ

$$(p, q) \cdot (e^F(a), T_a{}^F g(a)) = (p, q) \cdot (f(a), g(a))$$

so $0 \geq \inf (p, q) \cdot \psi^F(a)$ for each $a \in A$. Hence, $0 = \inf (p, q) \cdot \psi^F(a)$ and this means that, for each $F \in \mathscr{F}$,

(3) a.e. in A, $(p, q) \cdot (e^F(a), T_a{}^F g(a)) \leq (p, q) \cdot (x, s)$ for any pair (x, s) such that $x \succ_a{}^F f(a)$, $s \in S_a{}^F$.

Then, as in the proof of Theorem 2 one can show for almost all a in A, $(f(a), g(a))$ is maximal for $\succ_a{}^F$ in the budget set

$$\{(x, s) \in X_a{}^F \times S_a{}^F \mid p \cdot x + q \cdot s \leq p \cdot e^F(a) + q \cdot T_a{}^F s\}$$

for each $F \in \mathscr{F}$. This establishes that $(f, g) \in W^s(\mathscr{E}^s)$ and Theorem 1 implies that $(f, g) \in C^s(\mathscr{E}^s)$ when $\mathscr{E}^s = (\mathscr{E}, \sigma, \tau)$ and σ and τ are defined as above. ∎

REFERENCES

1. J. DRÈZE (Ed.), "Allocation and Uncertainty: Equilibrium and Optimality," Macmillan & Co., London, 1974.
2. J. M. GRANDMONT, On the short run equilibrium in a monetary economy, *in* "Allocation and Uncertainty: Equilibrium and Optimality" (J. Drèze, Ed.), Chap. 12, pp. 213–226, Macmillan & Co., London, 1974.
3. J. M. GRANDMONT AND G. LAROQUE, Money in the pure consumption loan model, *J. Econ. Theory* **6** (1973), 382–395.
4. J. M. GRANDMONT AND G. LAROQUE, On money and banking, *Rev. Econ. Stud.* **42** (1975), 207–236.
5. S. GROSSMAN, A characterization of the optimality of equilibrium in incomplete markets, *J. Econ. Theory* **15** (1977), 1–15.
6. S. GROSSMAN AND O. HART, A theory of competitive equilibrium in stock market economies, *Econometrica*, in press.
7. F. HAHN, Equilibrium with transaction costs, *Econometrica* **39** (1971), 417–439.

8. F. Hahn, On transaction costs, inessential sequence economies and money, *Rev. Econ. Stud.* **40** (1973), 449–461.
9. O. Hart, On the optimality of equilibrium when markets are incomplete. *J. Econ. Theory* **11** (1975), 418–443.
10. W. Heller, The holding of money balances in general equilibrium, *J. Econ. Theory* 7 (1974), 93–108.
11. W. Hildenbrand, "Core and Equilibria of a Large Economy," Princeton Univ. Press, Princeton, N.J., 1974.
12. M. Kurz, Equilibrium with transaction cost and money in a single market exchange economy, *J. Econ. Theory* 7 (1974), 418–452.
13. J. Niehans, Money in a static theory of optimal payments arrangements, *J. Money Credit Banking* **1** (1969), 706–726.
14. J. Niehans, Money and barter in general equilibrium with transaction costs, *Amer. Econ. Rev.* **61** (1971), 773–783.
15. J. Ostroy, The informational efficiency of exchange, *Amer. Econ. Rev.* **63** (1973), 597–610.
16. J. Ostroy and R. Starr, Money and the decentralization of exchange, *Econometrica* **42** (1974), 1093–1113.
17. R. Radner, Existence of equilibrium of plans, prices and price expectations in a sequence of markets, *Econometrica* **40** (1972), 189–304.
18. R. Starr, The price of money in a pure exchange economy with taxation, *Econometrica* **42** (1974), 45–54.
19. D. Starrett, Inefficiency and the demand for money in a sequence economy. *Rev. Econ. Stud.* **40** (1973), 437–448.

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On Labour Supply and the Measurement of Inequality*

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Traditional measure of inequality (the Gini coefficient, Atkinson measure, for example) are designed to produce a measure of the degree of inequality in the distribution of a single commodity-income or wealth. These measures are useful for capturing concern about the inequality of consumption opportunities conferred by the possession of different amounts of disposable income. However if we are concerned with the effect on inequality of various policies which will influence the way in which the income is generated, then the single-commodity framework is strictly only applicable if we assume either that all individuals face the same wage rate (so that differences in income arise solely through differences in unearned income) or that for some reason everyone works the same fixed number of hours. Neither of these cases is very interesting and there have been a number of proposals recently as to how inequality should be measured when there are differences in earning ability and labour supply is freely variable (Allingham [1], Stiglitz [7]). In particular, Stiglitz has argued that when appropriate account is taken of labour supply, then the resulting measure will typically be lower than the traditional measures and indeed could sometimes be negative.[1]

Now, as I shall show in the paper, it is important to distinguish the case in which redistribution occurs through a redistribution of the initial endowments of the unearned income and labour skills in the economy, from the case in which the endowments of labour skills are fixed and all that society can do is to reallocate the individual consumption bundles (of income and leisure). While the Stiglitz measure was developed in the former context, it can also be applied in the latter. What I shall show is that when the Stiglitz measure is interpreted in the latter context then the negativity of his measure arises from the confusion of an efficiency loss with a distributional loss. I propose an alternative measure which removes this efficiency loss and which is consequently always non-negative. I then show that while this alternative measure need not always be greater than the traditional measure, for the

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[1] The measure proposed by Allingham is very similar to that proposed by Stiglitz and in particular will be negative whenever the Stiglitz measure is negative. I shall point out in the text where Allingham's proof that his measure is always positive goes wrong.

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empirical estimates of labour supply which have been typically found, it will indeed be greater.

Turning to the context in which labour endowments can be redistributed, I show that here both the Stiglitz measure and the Allingham measure can be negative and that when this is the case it is because the benchmark of equality they select is again not efficient. Here the problem is more complex because in these cases there is no fair and efficient allocation, a result which extends some recent work by Varian [9] and Pazner and Schmeidler [5] on fair allocations. Moreover, when one adopts a fairly natural procedure to avoid this non-existence problem, it is found that the resulting feasible set is not convex, and that only under the extreme inequality aversion of the Rawlsian maximin criterion will the fair and efficient allocation be welfare maximising. Thus even comparing efficient allocations it can turn out that welfare based measures of inequality are negative. I show however that these findings are a consequence of the assumption employed by Allingham and Stiglitz that an individual's skill endowment is indivisible. When perfect divisibility of skills is allowed, the resulting welfare measure of inequality is again non-negative.

The plan of the paper is that after setting out the basic model in the next section, I shall then, in the next two sections, consider in turn the analysis of the case in which endowments of labour skills are fixed and the case in which they are variable. The less important details of proofs are confined to an Appendix.

1. THE MODEL

There is a single produced good, called income, which serves solely for consumption, the only input to production being labour. There is, however, an infinite variety of labour skills. Production takes place under an infinite elasticity of substitution constant returns to scale production function, so that the wage for each skill level is independent of the amounts of all types of labour employed. We can therefore let the single variable, n, measure both skill level and the wage rate which that skill commands. Each individual has the same *strictly concave* utility function which depends only on the amount of income and the number of hours of leisure he consumes (so individuals do not care about the type of job they do). Both consumption and leisure are assumed to be normal.

Faced with a wage rate, n, an individual will choose his hours of work so as to

$$\operatorname*{Max}_{c,l\geqslant 0} u(c,l) \qquad \text{s.t.} \quad c + nl \underset{l\leqslant 1}{\leqslant} n \tag{1}$$

and will consequently enjoy utility $v(n, 0)$, where $v(\cdot)$ is the indirect utility function

$$v(n, s) = \max_{c,\, l \geq 0} u(c, l) \qquad \text{s.t.} \quad \underset{l \leq 1}{c + nl \leq n + s} \tag{2}$$

For future purposes it will be useful to distinguish

a) the elasticity of the marginal utility of income, μ, defined by

$$\mu(c, l) = -\frac{c \cdot u_{11}}{u_1} \tag{3}$$

and

b) the elasticity of the marginal utility of *full income*, ν, defined by

$$\nu(n, s) = -\frac{(n + s)v_{22}}{v_2}{}^{2} \tag{4}$$

Given the concavity of $u(\cdot)$, both ν and μ are positive.

If $H(n, s)$ is the labour supply function arising from (2), I will define the price, income and substitution elasticities by

$$\eta_1 = \frac{nH_1}{H}, \quad \eta_2 = \frac{(n + s)H_2}{H}{}^{3}, \quad \eta_1{}^c = \frac{nH_1{}^c}{H} \tag{5}$$

where H^c is the compensated supply function.

If skills are distributed in the population according to the density function $f(n)$, then in the initial position social welfare is

$$\bar{v} = E(v(n, 0)) = \int v(n, 0) f(n)\, dn \tag{6}$$

In this initial position, income distribution is given by the function

$$c(n) = nH(n, 0) \tag{7}$$

with mean

$$\bar{c} = E(c(n)) \approx c(\bar{n}) = \bar{n}H(\bar{n}, 0) \tag{8}$$

Following Stiglitz, I will give the following interpretation to the Atkinson measure of inequality for the income distribution $c(n)$. I will take it that the Atkinson measure implicitly assumes that everyone works the same number of hours $\bar{H} \approx H(\bar{n}, 0)$. Under this assumption, social welfare is

$$\bar{u} = E\{u(c(n), 1 - \bar{H})\} \tag{9}$$

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[2] An alternative means of defining an elasticity of marginal utility of income would have been $\nu' = v_{22}/v_2\, s$. But this runs into difficulties when s is zero or negative.

[3] The justification for this definition is the same as in Footnote 2.

Given the concavity of $u(\cdot)$, then if we to give everyone the same income $\bar{c}$ (holding hours of work at $\bar{H}$) there would be an increase in social welfare. We can then deduct a certain amount of income, T, from everyone and bring social welfare back to its original level. The standard Atkinson measure then expresses T as a percentage of $\bar{c}$. However, when we turn to other measures of inequality, we will allow income to vary under redistribution, and so to be able to make valid comparisons, it will be better to measure inequality as a percentage of the value of initial endowments, $\bar{n}$, which will, given our assumptions, remain constant under all rearrangements of the economy's resources.

Formally, then, the Atkinson measure, α, is given by

$$u(\bar{c} - \alpha\bar{n}, 1 - \bar{H}) = \bar{u} \tag{10}$$

Later in the paper I will want to compare the Atkinson measure with other measures of inequality, and since direct comparison is difficult, I will follow Stiglitz in making the comparisons by means of approximation formula for the various measures. In particular, I will use the approximation for α which Stiglitz derives and which is given by

$$\alpha \approx \frac{\bar{H}}{2} \mu s_c^2 \tag{11}$$

where $\mu = \mu(\bar{c}, 1 - \bar{H})$, and, here, and throughout the paper I will use the notation

$$s_x^2 = \frac{E\{(x(n) - \bar{x})^2\}}{\bar{x}^2} \tag{12}$$

where $x(\cdot)$ is some function of n.

It is obvious from the way in which I set up the Atkinson measure that it is rather contrived, and this for two reasons. Firstly the initial value of social welfare $\bar{u}$, will not, in general, correspond to the true value of social welfare $\bar{v}$. Secondly, there is no justification given for taking the position in which everyone has income and leisure $\bar{c}$, $1 - \bar{H}$ respectively as the benchmark of perfect equality. We simply held hours of work artibrarily at $\bar{H}$ and given the constraint could only really mean by equality a position in which everyone gets the same income c.

In the next two sections I will explore a number of ways of meeting these objections.

2. REDISTRIBUTION WITH FIXED ENDOWMENTS OF LABOUR SKILLS

As mentioned in the introduction it is important to recognise that when trying to measure inequality in the context of the model outlined in the previous section, there are two distinct types of comparison that the measure could be designed to capture. Under the first, one is trying to form a judgement about how unequally society's initial endowments are distributed relative to some norm or benchmark of a perfectly equal distribution. However even if the distribution of endowments is held fixed, there are a variety of different allocations of consumption and leisure to different individuals which are compatible with this given distribution of endowments, and one can ask how unequally a given allocation is distributed relative to some benchmark allocation of perfect equality. This benchmark will of course itself have to be feasible relative to the given distribution of endowments. It is with comparisons of this type that the present section will be concerned.[4]

Throughout this section, therefore, I will take the set of feasible allocations to be defined as follows. An *allocation* is a pair of functions $c: R \rightarrow R$, $l: R \rightarrow R$, with $c(n) \geq 0$ all n, $0 \leq l(n) \leq 1$ all n. The feasible set, $\mathscr{F}$, is then defined as

$$\mathscr{F} = \{(c, l) \mid \int (c(n) + nl(n) - n) f(n)\, dn \leq 0\} \tag{13}$$

The initial allocation $(nH(n, 0), 1 - H(n, 0))$ is clearly feasible, and we now have to decide which other feasible allocation to choose as the benchmark of perfect equality.

One benchmark which one might think of trying is a position in which everyone has the same income $\bar{c}$, and the same leisure $\bar{l}$ (and consequently the same utility $u(\bar{c}, \bar{l})$). Given the feasibility requirement of (13), the choice of $\bar{c}, \bar{l}$ is constrained by the budget constraint

$$\bar{c} + \bar{n}\bar{l} \leq \bar{n}, \tag{14}$$

It is then natural to choose that particular combination of $(\bar{c}, \bar{l})$ which maximises $u(\cdot)$ subject to (14). Thus in the benchmark position social welfare is just $v(\bar{n}, 0)$. We could then measure inequality as the percentage of initial endowments which would have to be imposed as a lump-sum tax to bring social welfare from the benchmark level to the initial level $\bar{v}$. Formally, then, this measure of inequality, which I denote by β, is given by

$$\bar{v} = v(\bar{n}, -\beta\bar{n}) \tag{15}$$

Now β, as defined by (15), is precisely the measure of inequality proposed by Stiglitz, though, as we shall see in the next section he gave it a different interpretation.

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[4] It should be noted that it is precisely within the context of a fixed distribution of labour skills that most of the literature on optimal income taxation has been developed. The framework for measuring inequality employed in this section may therefore be the most appropriate for comparing the distributional effects of different income tax schedules. Such an approach is developed more fully in a subsequent paper (Ulph [8]).

As Stiglitz shows, β can be negative, and it is easy to see technically why this is so: for, from (15),

$$\beta \gtreqless 0 \text{ as } v(\bar{n}, 0) \gtreqless \int v(n, 0)\, f(n)\, dn, \tag{16}$$

and while $u(\cdot)$ is strictly concave, there is nothing to guarantee that $v(\cdot)$ is concave in the wage rate. Indeed, from the well-known relation

$$v_1 = v_2 H \tag{17}$$

it follows that

$$v_{11} = v_2 H_1 + H v_{12} \tag{18}$$

and although v_{12} is negative under the assumptions made about $u(\cdot)$ in Section 1, if the price elasticity of labour supply is sufficiently large and positive then $v_{11} > 0$ and $v(\cdot)$ will be convex (at least over a region). In his counter-example, Stiglitz employs a utility function with infinite elasticity of substitution, and since this generates (locally) an infinite price elasticity of labour supply we can see, from (18), that this can certainly produce a negative value of β.

On the other hand, it is clear from the interpretation given to β in this section why the measure is inadequate, for the benchmark position is not Pareto-efficient while the initial position certainly is. Thus the move from the initial position to the benchmark involves an efficiency loss as well as a distributional gain, and both of these are being combined in β. The negativity of β simply reflects the fact that the efficiency loss outweighs the distributional gain. However, even if $v(\cdot)$ is concave and β is positive, there is still an efficiency loss, and β is undermeasuring the degree of inequality.

This suggests that the correct procedure for measuring inequality is as follows. From the initial position impose a series of lump-sum taxes and subsidies so as to move to the point on the efficiency frontier at which everyone has the same utility. The aggregate tax revenue so raised will be zero and, as long as the initial position was one of inequality, social welfare will increase. (For reasons to be discussed below, this latter statement requires proof, which is given in the proof of Theorem 1). Now impose taxes on everyone to bring them from this position of equality to the position in which everyone has utility level $\bar{v}$. Unlike the Atkinson and Stiglitz measures, these taxes will be different for individuals of different abilities, but nevertheless we can still express the measure of inequality as the ratio of the total tax raised to the value of initial endowments.

Formally, then define the function $t(n, z)$ by

$$v(n, -t(n, z)) = z \tag{19}$$

$t(\cdot)$ is thus the lump-sum tax (subsidy if t is negative) required to take an individual of ability n to utility level z.

The measure of inequality proposed above, which I will denote by γ, is therefore defined by

$$\gamma \bar{n} = \int t(n, \bar{v}) f(n)\, dn \tag{20}$$

I now want to prove that $\gamma \geq 0$, with $\gamma = 0$ if and only if in the initial position everyone has the same utility. The latter result follows obviously from (2) and the definition of $t(\cdot)$ in (19), while to prove the former it is clearly necessary and sufficient to prove that if in the initial position there is some inequality then a movement along the efficiency frontier to the point of perfect equality will increase social welfare. Now it is well-known that with a feasible set as defined by (13), when social welfare is maximized the more able are made worse off than the less able, so consequently there are many points on the efficiency frontier from which a move to the point of perfect equality will entail a reduction in social welfare.[5] It is therefore necessary to prove that the initial position is not such a point. A key result in proving this is contained in the following.

LEMMA 1. *Let $\tau(n)$ be a differentiable series of lump-sum taxes (subsidies for negative τ) such that $\alpha(n) = v(n, -\tau(n))$ is non-decreasing in n. Then*

$$\frac{d}{dn} v_2(n, -\tau(n)) \leqslant 0 \tag{21}$$

with equality if and only if $\alpha' = 0$ and $H_2 = 0$.

Proof.

$$\frac{d}{dn} v_2(n, -\tau(n)) = v_{21} - v_{22}\tau'(n) \tag{22}$$

since $\alpha(n)$ is non-decreasing

$$\alpha'(n) = v_1 - v_2\tau'(n) \geqslant 0 \tag{23}$$

Using (17), (23) becomes

$$\tau'(n) \leqslant H. \tag{24}$$

Substituting (24) into (22) yields

$$\frac{d}{dn} v_2(n, -\tau(n)) \leqslant v_{21} - v_{22}H \tag{25}$$

Differentiating (17) gives

$$v_{21} = v_2 H_2 + v_{22} H$$

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[5] The lack of symmetry of the utility-possibility frontier makes the measure γ inapplicable in the more general case where there is measured income and where, in principle, the initial allocation can lie anywhere on the utility possibility frontier. A procedure for generalising the measure to circumvent this difficulty is discussed in Ulph [8].

Hence

$$\frac{d}{dn} v_2(n, -\tau(n)) = v_2 H_2 < 0 \tag{26}$$

Since $v_2 > 0$, the lemma is proved.

THEOREM 1. *$\gamma \geqslant 0$ with $\gamma = 0$ if and only if $v(n, 0) = \bar{v}$ all n.*

Proof. The proof of this theorem is neater for the case of all finite population than for a continuous distribution though exactly the same structure of proof will go through in both cases. Accordingly let there be N individuals indexed $i = 1,\dots N$ in such a way that n_i is non-decreasing.

If $v(n_i, 0) = \bar{v}$ all i, then $t(n_i, \bar{v}) = 0$ all i and $\gamma = 0$. Accordingly assume there is a v_0, K such that

$$v(n_i, 0) = v_0, \quad i = 1,\dots K, \quad v(n_j, 0) > v_0 \quad j = K+1,\dots N$$

In particular, this implies that $H(n_{K+1}, 0) > 0$. For if $H(n_{K+1}, 0) = 0$, $H(n_i, 0) = 0$ $i = 1,\dots K$ and $v(n_{K+1}, 0) = u(0, 1)$ — a contradiction. Furthermore, $H(n_{K+1}, 0) > 0$ implies $H_2(n_{K+1}, 0) < 0$.

Impose a lump-sum tax of $X > 0$ on individual $K + 1$ and use the proceeds to make lump-sum subides $X_1, \dots, X_K$ to the first K individuals in such a way that

$$\sum_{i=1}^{K} X_i = X, \quad v(n_i, X_i) = v(n_j, X_j) \qquad i, j = 1,\dots, K.$$

Let $W(X) = \sum_{i=1}^{K} v(n_i, X_i) + v(n_{K+1}, -X)$

Then

$$W'(X) = \sum_{i=1}^{K} v_2(n_i, X_i)\frac{\partial X_i}{\partial X} - v_2(n_{K+1}, -X)$$

Apply Lemma 1 and we find that

$$\begin{aligned} W'(X) &\geqslant v_2(n_1, X_1) \sum_{i=1}^{K} \frac{\partial X_i}{\partial X} - v_2(n_{K+1}, -X) \\ &= v_2(n_1, X_1) - v_2(n_{K+1}, -X) \end{aligned}$$

Apply Lemma 1 again and we find $W'(0) > 0$, and $W'(X) > 0$ as long as $v(n_{K+1}, -X) \geqslant v(n_i, X_i)$.

Thus increasing X steadily from zero to the point of perfect equality between the first $K + 1$ individuals will strictly increase social welfare. Now simply repeat the argument until $K = N$, and at each stage the redistribution will be non-decreasing in social welfare. Since all these moves involve pure redistribution they all take place across the efficiency frontier. This proves that as long as there is inequality initially then moving to the point of perfect equality on the efficiency frontier is strictly welfare improving. The rest of the proof of the theorem is obvious. As argued above, the fact that the Stiglitz measure does not use a Pareto-efficient point as benchmark will mean that even when the Stiglitz measure is positive it will still under-measure the degree of inequality. This is now proved in

THEOREM 2. *$\gamma \geq \beta$ with $\gamma = \beta$ if and only if $\gamma = \beta = 0$.*

Proof. If we compare (15) and (19) we see that

$$\beta\bar{n} = t(\bar{n}, \bar{v}) \tag{27}$$

Differentiate (19) w.r.t. n and we find $v_1 - v_2 t_1 = 0$ i.e.,

$$t_1 = H^c(n, z) \tag{28}$$

Differentiate (28) again w.r.t. n and we find

$$t_{11} = H_1^c \geq 0$$

with equality only if $H^c = 0$. Hence t is strictly convex in n as long as $H^c(n, z)$ is positive. Thus $t(\bar{n}, \bar{v}) \leq \int t(n, \bar{v}) f(n)\, dn$ with equality if and only if either everyone has the same ability (and so the same utility) or those with lower ability do no work, so once again everyone has the same utility.

I turn now to the claim by Stiglitz that the traditional Atkinson measure overstates the degree of inequality. Since the Atkinson measure ,α, is constructed in a rather different way from the measures γ and β, I have not been able to obtain simple comparisons and results such as those contained in Theorem 2, and so have adopted two other approaches to making the comparisons.

The first is that employed by Stiglitz of using approximation formulae for each of the measures. An approximation formula for α has already been given in (11). As Stiglitz shows, we can obtain in an exactly anologous way the following approximation for β

$$\beta \approx -\frac{1}{2}\frac{v_{11} \cdot n}{v_2} s_n^2 \tag{29}$$

To show how this relates to α it is necessary to relate the standard deviation in the wage rate to the standard deviation in consumption, and the elasticity of the marginal (indirect) utility of the wage to the elasticity of the marginal utility of consumption.

It is intuitively obvious, (and can be easily proved formally) that that

$$s_c^2 = s_n^2(1 + \eta_1)^2$$

so that the standard deviation in consumption is greater equal or less than that in the wage according as those with the higher wage work harder, just as hard, less hard than those with a lower wage. Comparing the elasticities of marginal utility is a more technical matter, and I show in the Appendix that when the relevant substitutions are made (29) becomes

$$\beta = \frac{\alpha}{(1+\eta_1)^2} - \frac{\bar{H}}{2}\left(\frac{\eta_1}{1+\eta_1}\right)^2 \frac{s_c{}^2}{\eta_1{}^c} \tag{30}$$

This formula suggests that if $\eta_1 = 0$ then $\beta = \alpha$. At first sight this may seem to be a correct result, since if $\eta_1 = 0$ the assumption implicitly made in constructing the Atkinson measure (that everyone works the same number of hours) is valid. However, while it is certainly true that when $\eta_1 = 0$ both measures employ the same benchmark position ($\bar{c} = c(\bar{n})$, $\bar{H} = H(\bar{n})$), and impose taxes to bring social welfare back to the same level ($\bar{u} = \bar{v}$), under the Atkinson measure the individual is not allowed to adjust his hours of work, so it should require a smaller lump-sum tax to reduce average welfare to its original level than is the case with the Stiglitz measure. Thus the Stiglitz measure ought to be greater than the Atkinson measure for this case.

(30) also suggests that if $\eta_1 > 0$ then $\beta < \alpha$. As most empirical estimates suggest $\eta_1 < 0$, it is particularly interesting to see what happens for this case, and this is most usefully done by re-writing (30) to obtain

$$\beta = \alpha\left\{1 - \left(\frac{\eta_1}{1+\eta_1}\right)^2\left(1 + \frac{1}{\mu\eta_1{}^c} + \frac{2}{\eta_1}\right)\right\}$$

Thus $\beta > \alpha$ iff

$$1 + \frac{1}{\mu\eta_1{}^c} + \frac{2}{\eta_1} < 0 \tag{31}$$

Now if consumption is a normal good a rise in the wage rate will increase consumption, that is

$$\frac{\partial}{\partial n}(nH(n, s)) > 0$$

which implies $\eta_1 > -1$. Thus, when $\eta_1 < 0$

$$\frac{-2}{\eta_1} - 1 > 0, \tag{32}$$

and (31) can be written

$$\mu > -\frac{1}{\eta_1{}^c(2/\eta_1 + 1)} \tag{33}$$

so that $\beta > \alpha$ as long as inequality aversion is not too small.

To see concretely what (33) implies, let us substitute some empirical estimates of η_1^c and η_1. I will use those given in Brown, Levin and Ulph (2). These show estimates ranging from $\eta_1 = -0.107$, $\eta_1^c = 0.204$ to $\eta_1 = -0.308$ $\eta_1^c = 0.405$. In the first case (33) becomes $\mu > 0.277$ while in the second it becomes $\mu > 0.449$. This suggests that for these labour supply estimates almost any reasonably egalitarian ethic would make the Stiglitz measure greater than the Atkinson measure.

Turning now to γ, it can be shown that

$$\gamma \approx \beta + \frac{\bar{H}}{2}\eta_1^c s_n^2 \tag{34}$$

$$\alpha\left(1 - \frac{\eta_1^2}{(1+\eta_1)^2}\left(1 + \frac{2}{\eta_1} + \frac{\eta_1^2 - (\eta_1^c)^2}{\mu\eta_1^c}\right)\right) \tag{35}$$

(34) confirms that $\gamma > \beta$ and suggests that the size of the difference depends on the mean hours of work, the substitution elasticity and the size of the spread in the wage rates. This is just what we would have expected.

We see from (35) that

$$\gamma \gtrless \alpha \text{ as } 1 + \frac{2}{\eta_1} + \frac{\eta_1^2 - (\eta_1^c)^2}{\eta_1^c} \gtrless 0$$

Now if $\eta_1 > 0$, $|\eta_1| < |\eta_1^c|$ so $\eta_1^2 - (\eta_1^c)^2 < 0$ so

$$\gamma > \alpha \text{ as } \mu < \frac{(\eta_1^c)^2 - \eta_1^2}{\eta_1^c(1 + 2/\eta_1)}.$$

That is the Atkinson measure undermeasures inequality only for "small" values of inequality aversion.

If $\eta_1 < 0$, there are two cases to consider.

(a) $|\eta_1| < |\eta_1^c|$ in which case $\gamma > \alpha$ for all values of μ.

(b) $|\eta_1| > |\eta_1^c|$ in which case $\gamma > \alpha$ except for "small" values of μ.

Going back to the empirical estimates referred to above, we see that they are consistent with case (a) which suggests that the traditional measure does underestimate the degree of inequality.

Now we saw above that the approximation formulae can go wrong in at least one case, so as a check on results I performed some direct calculations of the various measures for a variety of cases. Thus I have taken for $f(n)$ the lognormal distribution used by Mirrlees [3] in his calculations of optimal income tax schedules. I used a variety of utility functions of the C.E.S. type. Thus I set

$$u = \frac{1}{1-\pi}(ac^{-\lambda} + (1-a)l^{-\lambda})^{-(1-\pi)/\lambda}$$

and chose three different values for the elasticity of substitution $\epsilon = 1/1 + \lambda$, namely $\epsilon = 0.5, 1.0, 1.5$. These were chosen to generate the three cases $\eta_1 = 0$, $\eta_1 > 0$ respectively. Similarly, given the results emerging from the approximation formulae, it seemed important to try a range of values for π, and the three selected were $\pi = 2.0, 3.0, 10.0$. The first two were selected as being in line with the values of inequality aversion found by Stern [6], while the latter value was chosen more or less at random to see if it would pick up any of the reversals of ranking of the measures discovered by approximation formulae. Finally a was set at 0.25 so that in the case $\epsilon = 1$ people could be working for a quarter of the week. It would probably have been more realistic to set a so that people were working an average of 42 hours a week when $\epsilon = 0.5$. The results are presented in Table 1. All the measures are presented as percentages of the value of initial endowments. If it helps to think of these as percentages of GNP then the figures should be multiplied by the values of $\bar{n}/\bar{c}$ given in the bottom row of the Table.

TABLE 1

(Measures Are in Percentages)

ϵ	0.5			1.0			1.5		
π	2.0	3.0	10.0	2.0	3.0	10.0	2.0	3.0	10.0
α	2.8	4.1	11.6	1.4	2.2	5.2	0.6	1.1	2.4
β	5.2	7.3	20.9	8.3	9.0	12.1	−0.2	0.2	1.0
γ	6.4	8.5	21.9	9.7	10.5	13.6	0.9	1.4	2.2
$\bar{n}/\bar{c}$		2.1			4.0			8.5	

The results confirm the findings of the approximation formulae very well, apart from the one case noted already in which $\eta_1 = 0$ ($\epsilon = 1$). Thus when $\epsilon = 0.5$ ($\eta_1 < 0$) $\beta > \alpha$ for all values of π, though it may be that for much lower values we would have found $\beta < \alpha$. Similarly when $\epsilon = 1.5$ ($\eta_1 > 0$) then $\beta < \alpha$ for all values of π. Indeed, even with this value of ϵ it is possible to generate a negative value for β. On the other hand when $\epsilon = 1.0$, $\beta > \alpha$ as we would have expected, though this is the case where the approximation formula goes wrong.

$\gamma > \beta$ throughout as we know to be case from Theorem 1 and (34). Notice also that for any given ϵ the difference between γ and β is pretty well independent of π — which is what the approximation formula (34) predicts. The difference is also constant across the Table, and if this seems odd at first sight, notice that in doubling the elasticity of substitution in going from the first to the second part of the Table the average hours of work have at the same time approximately halved. Hence $\bar{H}\eta_1{}^c$ has stayed more or less constant and this, according to the formula (34) should result in $\gamma - \beta$ staying constant. Had I allowed a to vary across the Table to keep $\bar{n}/\bar{c}$ constant then in the first part of the Table the difference between γ and β would have approximately doubled.

$\gamma > \alpha$ throughout the Table except for the very last entry where $\eta_1 > 0$ and π is large. Again this confirms the discussion above. For what is probably the most interesting case, $\epsilon = 0.5$, it appears that the traditional measure gives only about half the correct figure for inequality, though this order of magnitude of difference could be very sensitive to the assumptions made.

To conclude, the measure proposed by Stiglitz, in the interpretation given here, confuses efficiency losses with distributional losses. When these efficiency losses are removed the resulting measure is non-negative and, while not always greater than the traditional Atkinson measure, it is so for many of the cases that are of empirical interest.

3. INITIAL ENDOWMENTS VARIABLE

In the last section I considered the problem of constructing a measure to reflect judgements about how unequally a given allocation of consumption and leisure was distributed relative to a set of allocations constrained by the initial endowments of labour skills in the population. In this section I will be concerned with the construction of a measure of how unequally these skills are distributed in the population.

For expositional purposes, I will confine attention in this section to the case of a finite population. As in the previous section it is important to begin by specifying precisely what alternative allocations are open to society. A fairly natural set, employed by both Stiglitz and Allingham, is

$$\mathscr{F}_1 = \left\{ ((c_1, l_1) \cdots (c_N, l_N)) \mid c_i = m_i H(m_i, 0),\ l_i = 1 - H(m_i, 0), \right.$$
$$\left. \sum_{i=1}^{N} m_i = \sum_{i=1}^{N} n_i \right\}$$

That is, is it the set of allocations emerging as equilibria for the economy as individuals are given different skill levels but with the mean skill level remaining constant.

It then seems natural to take as benchmark of perfect equality the point in which everyone has consumption $\hat{c}$ and leisure $\hat{l}$, where

$$\hat{c} = \bar{n}H(\bar{n}, 0),\ \hat{l} = 1 - H(\bar{n}, 0) \qquad \text{where} \quad \bar{n} = 1/N \sum_{i=1}^{N} n_i \tag{36}$$

One can then measure inequality by asking what percentage of the value of initial endowments could be taxed away to take social welfare from that prevailing in the benchmark position to that in the initial position. If this tax is imposed as a lump-sum tax, we have once again the Stiglitz measure β defined by

$$v(\bar{n}, -\beta\bar{n}) = \bar{v}$$

while if the tax is imposed as an ability tax we have the Allingham measure (which I denote by β') defined by

$$v((1-\beta')\bar{n}, 0) = \bar{v} \qquad (37)$$

Now we saw in the last section that β could be negative because an efficiency loss was being confused with a distributional loss. Since the formal definition of β is exactly the same as before, it follows that β can be negative in the interpretation given here, and it turns out that although the reason for the negativity is similar, in this case it is harder to remove.

The problem arises if there exists some $\hat{n} > 0$ such that $H(\hat{n}, 0) = 0$ (which implies $H(n, 0) = 0$ for all $n < \hat{n}$) and if $\bar{n} \leq \hat{n}$. The first condition depends on the nature of the utility function, the second on the nature of the initial distribution. Suppose both of these conditions hold, and consider the point of perfect equality in which $n_i = \bar{n}$ for all i. Now raise one person's ability level above $\hat{n}$ making corresponding reductions in everyone else's. Those whose whose skill level has been reduced will be no worse off than before (they were doing no work and consuming no income before and continue to do so), while the person whose skill level has been raised will be better off. The benchmark of perfect equality is therefore not Pareto efficient, so moving to it may well cause a welfare loss, and hence a negative value of β. However, the problem is more serious than in the previous section, because what the above argument has shown is that for these cases there is no point of perfect equality on the utility possibility frontier. In other words the problem is that there is no fair and efficient allocation. This is a different kind of non-existence result to those already noted in the literature[6], where it is assumed that initial labour endowments are fixed, and where non-existence arises only in models where individuals have different preferences. It should be noted that the non-existence result found here arises even though all individuals have identical preferences.

Two points should be noted about this result.

Firstly, since the reasons for β being negative in this case lie in the fundamental nature of the efficiency frontier, it is clear that $\beta < 0$ if and only if $\beta' < 0$. So the Allingham measure can also be negative. This contradicts

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[6] See, for example, Pazner [4], Pazner and Schmeidler [5], Varian [9], [10].

Allingham's "proof" that his measure is always non-negative. Allingham's mistake is in his assertion at the outset of his paper that the indirect utility function is always increasing in the wage. It will not be increasing if H is zero which is precisely the situation that arises in the example discussed above.

Secondly, the assumptions lying behind the measures β and β' above are exactly the reverse of those employed in the previous section, in that while labour endowments are now assumed variable, it is implicit in the construction that income cannot be redistributed. This suggests that one way out of the non-existence problem discovered above will be to allow income to be redistributed.

Thus suppose we take as feasible set the set

$$\mathscr{F}_2 = \Big\{(c_1, l_1) \cdots (c_N, l_N) \mid 0 \leq c_i, l_i, m_i, l_i \leq 1, \\ \sum_i (c_i + m_i l_i - m_i) \leq 0 \sum_i m_i = \sum n_i\Big\}$$

$\mathscr{F}_2$ is therefore the union of feasible sets of the kind considered in the previous section, but where we now allow any reallocation of the initial wage rates which gives the same average wage rate and no one a negative wage. Since for any given redistribution of the wage rates there is a fair and efficient allocation, and since the efficiency frontier of $\mathscr{F}_2$ is the outer envelope of efficiency frontiers as we sweep over different wage distributions, it follows that there is a fair and efficient allocation in $\mathscr{F}_2$ which we can use as a benchmark of perfect equality with respect to which we can compare the initial distribution.

Before considering how to make such a comparison, it is worth characterising the efficient allocations of $\mathscr{F}_2$. This is done through the following lemmas.

LEMMA 2. *If* $(\hat{c}_1, \hat{l}_1) \cdots (\hat{c}_N, \hat{l}_N)$ *is an efficient allocation, then* $\forall i, j$, $i \neq j$ $\hat{l}_i < \hat{l}_j \Rightarrow \hat{l}_j = 1$, $\hat{m}_j = 0$

Proof. Suppose not so, and there is i, j with $\hat{l}_i < \hat{l}_j < 1$. If $\hat{m}_j = 0$ we have a contradiction, since we can then raise l_j to 1 and obtain a Pareto-superior allocation. Hence $\hat{m}_j > 0$. But then we could set $\hat{m}_j$ to 0, increase $\hat{m}_i$ accordingly and (holding $\hat{l}_i$, $\hat{l}_j$ constant) produce strictly more consumption, again a contradiction. Hence $\hat{l}_j = 1$ and $\hat{m}_j = 0$.

LEMMA 3. *If* $(\tilde{c}_1, \bar{l}_1) \cdots (\tilde{c}_N, \bar{l}_N) \in \mathscr{F}_2$ *and* $\bar{l}_i = \bar{l}_j = \bar{l}$, *for some* $\bar{l}$, $0 \leq \bar{l} < 1$, *and for some* i, j, $i \neq j$, *then the allocation is not efficient.*

Proof. Suppose the allocation is efficient and let $\tilde{m}_i$, $\tilde{m}_j$ be the wages set for i and j. We must have $\tilde{m}_i > 0$, $\tilde{m}_j > 0$, since otherwise we could increase welfare by increasing leisure with no loss to consumption. But now consider the alternative allocation which holds everything constant for $k \neq i, j$, sets $m_j = 0$, $l_j = 1$, $c_j = \tilde{c}_j$, $m_i = \tilde{m}_i + \tilde{m}_j$, $c_i = \tilde{c}_i$, $l_i = \tilde{l}_i$. By construction this is still a feasible allocation, but is preferred by j and is just as good for i.

Lemmas 2 and 3 demonstrate that all the efficient allocations in $\mathscr{F}_2$ are generated by giving all the skills in society to one individual, have him do all the work and re-distribute the income so earned via a system of lump-sum subsidies. This is certainly not surprising given the assumption of infinite substitutability between labour skills.

It is worth asking which individual in society does all the work.

LEMMA 4. *If $(u_1 \cdots u_N)$ is any point on the utility possibility frontier, and if individual k has the entire endowment of skills, then $u_k \leqslant u_i$ for all i.*

Proof. Suppose $(u_1 \cdots u_N)$ is on the efficiency frontier (in utility space), and that individual k has the entire skill endowment. Then there are lump-sum taxes (subsidies for negative t_i), $t_1 \cdots t_N$ such that

$$u_i = v(0, -t_i) \qquad i \neq k$$

$$u_k = v(M, -t_k) \qquad \text{where} \quad M = \sum_{i=1}^{N} n_i$$

$$\sum_{i=1}^{N} t_i = 0$$

If this is an efficient allocation, then it must be the case that no Pareto-gain can be made by transferring the skill endowment between individual k and individual i. This is true if and only if

$$t(0, u_i) + t(M, u_k) \geqslant t(M, u_i) + t(0, u_k)$$

where $t(\cdot)$ is the tax function defined in (19). This is true if and only if

$$t(M, u_k) - t(0, u_k) \geqslant t(M, u_i) - t(0, u_i) \tag{38}$$

Now it is easily checked that $t_1 = H^c(w, u)$ and $t_{12} = \partial H^c/\partial u \leqslant 0$, so the inequality in (38) can hold if and only if $u_k \leqslant u_i$, which proves the lemma.

Thus the individual who has the total endowment of skill is always worst off (though not necessarily uniquely so). The implications of this can be seen most sharply by considering the utility-possibility frontier in the two-person case. This is defined by

$$t(M, u_1) + t(0, u_2) = 0 \tag{39}$$

for $u_1 \leqslant u_2$ and by $t(M, u_2) + t(0, u_1) = 0$ for $u_2 \leqslant u_1$.

Differentiate (39) totally w.r.t. u_1 and we have

$$t_2(M, u_1) + t_2(0, u_2)\frac{du_2}{du_1} = 0,$$

i.e.

$$\frac{du_2}{du_1} = -\frac{t_2(M, u_1)}{t_2(0, u_2)} < 0,$$

while

$$t_2(0, u_2)\frac{d^2u_2}{du_1^2} = -\left(t_{22}(M, u_1) + t_{22}(0, u_2)\left(\frac{du_2}{du_1}\right)^2\right)$$

Since $t_2 < 0$ and $t_{22} < 0$ it follows that $d^2u_2/du_1^2 < 0$.

The important point to note is that when $u_1 = u_2$ then, bearing in mind that $t_2 < 0$, $t_{12} < 0$,

$$\left|\frac{du_2}{du_1}\right| > 1,$$

where this derivative is now to be interpreted as a left-hand derivative of the utility possibility frontier as it crosses the 45° line. By symmetry the right-hand derivative has $|du_2/du_1| < 1$, at this point. It follows then that no differentiable, symmetric social welfare function will treat the point of perfect equality as the point of maximum social welfare, and indeed this point will only be welfare maximising under something like a Rawlsian social welfare function. Put somewhat differently, for almost all social welfare functions, there will be utility distributions on the efficiency frontier such that moving from these distributions to the point of perfect equality will entail a loss in social welfare. Consequently any welfare-based measure of inequality will be negative for some initial distributions.

Thus with this particular approach to measuring inequality when initial endowments can be redistributed the possibility of a negative measure of inequality seems to be far more deeply rooted than in the case when labour endowments were fixed.

One way out of this difficulty comes by noticing that implicit in the construction of the measures used so far in this section, is the assumption that skills are "lumpy" in that any individual can have only one type of skill. While this may be a fairly natural description of individuals in certain circumstances, if we are thinking more of a lifecycle approach to the individual it is more natural to think of the individual as being able to work at different skill levels for different proportions of his life.

Accordingly this suggests we use as benchmark position of equality that redistribution of initial endowments of labour skills in which each individual can work at each skill level for a proportion of his time allocation equal to the proportion of the total population endowed with that particular skill.

More formally, suppose that there are K different skills and that individual i's endowment of labour skills is given by the vector $\pi_i = (\pi_{i1}, \dots, \pi_{iK})$ where π_{ik} is the proportion of time that individual i can devote to working at skill level k. This formulation of the individual endowments will include as special case the model used at the beginning of this section in which π_i was a unit vector for each i. The benchmark of equality proposed above would therefore be the vector $\bar{\pi}$ in which

$$\bar{\pi}_k = 1/N \sum_{i=1}^{N} \pi_{ik}, \qquad k = 1, \dots, K$$

i.e.,

$$\bar{\pi} = 1/N \sum_{i=1}^{N} \pi_i$$

Given a vector of prices $(n_1, \dots, n_K)$ for each of the skills, and an initial endowment vector π, we can define the indirect utility function $V(\pi)$ by

$$V(\pi) = \operatorname{Max} u\left(c, 1 - \sum_{k=1}^{K} H_k\right) \qquad \text{s.t. } c \leqslant \sum_{k=1}^{K} n_k H_k$$
$$H_k \leqslant \pi_k, \qquad k = 1, \dots, K$$

It is easily checked that as long as $u(\cdot)$ is strictly concave, so too is $V(\cdot)$ and hence if $(\pi_1, \dots, \pi_N)$ is an efficient and inegalitarian initial endowment vector, then the move to the point of perfect equality is welfare-improving. Hence any welfare-based measure of inequality using $\bar{\pi}$ as benchmark will always be non-negative, and will be zero if and only if initially there is perfect equality. Rather than develop such a measure formally let me conclude this section with four remarks about this alternative approach to measuring inequality.

Firstly this way of viewing labour endowments is now precisely analogous to the way one could handle endowments of other commodities, where it is natural to think of pursuing equality by giving everyone a small share in every commodity rather than one unit of some "average" commodity.

Secondly what the measure stresses is that it is not the diversity of skills *per se* that generates inequality, but rather the concentration of these skills in a few individuals, and it is essentially this concentration that the measure is picking up. Put another way, this measure is not designed to tell whether one overall distribution of skills in a population is more equal than another, but rather whether one dispersion of a given overall allocation of skills amongst different individuals is more equal than another dispersion of the same skills.

Thirdly, it is important to notice that any arbitrary dispersion of skills in a population need not be efficient. Consequently in comparing the initial position to the benchmark position, it is important to remove efficiency gains.

Finally, it is important not to overplay the similarity between labour and other commodities. For while skills may not be as lumpy as the earlier model required, learning by doing and other factors suggest that there may be some minimum of time people need to spend at particular skill levels to enable them to perform effectively at that level. Hence skill times may not be as infinitely divisible as this second measure has required. Moreover, while indivisibility of capital in production does not, in principle, prevent the ownership of capital being infinitely divisible, we accept that people cannot have a share in the labour skills of others.

I conclude that it is still an open question as to how inequality in labour endowments is to be measured. One approach that could be explored is some combination of the two measures explored in this section. That is one could allow that skills could be split up amongst individuals subject to individuals having to spend some minimum proportion of time at a particular skill level if they are to have any endowment of that skill at all. Alternatively one could try to model more explicitly how individual skills could be altered through, for instance, education. In either case, it should be clear from the discussion of this section that it is important to consider carefully the nature of the feasible set before constructing the measure.

APPENDIX

I wish to relate the Stiglitz measure

$$\beta \approx -\frac{1}{2}\frac{v_{11}}{v_2}\bar{n}s^2{}_n \tag{A.1}$$

to the Atkinson measure

$$\begin{aligned}\alpha &\approx \frac{H}{2}\mu s_c{}^2 \\ &= -\frac{1}{2}\frac{u_{cc}}{u_c}\bar{n}\bar{H}\cdot\bar{H}\cdot s_c{}^2\end{aligned} \tag{A.2}$$

I begin by substituting in (A.1) the following relationship (derived in Stiglitz (6))

$$v_{11} = v_{22}H^2 + v_2(H_1 + H_2H) \tag{A.3}$$

to obtain

$$\beta = -\frac{1}{2}\frac{v_{22}}{v_2}\bar{n}\cdot H^2 s_n{}^2 - \frac{\bar{n}}{2}s_n{}^2(H_1 + H_2H) \tag{A.4}$$

I now use the following

LEMMA

$$\frac{v_{22}}{v_2} = \frac{u_{cc}}{u_c} + \frac{H_2^2}{H_1^c} \tag{A.5}$$

Proof. We have

$$u_c = v_2 \tag{A.6}$$

Differentiate (A.6) w.r.t. s to obtain

$$u_{cc}c_s + u_{cl}l_s = v_{22} \tag{A.7}$$

while from the budget constraint

$$c_s + nl_s = 1 \tag{A.8}$$

Substitute (A.8) into (A.7) to obtain

$$u_{cc} - (u_{cl} - wu_{cc})H_2 = v_{22} \tag{A.9}$$

Now differentiate (A.6) w.r.t. n to obtain

$$u_{cc}H - (u_{cl} - wu_{cc})H_1 = v_{12} = v_{22}H + v_2H_2 \tag{A.10}$$

(A.10) − H × (A.9) then gives

$$-(u_{cl} - wu_{cc})H_1^c = v_2H_2 \tag{A.11}$$

Substitute (A.11) into (A.9) and we have (A.5).

Now substitute (A.5) into (A.4) and we have

$$\begin{aligned}\beta &= -\frac{1}{2}\frac{u_{cc}}{u_c}\bar{n}H^2{s_n}^2 - \frac{1}{2}\bar{n}H^2{s_n}^2\frac{{H_2}^2}{H_1^c} - \frac{\bar{n}}{2}{s_n}^2(H_1 + H_2H)\\ &= -\frac{1}{2}\frac{u_{cc}}{u_c}\bar{n}H^2{s_n}^2 - \frac{1}{2}\frac{\bar{n}{s_n}^2(HH_2 + H_1^c)^2}{H_1^c}\\ &= -\frac{1}{2}\frac{u_{cc}}{u_c}\bar{n}H^2{s_n}^2 - \frac{1}{2}\bar{n}{s_n}^2 \cdot \frac{{H_1}^2}{H_1^c}\end{aligned} \tag{A.12}$$

Now use the relationship, again derived in Stiglitz (2),

$${s_c}^2 = {s_n}^2(1 + \eta_1)^2$$

and we have

$$\beta = \frac{\alpha}{(1 + \eta_1)^2} - \frac{\bar{H}}{2}\left(\frac{\eta_1}{1 + \eta_1}\right)^2\frac{{s_c}^2}{\eta_1^c}$$

which is (30) of the text.

REFERENCES

1. M. G. ALLINGHAM, "The Measurement of Inequality," *Journal of Economic Theory* **5** (1972), 163–196.
2. C. V. BROWN, E. LEVIN AND D. T. ULPH, "Estimates of Labour Hours Supplied by Married Male Workers in Great Britain," *Scottish Journal of Political Economy* (1976).
3. J. A. MIRRLEES, "An Explanation in the Theory of Optimum Income Taxation," *Review of Economic Studies* **38** (1971), 175–208.
4. E. PAZNER, "Pitfalls in the Theory of Fairness," *Journal of Economic Theory* **14** (1971), 458–466.
5. E. PAZNER AND D. SCHMEIDLER, "A Difficulty in the Concept of Fairness," *Review of Economic Studies* **41** (1974), 441–443.
6. N. H. STERN, "On the Specification of Models of Optimum Income Taxation," *Journal of Public Economics*, **6** (1976), 123–162.
7. J. E. STIGLITZ, "Simple Formulae for Optimal Income Taxation and the Measurement of Inequality," *Journal of Economic Theory* (forthcoming).
8. D. T. ULPH, "Labour Supply, Taxation and the Measurement of Inequality," mimeo, 1978.
9. H. R. VARIAN, "Equity, Envy and Efficiency," *Journal of Economic Theory* **9** (1974), 63–91.
10. H. R. VARIAN, "Distributive Justice, Welfare Economics and the Theory of Fairness," *Philosophy of Public Affairs* **4** (1975), 223–247.

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On the Theory of Dynamic Input–Output Models with Different Time Profiles of Capital Construction and Finite Life-Time of Capital Equipment

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1. BACKGROUND

There exists a well developed theory of "classical" or standard dynamic input-output models with uniform construction or gestation periods for all sectors and types of capital equipment, and with infinite durability of capital. In practice different construction periods are very important, ranging from, say, one year up to some seven or eight years. For some sectors finite durability of capital equipment is also important, although, for many purposes, perhaps less so than the differences in constructions period. In practical planning dynamic input-output models with different construction periods seem to have been used mainly in Eastern Europe, but examples can also be found in some developing countries. Experience particularly in Eastern Europe suggests that the length of the construction periods is of considerable importance for the growth potential of an economy. In this paper we shall be concerned particularly with the growth aspect, although construction periods are obviously important also for the generation of possible cycles.

In addition to the formulation of the dynamic system in terms of quantities of inputs and outputs, we shall also study the corresponding price system. In connection with the more complicated dynamic pattern in the present context, it will be elucidating to consider the price version of the input-output model as a system of equations representing investment calculations rahter than equations representing price fixation on the basis of cost calculations.

As far as possible we shall exploit results for the simplest case of a dynamic input-output model. For convenience we put down the equations for this model.

Let there be n sectors, and let x_{it} and y_{it} represent total output and final demand for sector i in period t. Let a_{ij} be the current input coefficients

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and b_{ij} be the coefficients for capital stock produced in sector i and used in sector j. The ordinary input-output equations are then

$$x_{it} - \sum_{j=1}^{n} a_{ij}x_{jt} - \sum_{j=1}^{n} b_{ij}(x_{j,t+1} - x_{jt}) = y_{it} \qquad (i = 1,\dots, n). \tag{1.1}$$

In order to study the growth possibilities in such a system, it is useful to study first the corresponding homogeneous system, i.e. the system for which all $y_{it} = 0$. For this system we have a balanced growth path with a growth rate λ, i.e. a growth path at which

$$x_{it} = x_i(1 + \lambda)^t \tag{1.2}$$

where the composition of output $x_1, \dots, x_n$ and the growth rate λ must satisfy the following equations:

$$x_i - \sum_{j=1}^{n} a_{ij}x_j - \lambda \sum_{j=1}^{n} b_{ij}x_j = 0 \qquad (i = 1,\dots, n). \tag{1.3}$$

The value of λ is determined as the real positive root of the equation

$$|I - A - \lambda B| = 0 \tag{1.4}$$

where I is the unit matrix, A is the matrix of a_{ij} and B is the matrix of b_{ij}.

The value of λ represents the growth potential of the economy in the sense that it is the growth rate which is achievable if all outputs are used as inputs into production again (as current input and investment). If there is some non-productive consumption, represented by $y_{it} > 0$, then the growth rate will be some proportion of λ determined by the savings ratio.

The price equations corresponding to this system is

$$p_i = \sum_{j=1}^{n} p_j a_{ji} + r \sum_{j=1}^{n} p_j b_{ji} \qquad (i = 1,\dots, n) \tag{1.5}$$

where $p_1, \dots, p_n$ are prices and r is the interest rate. The value of r is determined by an equation similar to (1.4), and we have $r = \lambda$. This equality represents the fact that the rate of interest is equal to the rate at which capital is able to grow if all outputs are ploughed back as productive inputs.

For some purposes it is useful to have the following formula for the change in λ (and r) when the coefficients a_{ij} and b_{ij} are changed:

$$d\lambda = -\frac{\sum_{i,j}(p_i x_j da_{ij} + \lambda p_i x_j db_{ij})}{\sum_{i,j} p_i b_{ij} x_j} \tag{1.6}$$

This formula can be derived very simply by differentiating the equations in (1.3), multiplying the resulting equations by p_i, taking the sum over i, and finally using Eq. (1.5). (A. Bródy [2] uses this formula for assessing the consequences for numerical computations of λ of errors in the data for a_{ij} and b_{ij}. However, the formula also has important economic interpretations, giving directly the consequences for the growth rate — and the interest rate — of various sorts of combined changes in the coefficients of the input-output system.)

Eq. (1.3) and (1.4) above were written down for the case of a closed model, i.e. the case in which the elements y_{it} in (1.1) are zero. If we interpret the elements y_{it} in (1.1) as consumption, then we could again study the growth rate under proportional growth, i.e. growth in which all elements of output and consumption grow at the same rate. Then the growth rate will be equal to the investment or savings ratio multiplied by the intrinsic growth rate λ determined by (1.4), when the investment or savings ratio is calculated in terms of total income and total consumption evaluated by means of the equilibrium prices determined by the price equations (1.5). For an elaboration of this observation, see L. Johansen [6].

The equations of the model as written above do not specify labour explicitly. The model could be conceived as a pure capital accumulation model, with labour abundance, or one may in a well-known way interpret one sector, say number n, as a labour sector, defining the coefficients on the row $a_{n1},\ldots, a_{nn}$ as labour input coefficients, and the column coefficients $a_{1n},\ldots, a_{nn}$ as consumption requirements per unit of work. The price p_n would then be the minimum wage rate necessary to cover the corresponding consumption expenditures. (If we interpret sector n in this way, and still have final demand items $y_{1t},\ldots, y_{nt}$, then these items are final demand categories in excess of necessary consumption, for instance interpreted as consumption out of non-wage income.)

This remark on the treatment of labour applies also to the more general model set out in the following sections.

2. THE MODEL WITH DIFFERENT TIME PROFILES FOR CAPITAL CONSTRUCTION AND FINITE LIFE-TIME OF CAPITAL EQUIPMENT

Let b_{ij} represent the amount of capital equipment produced in sector i and needed per unit of output in sector j, as before. However, we now decompose b_{ij} into elements $b_{ij1}, b_{ij2},\ldots, b_{ijT}$ such that

$$b_{ij1} + b_{ij2} + \cdots + b_{ijT} = b_{ij}\,.$$

Here $b_{ij\theta}$ is input in capital construction which must be delivered θ periods before the piece of production capacity, of which it is a part, is to be ready for use, for $\theta = 1,\ldots, T$. T is accordingly the longest gestation period, and the distribution of b_{ij} over $b_{ij1}, \ldots, b_{ijT}$ gives the time profile of this particular input into capital construction. We could of course introduce T_{ij} for the longest gestation period for deliveries belonging to b_{ij}. However, we get a simpler notation system by using T for the longest gestation period for the whole system, implying that $b_{ij\theta} = 0$ for $\theta > T_{ij}$.

On the output side we let capacity created in sector i in a certain period last for S_i periods. In other words, the capacity remains constant for S_i periods, and then suddenly drops to zero. It is possible to extend the model so as to cover also more general cases (and this has recently been done by M. Åberg and H. Persson [8]).

In setting out the quantity equations for this model it is convenient to write down separately equations for output and equations for changes in capacity. As above we let x_{it} represent output from sector i in period t. We let production capacity in the same sector in the same period be K_{it}, so that $x_{it} \leq K_{it}$. Furthermore, production capacity to be completed so as to be ready for use in period t will be denoted by k_{it}. This is a gross concept, representing net addition to capacity as well as replacement investment. Instead of (1.1) we now have the following equations for output:

$$x_{it} - \sum_{j=1}^{n} a_{ij}x_{jt} - \sum_{\theta=1}^{T} \sum_{j=1}^{n} b_{ij\theta}k_{j,t+\theta} = y_{it} \qquad (i = 1, \ldots, n). \tag{2.1}$$

The double sum in this formulation represents deliveries to capacity which is to be ready for use 1, 2,..., T periods ahead.

Total capacity in period t is determined as total capacity in the preceding period, $t - 1$, plus gross addition to capacity, k_{it}, minus capacity which is becoming obsolete in period t, which is the same as capacity created S_i periods earlier, i.e. $k_{i,t-S_i}$. This gives the following equation

$$K_{it} = K_{i,t-1} + k_{it} - k_{i,t-S_i} \qquad (i = 1, \ldots, n). \tag{2.2}$$

We now want to study balanced growth paths with full capacity utilization, i.e. we assume

$$x_{it} = K_{it} \qquad (i = 1, \ldots, n) \tag{2.3}$$

so that

$$x_{it} = x_{i,t-1} + k_{it} - k_{i,t-S_i} \qquad (i = 1, \ldots, n). \tag{2.4}$$

Our system now consists of (2.1) and (2.4) with x_{it} and k_{it} as variables.

We again study the homogeneous system and consider especially the possibility of proportional growth in all sectors. This requires both output and capacity creation to grow at the same rate, i.e.

$$x_{it} = x_i(1 + \lambda)^t, \; k_{it} = k_i(1 + \lambda)^t \qquad (i = 1, \ldots, n). \tag{2.5}$$

Inserting this in Eq. (2.1) and (2.4) we obtain, after cancellation of the factor $(1+\lambda)^t$:

$$x_i - \sum_{j=1}^{n} a_{ij}x_j - \sum_{\theta=1}^{T}\sum_{j=1}^{n} b_{ij\theta}k_j(1+\lambda)^{\theta} = 0, \tag{2.6}$$

$$x_j = x_j(1+\lambda)^{-1} + k_j - k_j(1+\lambda)^{-S_j}. \tag{2.7}$$

We can now use Eq. (2.7) to express k_j in terms of x_j. (We assume throughout that $\lambda > 0$.) Inserting the resulting expression into (2.6) we have

$$x_i - \sum_{j=1}^{n} a_{ij}x_j - \lambda \sum_{j=1}^{n}\left[\sum_{\theta=1}^{T} b_{ij\theta}\frac{(1+\lambda)^{S_j+\theta-1}}{(1+\lambda)^{S_j}-1}\right]x_j = 0. \tag{2.8}$$

This system of equations is comparable to (1.3) for the standard dynamic input-output model. It reduces to this form if $T = 1$ and $S_j \to \infty$. Then the sum in brackets in (2.8) contains only one element which tends to b_{ij1} when $S_j \to \infty$.

For the further consideration of system (2.8) it is convenient to introduce a sort of total capital coefficients defined (for $\lambda \neq 0$) by

$$b^*_{ij} = b^*_{ij}(\lambda) = \sum_{\theta=1}^{T} b_{ij\theta}\frac{(1+\lambda)^{S_j+\theta-1}}{(1+\lambda)^{S_j}-1}. \tag{2.9}$$

We write $b^*_{ij}(\lambda)$ to signify that these are functions of the growth rate λ. With these symbols introduced the system (2.8) can be written as

$$x_i - \sum_{j=1}^{n} a_{ij}x_j - \lambda\sum_{j=1}^{n} b^*_{ij}(\lambda)x_j = 0. \tag{2.10}$$

Corresponding to the determination of λ by (1.4) in the standard case λ is now to be determined by

$$|I - A - \lambda B^*(\lambda)| = 0 \tag{2.11}$$

where $B^*(\lambda)$ is the matrix of the coefficients defined by (2.9) as functions of λ.

The introduction of the "coefficients" $b^*_{ij}(\lambda)$ is useful because it makes the system similar to the system for the standard case, the "only" difference being that the constant matrix B in the standard case is replaced by a matrix $B^*(\lambda)$ whose elements depend on λ. One step in the same direction is taken in papers by V. Z. Belenkii et al. [1] and by V. A. Volkonskii [7] who introduce similar coefficients which reflect the time profiles of capital construction. They are somewhat more general in that they assume different growth rates for different sectors, but less general in that they do not allow for different durabilities. On the other hand W. F. Gossling [3, 4] uses a similar type of "coefficients" to reflect replacement needs in the case of finite durabilities of capital equipment, but does not allow for different time profiles of capital construction.

Some further comments on the economic interpretation of the "coefficients" $b_{ij}^*(\lambda)$ will be given in section 4.

The system now described will determine a unique value of the growth rate λ. If we are able to solve (1.4) numerically for λ, then it will also be a simple matter to solve (2.11), at least approximately, by iteration or interpolation. Some further observations on these aspects are given in the appendix notes.

From the formulation just given one can see the direction of the effect of the time profiles introduced in the present formulation of the dynamic input-output model. Consider first the capital construction profile. Suppose that we have the correct solution for a special set of profiles. Then increase the production or gestation periods by shifting some parts of the total coefficients b_{ij} towards longer lags, keeping the totals fixed as given by $b_{ij} = \sum_\theta b_{ij\theta}$. If we calculate $b_{ij}^*(\lambda)$ for the new profile of construction periods, but with the old value of λ, then it follows from (2.9) that the values of $b_{ij}^*(\lambda)$ will increase, i.e. the elements of the matrix $B^*(\lambda)$ in (2.11) will increase. The old value of λ does then no longer satisfy Eq. (2.11), and the new correct value must be lower than the previous one. (The formal proof is given in section 5.) In other words, extentions of construction periods diminish the growth rate.

Consider next the durabilities of capital equipment S_j. If some S_j is increased, then all $b_{ij}^*(\lambda)$ for this j decline, for the initially given value of λ. Then this value of λ does no longer satisfy (2.11), and the new solution for λ will give a higher value than the previous one. In other words, increases in the durability of capital equipment increase the growth rate.

3. PRICES, INTEREST RATE, AND INVESTMENT PROFITABILITY

In the standard case the price of the dynamic input-output model can be written down in the form (1.5), simply as current input cost plus an interest rate applied to the value of capital stock in each sector. In the case of different construction periods and finite lifetime of capital equipment this cannot be done so simply, because we cannot write down a simple expression for the value of capital per unit of output as done by $\sum_j p_j b_{ji}$ in (1.5). It seems better to approach the price problem via the investment profitability calculations. The equilibrium requirement now is that the present value of investment outlays and the present value of future revenue achieved by establishing a piece of production capacity in a sector should be in balance.

Now consider sector i. For the comparison of present value of outlays with the present value of revenues it is immaterial to which point if time we refer the various items by discounting. We consider the point of time at which a piece of capacity is being completed and ready for use. For establishing a capacity of one unit in sector i we must invest an amount $\sum_j p_j b_{ji1}$ one period earlier, an amount $\sum_j p_j b_{ji2}$ two periods earlier, etc. The value of this stream of investment outlays, calculated at the point of time when the capacity is completed, will be

$$\sum_{\theta=1}^{T} \sum_{j=1}^{n} p_j b_{ji\theta}(1+r)^{\theta}. \tag{3.1}$$

When the capacity has been installed, it will produce one unit of output per period for S_i periods, beginning with the immediately commencing period. The revenue earned per period will be $(p_i - \sum_j p_j a_{ji})$, so that the present value of the stream of revenues will be

$$\begin{aligned}\left(p_i - \sum_{j=1}^{n} p_j a_{ji}\right)&\left[1 + \frac{1}{1+r} + \left(\frac{1}{1+r}\right)^2 + \cdots + \left(\frac{1}{1+r}\right)^{S_i-1}\right] \\ &= \frac{(1+r)^{S_i}-1}{r(1+r)^{S_i-1}}\left(p_i - \sum_{j=1}^{n} p_j a_{ji}\right).\end{aligned} \tag{3.2}$$

Equilibrium now requires that (3.1) and (3.2) be equal:

$$\sum_{\theta=1}^{T} \sum_{j=1}^{n} p_j b_{ji\theta}(1+r)^{\theta} = \frac{(1+r)^{S_i}-1}{r(1+r)^{S_i-1}}\left(p_i - \sum_{j=1}^{n} p_j a_{ji}\right). \tag{3.3}$$

This formula can be reorganized so as to make it comparable with the formulas for the volume system. We obtain

$$p_i - \sum_{j=1}^{n} p_j a_{ji} - r \sum_{j=1}^{n} p_j \left[\sum_{\theta=1}^{T} b_{ji\theta} \frac{(1+r)^{S_i+\theta-1}}{(1+r)^{S_i}-1}\right] = 0. \tag{3.4}$$

This is comparable with (2.8). Introducing the notations from (2.9) we can write the equations as

$$p_i - \sum_{j=1}^{n} p_j a_{ji} - r \sum_{j=1}^{n} p_j b^*_{ji}(r) = 0 \tag{3.5}$$

corresponding to (2.10). For the determination of the rate of interest r this yields the same equation as for the determination of the growth rate λ in the volume system, i.e.

$$|I - A - rB^*(r)| = 0. \tag{3.6}$$

This means that we have the rate of interest equal to the growth rate, i.e. $r = \lambda$, just as for the standard system set out in the introductory section.

We may conclude as follows: By requiring present value calculations for investments to be in balance, we have obtained a system of price equations which stand in relation to the volume equations in exactly the same way as for the price equations in the standard dynamic input-output model. We have also obtained a way of representing the generalized system, allowing for different time profiles of capital construction and finite life-time of capital equipment, which is exactly the same as for the standard system, except for the fact that we have to replace the constant capital input coefficients b_{ij} by "coefficients" $b^*_{ij}(\lambda)$ or $b^*_{ij}(r)$ which are not constant, but depend on the growth rate or the rate of interest.

4. ALTERNATIVE DERIVATION OF THE PRICE EQUATIONS

As already pointed out we do not have a simple measure of capital value, independent of the rate of interest, in the present model. Furthermore, if we consider a piece of production capacity through its history, it will begin to increase in value from the first investment; it will reach the maximal value at the point of time when it is ready for use, and from then on decline in value towards the expiration of its life-time. Under these circumstances we cannot build up a price equation from the cost side by taking costs into account simply by multiplying capital value by the rate of interest. This is why we preferred the investment calculation point of view used in preceding section.

However, as an alternative it is possible to build up the price equations from cost considerations by using an annuity method of calculating interest charges and amortization. We then write the price equations in the form

$$p_i = \sum_{j=1}^{n} p_j a_{ji} + D_i \tag{4.1}$$

where D_i represents capital costs.

We now interpret D_i as a constant amount every year over the life-time of the capacity such that its present value is equal to the investment made. The value of capital invested per unit of capacity in sector i, calculated at the point of time when the capacity is ready for use, is

$$J_i = \sum_{\theta=1}^{T} \sum_{j=1}^{n} p_j b_{ji\theta}(1+r)^{\theta}. \tag{4.2}$$

Accordingly we get the following relation between D_i and J_i :

$$D_i\left[1 + \frac{1}{1+r} + \cdots + \left(\frac{1}{1+r}\right)^{S_i-1}\right] = J_i\,, \tag{4.3}$$

which gives

$$D_i = \frac{r(1+r)^{S_i-1}}{(1+r)^{S_i}-1}\, J_i\,. \tag{4.4}$$

If we combine this with (4.2) and insert in (4.1), then we get precisely the price equations (3.4).

If we add the amounts D_i over the life-time of the equipment, then we get more than J_i. We can consider D_i as covering both amortization and interest charges. The essential thing is that the total annuity is worth J_i, but we might consider D_i as made up of a declining interest charge and an increasing amortization amount, according to the "constant total annuity" principle of amortization (see e.g. J. Lesourne [5, pp. 248 49]).

We may conclude this section by giving yet another interpretation which is perhaps less convincing as a representation of cost accounting, but which may nevertheless be of interest because of the light it sheds on the interpretation of the capital coefficients $b^*_{ij}(\lambda)$ or $b^*_{ij}(r)$.

Consider first a breakdown of the investment J_i defined by (4.2) according to sectors of origin j, i.e. consider

$$p_j \sum_{\theta=1}^{T} b_{ji\theta}(1+r)^{\theta-1}. \tag{4.5}$$

This is the value of capital produced in sector j which must be invested in sector i per unit of output from sector i. (As a minor deviation from (4.2) we have here calculated the value of the investment at the point of time of the last input into the project, i.e. one period before it is actually used. This is the reason for the exponent $\theta - 1$ instead of θ. In the present context this provides a better connection with previous formulas.) However, the capacity created lasts only for S_i periods. Thus, if we want to sustain the production of one unit of output from sector i for an indefinite time, then this investment must be repeated every S_i period. In this sense we may say that the capital requirement for capital produced by sector j and used in sector i, per unit of output from sector i, will be

$$\begin{aligned} &p_j \sum_{\theta=1}^{T} b_{ji\theta}(1+r)^{\theta-1}\left[1+\left(\frac{1}{1+r}\right)^{S_i}+\left(\frac{1}{1+r}\right)^{2S_i}+\cdots\right] \\ &\quad = p_j\left[\sum_{\theta=1}^{T} b_{ji\theta}(1+r)^{\theta-1}\right]\frac{1}{1-(1+r)^{-S_i}} = p_j b^*_{ji}(r). \end{aligned} \tag{4.6}$$

The last equality follows by a simple reorganization of formula (2.9). According to this, total capital value per unit of output in sector i in the sense now given will be

$$\sum_{j=1}^{n} p_j b^*_{ji}(r), \tag{4.7}$$

and the last term in the price equation (3.5) can be interpreted as interest on this extended capital concept.

Comparing the two approaches presented in this section, we may say that there is a choice between two viewpoints: We may consider capital costs as consisting of interest and amortization on existing capital equipment, or, we may consider capital costs as consisting of interest on the extended capital concept which is the present value of the stream of investments necessary to sustain a constant flow of output when an investment has to be repeated at certain intervals corresponding to the finite durability of capital equipment.

5. EVALUATION OF ALTERNATIVE TECHNIQUES

For the standard dynamic input-output model we have formula (1.6) for the effects of changes in the technological coefficients on the growth rate. We now seek a similar expression for the present case with different time profiles of capital construction and finite life-time of capital equipment. In connection with (1.6) we described how this formula could be derived by simple differentiation. If we apply the same procedure to formula (2.10) and use the price equations (3.5), then we get

$$d\lambda = -\frac{\sum_{i,j}(p_i x_j\, da_{ij} + \lambda p_i x_j\, db^*_{ij}(\lambda))}{\sum_{i,j} p_i b^*_{ij}(\lambda)\, x_j}. \tag{5.1}$$

In this formula the differential $db^*_{ij}(\lambda)$ must be evaluated so as to be expressed in terms of changes in the basic coefficients $b_{ij\theta}$. We must then also take into account that $db^*_{ij}(\lambda)$ depends on $d\lambda$. We have

$$db^*_{ij}(\lambda) = \sum_{\theta=1}^{T} \frac{\partial b^*_{ij}(\lambda)}{\partial b_{ij\theta}}\, db_{ij\theta} + \frac{\partial b^*_{ij}(\lambda)}{\partial\lambda}\, d\lambda \tag{5.2}$$

where $b^*_{ij}(\lambda)$ is defined by (2.9). (We use partial differentiation of $b^*_{ij}(\lambda)$ since $b^*_{ij}(\lambda)$ is a function of all $b_{ij\theta}$ as well as of λ, although we have for simplicity not indicated this explicitly in the notations.)

The partial derivatives of $b^*_{ij}(\lambda)$ with respect to $b_{ij\theta}$ are seen immediately from (2.9). The derivative with respect to λ can also be written down on the basis of (2.9), but does not simplify in any very convenient way. We therefore retain it as $\partial b^*_{ij}(\lambda)/\partial\lambda$. Then we get

$$d\lambda = -\frac{\sum_{i,j}(p_i x_j\, da_{ij} + \lambda p_i x_j \sum_{\theta=1}^{T} [(1+\lambda)^{S_j+\theta-1}/((1+\lambda)^{S_j}-1)]\, db_{ij\theta})}{\sum_{i,j}(p_i b^*_{ij}(\lambda)\, x_j + \lambda p_i(\partial b^*_{ij}(\lambda)/\partial\lambda)\, x_j)}. \tag{5.3}$$

The denominator in this expression can be shown to be positive. For this purpose introduce

$$\tilde{b}_{ij}(\lambda) = \lambda b^{*}_{ij}(\lambda). \tag{5.4}$$

Using the definition (2.9) this can be written as

$$\tilde{b}_{ij}(\lambda) = \sum_{\theta=1}^{T} b_{ij\theta}(1+\lambda)^{\theta} \Big/ \sum_{\tau=0}^{S_j-1} (1+\lambda)^{\tau} \tag{5.5}$$

from which it is easily seen that

$$\frac{\partial \tilde{b}_{ij}(\lambda)}{\partial \lambda} > 0. \tag{5.6}$$

On the other hand, we have

$$\frac{\partial \tilde{b}_{ij}(\lambda)}{\partial \lambda} = b^{*}_{ij}(\lambda) + \lambda \frac{\partial b^{*}_{ij}(\lambda)}{\partial \lambda}. \tag{5.7}$$

Then the positivity of the denominator in (5.3) follows.

The direction of the change in the growth rate by changes in the coefficients will accordingly be determined by the sign of the numerator. Thus, if we have a technological change affecting the input coefficients in a certain sector j, then the effect on λ is determined by the sign of

$$\sum_{i=1}^{n} p_i\, da_{ij} + \lambda \sum_{i=1}^{n} p_i \sum_{\theta=1}^{T} \frac{(1+\lambda)^{S_j+\theta-1}}{(1+\lambda)^{S_j}-1}\, db_{ij\theta}, \tag{5.8}$$

or, if we prefer, by the sign of

$$\frac{(1+r)^{S_j}-1}{r(1+r)^{S_j-1}} \sum_{i=1}^{n} p_i\, da_{ij} + \sum_{i=1}^{n} p_i \sum_{\theta=1}^{T} (1+r)^{\theta}\, db_{ij\theta}. \tag{5.9}$$

In this expression the first term is the present value (referring to the point of time when a piece of capacity is ready for use) of the changes in current input costs over the coming S_j periods, and the second term is the present value of the changes in investment costs. If the expression is positive, the value of the growth rate λ will decline, and vice versa. Thus, the effect of a change in technology on the growth rate can be indicated by a present value calculation of the implied cost changes, using the equilibrium prices and the equilibrium rate of interest. From this criterion it is clear, as pointed out before, that a change in the time profile of capital construction which shifts some capital input from a longer to a shorter gestation lag, while keeping the total capital input as defined by $\sum_{\theta} b_{ij\theta}$ constant, will increase the growth rate. However, for the exact numerical evaluation of the effect one must use the full expression (5.3).

6. THE MODEL WITH POSITIVE CONSUMPTION DEMAND. SAVINGS AND GROWTH

We now return to the model as given by (2.1)-(2.4) with positive final, non-productive demand, i.e. $y_{it} \geq 0$ for $i = 1,\dots, n$, and $y_{it} > 0$ for at least some i. We shall refer to y_{it} as consumption. (As already suggested, if there is a labour sector among $1,\dots, n$, then y_{it} could be interpreted as consumption out of non-wage income.)

When there is positive consumption, then the growth rate will no longer be the $\lambda = r$ previously considered. We now consider proportional growth of production and consumption at a rate γ. Inserting this and arranging the equations in the same way as we did in section 2, we now get

$$x_i - \sum_{j=1}^{n} a_{ij}x_j - \gamma \sum_{j=1}^{n} b^*_{ij}(\gamma)\, x_j = y_i \qquad (i = 1,\dots, n) \tag{6.1}$$

where $b^*_{ij}(\gamma)$ corresponds to $b^*_{ij}(\lambda)$ as defined by (2.9), only with the growth rate λ replaced by γ.

The system (6.1) is no longer homogeneous in $x_1,\dots, x_n$, and γ cannot be found by an equation like (2.11). Something must also be added in order to make the problem of finding γ determinate. This missing element must clearly be related to the saving or investment proportion in the economy. By inspection of (6.1) it seems plausible that, if we keep $x_1,\dots, x_n$ normalized to a certain level, and then increase consumption $y_1,\dots, y_n$, then the proportions between $x_1,\dots, x_n$ will have to be readjusted, and γ will have to be reduced in order to make the equations hold. (From (5.4–6) we know that $\gamma b^*_{ij}(\gamma)$ decreases if γ decreases.) The increase in consumption as compared with the level of production, means a reduced investment proportion. Accordingly, we should expect a higher consumption proportion to be associated with a lower rate of growth γ. To make these considerations more precise we have to bring the prices into the picture. We then use the equilibrium prices and the rate of interest as determined in section 3.

For simplicity we introduce the following notations:

$v_i = p_i - \sum_{j=1}^{n} p_j a_{ji}$ = gross value added per unit of output in sector i (profits if there is a labour sector among $1,\dots, n$);

$V = \sum_{i=1}^{n} v_i x_i$ = total gross value added;

$C = \sum_{i=1}^{n} p_i y_i$ = value of total consumption;

$J = V - C$ = value of gross investment.

Now multiply Eq. (6.1) by price p_i and take the sum over i. This gives

$$J = V - C = \gamma \sum_{i=1}^{n} \sum_{j=1}^{n} p_i b_{ij}^*(\gamma)\, x_j \,. \tag{6.2}$$

On the other hand, multiply through the equations in (3.5) – the equations determining the equilibrium prices and the interest rate – by x_i and take the sum over i. This gives

$$V = r \sum_{i=1}^{n} \sum_{j=1}^{n} p_i b_{ij}^*(r)\, x_j \,. \tag{6.3}$$

Combining (6.2) and (6.3) we get for the growth rate γ:

$$\gamma = \frac{J}{V}\, rR \text{ where } R = R(r, \gamma) = \frac{\sum_{i,j} p_i b_{ij}^*(r)\, x_j}{\sum_{i,j} p_i b_{ij}^*(\gamma)\, x_j} \,. \tag{6.4}$$

This expression consists of the gross investment quota J/V (investment in proportion to profits if there is a labour sector among 1,..., n) multiplied by the rate of interest r which reflects the productivity of capital, corrected by a term $R = R(r, \gamma)$ which in general depends on both r and γ and on prices and output composition.

For the standard case $S_j = \infty$ and $T = 1$, $b_{ij}^*(r)$ and $b_{ij}^*(\gamma)$ both reduce to a constant $b_{ij1} = b_{ij}$. Then the proportion to the right in (6.4) is identically equal to unity and we have

$$\gamma = \frac{J}{V}\, r \qquad (\text{for } S_j = \infty \text{ and } T = 1). \tag{6.5}$$

In this case there is no scrapping or depreciation. J/V is the investment proportion in total income, and the growth rate is the product of this investment proportion and the rate of interest r, which reflects capital productivity. Three points are worth observing in this connection.

1. The formula just given corresponds to the classical growth formula for one-sector models.

2. In deriving this growth formula we have used the equilibrium prices in calculating the value of gross investment J and total value added V. Accordingly, it appears that these equilibrium prices serve excellently in performing the aggregation, since (6.5) is an exact formula under the conditions stated when J and V are calculated by means of the equilibrium prices, i.e. there is no aggregation error.

3. An interesting corollary of this result is a *dynamic non-substitution theorem*. If there is a choice between different techniques in the various sectors and we want to maximize the growth rate γ under a fixed investment proportion J/V, then we should use the production techniques which maximize

the intrinsic growth rate $\lambda = r$. Criteria for this maximization follow from the methods of evaluation of alternative techniques given in section 5. The non-substitution aspect of this is that the technologies thus determined should be used regardless of the composition of consumption. (This non-substitution interpretation has been elaborated in L. Johansen [6].)

When the conditions stipulated for (6.5) do not hold, then the proportion to the right in (6.4) is variable. We may trace the connection between the gross investment quota J/V and the growth rate γ in the following way. Consider the system (6.1) and keep $y_1, \ldots, y_n$ fixed. For a given value of γ we can then solve the system for $x_1, \ldots, x_n$. (If we like we could require a normalization of $x_1, \ldots, x_n$ and readjust the level of $y_1, \ldots, y_n$.) By inserting the given value of γ and the resulting $x_1, \ldots, x_n$ into (6.4) we can calculate the implied gross investment quota J/V. By repeating this calculation for different values of γ we can trace a curve relating J/V and γ. For $\gamma = r$ it is clear that there is no room for $y_1, \ldots, y_n = 0$, so at this end we have $\gamma = r$ and $J/V = 1$. At the other end we have $\gamma = 0$. This case needs a special treatment since the definition (2.9) of $b_{ij}^*(\gamma)$ is not valid for this case. Clearly $b_{ij}^*(\gamma) \to \infty$ as $\gamma \to +0$. On the other hand, for the case $\gamma = 0$ we could replace $\gamma b_{ij}^*(\gamma)$ by

$$\tilde{b}_{ij}(0) = \frac{1}{S_j} \sum_{\theta=1}^{T} b_{ij\theta} \tag{6.6}$$

which follows from (5.5). We then get from (6.2) and (6.3)

$$\frac{J}{V} = \sum_{i,j} p_i \frac{\sum_\theta b_{ij\theta}}{S_j} x_j \Big/ r \sum_{i,j} p_i b_{ij}^*(r)\, x_j \qquad \text{for} \quad \gamma = 0. \tag{6.7}$$

In other words, we have a positive gross investment quota for a zero growth rate, which is of course as it should be.

The fact that $\gamma b_{ij}^*(\gamma)$ increases with γ (see (5.6)) suggests that there is a positive association between γ and J/V. We have

$$\sum_{i,j} p_i [\gamma b_{ij}^*(\gamma)]\, x_j = \frac{J}{V}\, r \sum_{i,j} p_i b_{ij}^*(r)\, x_j \,,$$

and for the increasing γ the left hand side would tend to increase while prices, r and $b_{ij}^*(r)$ remain the same. However, this consideration is not quite sufficient to establish a monotonous positive association since the composition of output $x_1, \ldots, x_n$ in general changes with the growth rate. It only establishes a presumption for normal cases.

The natural idea now is, of course, to introduce the concepts of *net* investment and *net* income and try ot establish an equation like

$$\gamma = \frac{\text{Net investment}}{\text{Net income}} \cdot r = sr = s\lambda \tag{6.8}$$

where net investment is gross investment J minus depreciation, net income is gross value added V minus depreciation, and s is introduced for the rate of saving out of net income. In fact, when r and γ are determined as explained before, then equation (6.8) can be shown to hold as a consequence of reasonable definitions of the concepts involved.

In order to see this it is necessary to introduce the notion of total value of existing capital. We write this as $K_t(r)$ for period t, and $K(r)$ for the base point of time. Since each piece of capital now lives through a history with a construction period of, in general, more than one time unit, and a productive life with gradually declining remaining life-time, the value of capital cannot be calculated independently of the rate of interest. This is signified by the notations just given. The value of capital will, of course, also depend upon prices, but this is true also for the standard case.

We may now define net income as

$$\text{Net income} = rK(r), \tag{6.9}$$

i.e. as the product of the interest rate and the value of capital stock. The rationale of this is that the rate of interest is equal to the intrinsic growth rate λ as shown before. This means that, if nothing is extracted for consumption, then the value of total capital stock would increase by $\lambda K_t(r) = rK_t(r)$ from one period to the next, and income is defined as the value which can be extracted from the system without reducing total wealth.

If we now introduce

$$I = \text{value of total net investment,}$$

and use C for consumption as introduced above, then the concepts should satisfy the following equation

$$C + I = rK(r). \tag{6.10}$$

Furthermore, since we are studying proportional growth at a rate γ, we must also have

$$I = \gamma K(r). \tag{6.11}$$

According to the definition of the rate of saving we have

$$I = rK(r) - C = srK(r). \tag{6.12}$$

From (6.11) and (6.12) follows (6.8) as sought for.

It follows from this development that the comments made in connection with (6.5) for the standard case also hold for the general case, including the "dynamic non-substitution theorem". In deriving these results for the general case, we have only used some reasonable requirements which the definitions of the aggregated concepts ought to satify, in addition to the fact that $r = \lambda$ is the rate at which wealth will increase if nothing is extracted for consumption. We have not yet shown in detail how the various concepts could be calculated. However, they are implied by the formulas we have already given.

Let D be total depreciation (at the base point of time). Then the net investment quota involved in (6.8) is $s = (J - D)/(V - D)$. Then (6.4) and (6.8) together imply

$$\frac{J}{V} = R \frac{J - D}{V - D} \tag{6.13}$$

where $R = R(r, \gamma)$ is defined in (6.4). This gives the following expression for depreciation, when we also use (6.4) to eliminate V:

$$D = J \frac{1 - R}{1 - (\gamma/r)}. \tag{6.14}$$

Net income will be

$$V - D = V \frac{1 - (\gamma/rR)}{1 - (\gamma/r)}. \tag{6.15}$$

From the definitional requirement (6.9) we have for the value of capital stock

$$K(r) = \frac{V}{r} \frac{1 - (\gamma/rR)}{1 - (\gamma/r)} \tag{6.16}$$

from which net investment according to (6.11) will be

$$I = V \frac{\gamma}{r} \frac{1 - (\gamma/rR)}{1 - (\gamma/r)} = J \frac{R - (\gamma/r)}{1 - (\gamma/r)} \tag{6.17}$$

where the last equality uses (6.4). It is easily seen that this is the same as $J - D$ as calculated from (6.14).

In the formulas above γ/rR should be replaced by J/V as given by (6.7) for the case of $\gamma = 0$.

The use of the formulas given above should perhaps be recapitulated. Suppose we want to calculate the growth rate corresponding to a given net investment quota. We must then first solve the price system and get the interest rate r (equal to the intrinsic growth rate λ). Then the growth rate γ follows simply from (6.8). Having established the growth rate γ we can use (6.1) to find output composition $x_1, \ldots, x_n$ for a given consumption composition $y_1, \ldots, y_n$. (We can normalize either the level of output or the level of consumption.) Having now both prices and outputs, r and γ, we can calculate $R = R(r, \gamma)$ according to the definition in (6.4). Value added V and gross investment J follow easily from the definitions, and then the other aggregated magnitudes, i.e. total depreciation D, net income $(V - D)$, the value of capital stock $K(r)$, and net investment follow from (6.14–17).

APPENDIX: NOTES ON THE SOLUTION OF THE CHARACTERISTIC EQUATION

The characteristic equation determining the intrinsic growth rate of the dynamic input-output model was given in section 2 in the following form

$$|I - A - \lambda B^*(\lambda)| = 0 \tag{A.1}$$

where the elements of the matrix $B^*(\lambda)$ are given by (2.9) in the text.

In this appendix we shall present some observations on the solution of (A.1). We are only interested in real and positive solutions. Furthermore, we shall exploit as far as possible what is known about the standard case in which we have a constant matrix B instead of $B^*(\lambda)$.

In order to study the existence and possible uniqueness of the solution of equation (A.1), it is convenient to introduce an additional variable z and replace (A.1) by the following system in two equations and two variables, λ and z:

$$|I - A - zB^*(\lambda)| = 0, \tag{A.2}$$

$$z = \lambda. \tag{A.3}$$

From (A.2) we may consider z as a function of λ, $z = g(\lambda)$. (It is known that there will exist one real positive solution z to (A.2) for a given λ if A is productive and irreducible, which we shall assume.) The function will be continuous for $\lambda > 0$ since z will be a continuous function of B^* and B^* is a continuous function of λ.

Let us first consider the case in which all $S_j = \infty$. Then we have

$$b^*_{ij}(\lambda) = \sum_{\theta=1}^{T} b_{ij\theta}(1 + \lambda)^{\theta-1}. \tag{A.4}$$

Then $b^*_{ij}(\lambda)$ is an increasing function of λ for all i, j for which some $b_{ij\theta} > 0$ for some $\theta > 1$; otherwise $b^*_{ij}(\lambda)$ is constant.

Now consider first $\lambda = 0$ in (A.2). The value $\lambda = 0$ raises no special problems in this case since $b^*_{ij}(\lambda)$ is well defined by (A.4) also for $\lambda = 0$. Then the system is the same as

$$|I - A - zB| = 0 \tag{A.5}$$

in which B is the matrix of

$$b_{ij} = \sum_{\theta=1}^{T} b_{ij\theta}. \tag{A.6}$$

This is the same as the characteristic equation of a dynamic input-output model of the standard type for which all investments in a project have been compressed into one period. For this system there exists one real positive solution z, i.e. we have $g(0) > 0$. If we now increase λ, then some or all elements of $B^*(\lambda)$ will increase while none will decline. It is then known that the value of the solution z to (A.2) will decline. (This also follows from (1.6).) Accordingly $g(\lambda)$ is a declining function of λ. It follows from this that there will exist one unique solution for λ in (A.1) corresponding to the unique solution for z and λ in (A.2–3) as illustrated in figure 1.

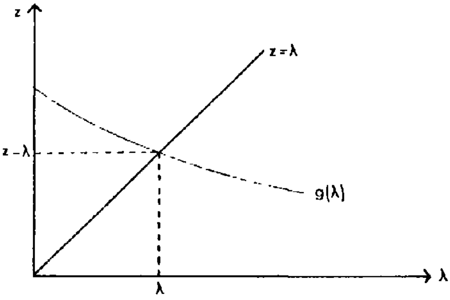

FIGURE 1

If we now go back to the general case in which we have $b_{ij}^*(\lambda)$ given by (2.9), then the reasoning will be more complicated. For a very small value of λ the value of $b_{ij}^*(\lambda)$ will be very large, and it will approach infinity as $\lambda \to +0$. It follows from this that $g(\lambda)$ will now tend to zero as $\lambda \to 0$. In other words, for very small values of λ, $g(\lambda)$ is an increasing function of λ. On the other hand, if we go on increasing λ, then sooner or later all $b_{ij}^*(\lambda)$ for which $b_{ij\theta} > 0$ for some $\theta > 1$ will start increasing, and then $g(\lambda)$ will be a declining function of λ. (This effect will dominate the effect of those $b_{ij}^*(\lambda)$ for which $b_{ij\theta} = 0$ for $\theta > 1$, since these $b_{ij}^*(\lambda)$ will tend to constants as λ increases.)

In this case the solution may be as illustrated in figure 2. In order to secure the existence of a solution $\lambda > 0$ in this case, the curve for $g(\lambda)$ must not lie below the line $z = \lambda$ throughout. In order to see when this will hold it is useful to rewrite Eq. (A.2) into the form

$$\left| I - A - \left(\frac{z}{\lambda}\right) [\lambda B^*(\lambda)] \right| = 0 \qquad \text{(A.7)}$$

or

$$| I - A - \rho \hat{B}(\lambda)| = 0 \text{ where } \rho = \frac{z}{\lambda} \qquad \text{(A.8)}$$

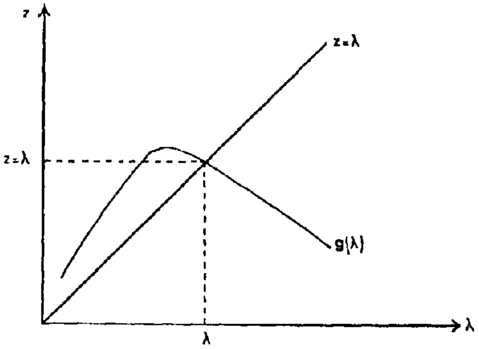

FIGURE 2

and where $\tilde{B}(\lambda) = \lambda B^*(\lambda)$ has positive elements defined by (5.4–5). It was also observed by (5.6) that the elements of $\tilde{B}(\lambda)$ increase with the value of λ, and at least some of them increase strictly. From these facts follows that ρ is a strictly declining function of λ.

From (5.5) we see that, when $\lambda \to 0$ the elements $\tilde{b}_{ij}(\lambda)$ tend towards

$$\tilde{b}_{ij}(0) = \frac{1}{S_j} \sum_{\theta=1}^{T} b_{ij\theta} \tag{A.9}$$

which are purely physical characteristics: investments of various sorts regardless of gestation periods, divided by the life-time of capital in the sector. ρ is a continuous function of the elements $\tilde{b}_{ij}$ and will tend towards a value ρ_0 determined by

$$|I - A - \rho_0 \tilde{B}(0)| = 0 \tag{A.10}$$

when λ tends to zero.

Now let us see what this means in connection with figure 2. If ρ_0 determined by (A.10) is $\rho_0 < 1$, then we will never get a solution $\lambda > 0$ since the function $g(\lambda) = z = \rho\lambda$ then starts from the origin by being below the ray $z = \lambda$, and will not intersect it for any $\lambda > 0$ since ρ is a declining function of λ. On the other hand if $\rho_0 > 1$, then $g(\lambda)$ will be above $z = \lambda$ in the figure immediately to the right of the origin, and since we know that $g(\lambda)$ will eventually decline, there will exist a solution $\lambda > 0$.

As a byproduct of this line of reasoning we also see that the solution for λ is unique. Since $\rho = z/\lambda$ is the slope of the ray from the origin to a point on the curve $z = g(\lambda)$ in figure 2, if there is an intersection between $z = \lambda$ and $z = g(\lambda)$ for some $\lambda > 0$, there cannot be another intersection to the right of the first one.

The treatment of the existence and uniqueness problem given above is partly influenced by the mathematical treatment of a generalized model by

M. Åberg and H. Persson [8]. They approach the problems in a somewhat different way, but their approach suggested the crucial role of the fact that the elements $\lambda b_{ij}^{*}(\lambda)$ increase with λ.

We saw above that the existence of a solution $\lambda > 0$ depends on the existence of a solution $\rho_0 > 1$ to Eq. (A.10). Introducing $\sigma = \rho_0 - 1$ we can write the equation as

$$|I - (A + \tilde{B}(0)) - \sigma\tilde{B}(0)| = 0. \tag{A.11}$$

We now have a solution $\lambda > 0$ if there is a positive solution for σ in (A.11). Such a solution exists if $A + \tilde{B}(0)$ is a productive matrix in the usual input-output sense. The interpretation of $A + \tilde{B}(0)$ is straightforward: it consists of the current input-output coefficients in A plus capital coefficients summed over gestation lags and averaged out over the life-time of capital equipment in the sector concerned, see (A.9).

From the formulation (A.2–3) it is clear that there would be no difficulty in solving the present problem numerically if we are able to solve the characteristic equation for the standard case of a constant matrix B. A natural idea is to solve the equations iteratively, i.e. to start with an initial guess about the value of λ, say λ_0, insert this in (A.2) and solve for z, next insert this value for λ and solve again for z, and so on. This would generate $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_q, \ldots$ according to

$$|I - A - \lambda_{q+1}B^{*}(\lambda_q)| = 0 \qquad (q = 0, 1, 2, \ldots). \tag{A.12}$$

This process would probably converge in most cases, but there seems to be nothing in what has been demonstrated above which shows that it will always converge from any initial guess λ_0. However, even if the process (A.12) should not work well, it would be easy to find the solution with any desired degree of accuracy by calculating the value of z for selected values of λ and then proceed by interpolations.

REFERENCES

1. V. Z. BELENKII, V. A. VOLKONSKII, AND N. V. PAVLOV, Dynamic input-output models in planning and price calculations and economic analysis, *Matekon* **10** (1973–74), 74–101.
2. A. BRÓDY, "Proportions, Prices and Planning," Akadémiai Kiadó, Budapest, and North-Holland, Amsterdam, 1970.
3. W. F. GOSSLING, "Productivity Trends in a Sectoral Macroeconomic Model," Input–Output Publishing Company, London, 1972.
4. W. F. GOSSLING, Relative prices and wages-bills, under steady growth-rates, *in* "Capital Coefficients and Dynamic Input–Output Models" (W. F. Gossling, Ed.) Input–Output Publishing Company, London, 1975.

5. J. Lesourne, "Economic Analysis and Industrial Management," Prentice–Hall, Englewood Cliffs., N.J., 1963.
6. L. Johansen, The rate of growth in dynamic input-output models: some observations along lines suggested by O. Lange and A. Bródy, in "Jahrbuch der Wirtschaft Osteuropas," Band 4, pp. 69–88, Günter Olzog Verlag, Munich, 1973.
7. V. A. Volkonskii, Industry price levels and the norm of effectiveness in the system of optimal planning, *Matekon* **12** (1975), 28–46.
8. M. Åberg and H. Persson, "A Dynamic Nonlinear Closed Input-Output Model," Working Paper, Department of Economics, University of Gothenburg, to appear.

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Notes, Comments, and Letters to the Editor

A General Equilibrium Model of Production with a Random Marginal Rate of Substituion*

This paper examines the qualitative effects of risk aversion in a two-sector general equilibrium model where uncertainty is due to random production in one of the sectors. Technology is more general than used previously and provides for a random marginal rate of substitution. Also, the possibility that risk is decreasing in factor usage is considered. Results show that earlier qualitative conclusions on the effects of changing risk may be reversed under these cases.

Recently, in this journal, Batra [1] has analyzed the qualitative effects of risk aversion in a two-sector general equilibrium model where uncertainty is due to random production in one of the sectors. By extending Rothenberg and Smith's [4] risk-neutral model, the effects of uncertainty on factor intensities, returns, and national income were explored invoking a small country assumption (i.e., price of output is fixed). Batra proves three basic results: (1) factors of production move from the uncertain industry as uncertainty is introduced; (2) if the uncertain sector is relatively capital intensive, then capital-labor ratios rise in both sectors; and (3) real returns to the intensive factor in the uncertain industry fall and returns to the intensive factor of the certain sector rise. His results are derived assuming risk aversion and a factor-neutral, multiplicative random disturbance in the production function of the uncertain sector,

$$X_1 = \epsilon F_1(K_1, L_1) = \epsilon L_1 f(k_1), \tag{1}$$

where X_1 is the output of industry 1; K_1 and L_1 are the quantities of capital and labor utilized in industry 1, respectively; F_1 denotes a transformation function; ϵ is a nonnegative random disturbance; and k_1 denotes the capital-labor ratio under the assumption of linear homogeneity. In this case the expected utility maximizer chooses K_1 and L_1 such that

$$\frac{w}{r} = \frac{F_{L1}}{F_{K1}} \tag{2}$$

where w/r is the labor-capital cost ratio, and alphabetic subscripts denote differentiation. Thus, the firm minimizes costs for each level of expected

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output, and uncertainty does not intrinsically affect the choice of technique. In other words risk preferences do not affect the marginal rate of substitution at the optimum (other than through changes in expected output).

In this paper technology is generalized such that the marginal rate of substitution is uncertain. Moreover, the possibility of decreasing risk via capital- (or labor-) intensive technology is incorporated in the analysis. It is found that all three of the above propositions are sensitive to the manner in which factors of production affect the probability distribution of production. In general, the condition that both factors of production have a positive marginal risk effect is sufficient to obtain Batra's results when the marginal rate of substitution is random; but some alternative cases are developed which may also be reasonable. An implicit assumption of Batra's work which is continued in this paper is that contingent (futures) markets are not operating with respect to production risks.

1. A MORE GENERAL STOCHASTIC PRODUCTION FUNCTION

The production function for the uncertain sector examined in this paper is of the form

$$X_1 = F_1(K_1, L_1) + G_1(K_1, L_1)\,\epsilon,\ E(\epsilon) = 0,\ E(\epsilon^2) < \infty \qquad (3)$$

where, without loss of generality, it is assumed that $G_1 > 0$. If $G_1 < 0$ at the optimum, then one can simply define a new random variable $\tilde{\epsilon} = -\epsilon$ and a new function $\tilde{G}_1 = -G_1 > 0$; the analysis of this paper thus applies for the production function $X_1 = \tilde{F}_1 + \tilde{G}_1\epsilon$.

It may also be noted that the specification in (3) may be viewed as an approximation of the general case where

$$X_1 = F_1^*(K_1, L_1, \epsilon).$$

With a small degree of uncertainty, a Taylor series expansion about $\epsilon = 0$ implies

$$X_1 = F_1^*(K_1, L_1, 0) + \epsilon \left.\frac{\partial F_1^*(K_1, L_1, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}$$

which suggests

$$F_1(K_1, L_1) = F_1^*(K_1, L_1, 0)$$

and

$$G_1(K_1, L_1) = \left.\frac{\partial F_1^*(K_1, L_1, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}$$

in (3). Note further, however, that (3) is more general than the specification used by Batra (without approximation) since if $G_1 = F_1$, (3) becomes

$$X_1 = F_1(K_1, L_1)(1 + \epsilon)$$

which is the special case examined by Batra. Many other interesting special cases are also apparent. For example, a strictly additive disturbance is obtained if G_1 is a constant. Appealing to the approximation, the case of random factors, e.g., $X_1 = \bar{F}_1(K\epsilon, L_1)$, is obtained with $F_1 = \bar{F}_1$ and $G_1 = \bar{F}_{K1}K$ where $K_1 = K\epsilon$.

A major advantage of (3) relates to the marginal impact of factor changes on risk. Focusing more specifically on the variance of output, $V(X_1)$, the marginal effect is

$$\frac{\partial V(X_1)}{\partial K_1} = 2G_1G_{K1}V(\epsilon) \gtreqless 0 \text{ as } G_{K1} \gtreqless 0$$

$$\frac{\partial V(X_1)}{\partial L_1} = 2G_1G_{L1}V(\epsilon) \gtreqless 0 \text{ as } G_{L1} \gtreqless 0.$$

Hence, unlike the usual specification in (1), factors of production may positively contribute to mean output while having a negative effect on the variance.[1]

The usual multiplicative specification implies that risk is increasing in all factors of production which have nonzero marginal products. The specification in (3), on the other hand, allows for random marginal productivities and allows risk to be decreasing in particular inputs. Several reasonable situations suggest that this flexibility may be needed. For example, with random employee strikes in developing economies, firms may diminish risk by increasing the capital-labor ratio. Also, in developing countries, capital intensification in agriculture often has the effect of diminishing output variability due to adverse weather conditions (Just and Pope [3]). As may easily be shown, the general class of production functions with multiplicative disturbances precludes such cases.

2. COMPETITIVE BEHAVIOR OF THE UNCERTAIN SECTOR

The assumption of linear homogeneity in (3), i.e., constant stochastic returns to scale (see, for example, Stiglitz [6]), implies that (3) can be written as

$$X_1 = L_1 f_1(k_1) + L_1 g_1(k_1)\,\epsilon.$$

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[1] For example, $\partial V(X_1)/\partial K_1$ in (1) is $2F_1F_{K1}V(\epsilon) > 0$ if $F_1, F_{K1} > 0$.

It is assumed that marginal productivity is positive ($F_{K1}, F_{L1} > 0$) and that every realization of the production function is concave, i.e., $X_1 = F_1 + G_1\epsilon$ is concave in K_1 and L_1 for every ϵ in the sample space.

Section 1 maximizes the expected utility of profit, $E[U_1(\Pi_1)] = E[U_1]$, by controlling input levels K_1 and L_1. Where $\Pi_1 = PX_1 - wL_1 - rK_1$, first-order conditions are:

$$E(U_1' \cdot \Pi_{K1}) = E\{U_1' \cdot [P(F_{K1} + G_{K1}\epsilon) - r]\} = 0 \tag{4}$$

$$E(U_1' \cdot \Pi_{L1}) = E\{U_1' \cdot [P(F_{L1} + G_{L1}\epsilon) - w]\} = 0. \tag{5}$$

From (4) and (5), one can deduce that

$$w = \frac{E[U_1' \cdot P(F_{L1} + G_{L1}\epsilon)]}{E(U_1')} \tag{6}$$

$$r = \frac{E[U_1' \cdot P(F_{K1} + G_{K1}\epsilon)]}{E(U_1')} \tag{7}$$

and

$$\frac{E(U_1' \cdot \epsilon)}{E(U_1')} = \frac{r - PF_{K1}}{PG_{K1}} = \frac{w - PF_{L1}}{PG_{L1}}. \tag{8}$$

Finally, using (8), it can be shown that

$$\Pi_{L1} = \Pi_{K1}\delta \tag{9}$$

where

$$\delta = \frac{w - PF_{L1}}{r - PF_{K1}} = \frac{G_{L1}}{G_{K1}} = \frac{g_1 - k_1 g_1'}{g_1'}$$

and, thus, δ provides a convenient means of expressing one derivative of the profit function in terms of the other.

Second-order conditions for a unique maximum in section 1 require

$$D = \left[\begin{array}{c|c} E[U_1' \cdot \Pi_{LL1} + \Pi_{L1}^2 U_1''] & E[U_1' \cdot \Pi_{KL1} + \Pi_{L1}\Pi_{K1}U_1''] \\ \hline E[U_1' \cdot \Pi_{KL1} + \Pi_{L1}\Pi_{K1}U_1''] & E[U_1' \cdot \Pi_{KK1} + \Pi_{K1}^2 U_1''] \end{array}\right] > 0,$$

and

$$E[U_1' \cdot \Pi_{LL1} + \Pi_{L1}^2 U_1''] < 0.$$

Assuming risk aversion ($U_1'' < 0$) and concavity of all possible realizations (all ϵ) of the production function (which implies concavity of Π), it can be readily verified that second-order conditions must hold. Furthermore, it can be shown that second-order conditions place no restrictions on the sign of δ; and, hence, the case of risk-reducing inputs can be meaningfully investigated.

3. COMPLETION OF THE GENERAL EQUILIBRIUM MODEL

Suppose now that industry 2 has nonrandom production denoted by

$$X_2 = F_2(K_2, L_2) = L_2 f_2(k_2)$$

where $k_2 = K_2/L_2$. First-order conditions are

$$w = f_2 - k_2 f_2' \tag{10}$$

$$r = f_2' \tag{11}$$

and second-order conditions are guaranteed by the concavity of F_2. It may be further noted that concavity of F_2 implies F_{LL2}, $F_{KK2} < 0$, or $f_2'' < 0$.

Given a competitive equilibrium, equations (6), (7), (10), and (11) indicate that

$$E[U_1' \cdot P(F_{L1} + G_{L1}\epsilon)] = F_{L2}E(U_1') \tag{12}$$

$$E[U_1' \cdot P(F_{K1} + G_{K1}\epsilon)] = F_{K2}E(U_1'). \tag{13}$$

Finally, assuming full employment, two resource constraints complete the model of the economy:

$$L_1 + L_2 = L \tag{14}$$

$$K_1 + K_2 = K = L_1k_1 + L_2k_2 \tag{15}$$

where L and K are total resource endowments of labor and capital, respectively.

4. THE IMPACT OF UNCERTAINTY ON THE ECONOMY

As in Batra, the major focus here is to determine the impact of marginal changes in risk on factor incomes and employment using the mean-preserving spread technique of Sandmo [5]. Defining

$$\hat{\epsilon}^* = \gamma\epsilon,$$

any change in the parameter γ has the effect of changing the spread of the distribution while leaving $E(\hat{\epsilon}) = 0$. Hence, the marginal impact of uncertainty can be found by differentiating (12)-(15) with respect to γ (and evaluating at $\gamma = 1$). Under linear homogeneity, it is convenient to consider these equations as functions of L_1, k_1, and k_2. The resulting implicit differentiation yields

$$E\Big[U_1'' \cdot P(F_{L1} + G_{L1}\epsilon)\frac{d\Pi_1}{d\gamma} + U_1' \cdot PG_{L1}\epsilon + PU_1' \cdot (F_{Lk1} + G_{Lk1}\epsilon)\frac{dk_1}{d\gamma} - F_{L2}U_1'' \cdot \frac{d\Pi_1}{d\gamma} - U_1' \cdot F_{Lk2}\frac{dk_2}{d\gamma}\Big] = 0 \tag{16}$$

$$E\Big[U_1'' \cdot P(F_{K1} + G_{K1}\epsilon)\frac{d\Pi_1}{d\gamma} + U_1' \cdot PG_{K1}\epsilon + PU_1' \cdot (F_{K1k} + G_{K1k}\epsilon)\frac{dk_1}{d\gamma} - F_{K2}U_1'' \cdot \frac{d\Pi_1}{d\gamma} - U_1' \cdot F_{Kk2}\frac{dk_2}{d\gamma}\Big] = 0 \tag{17}$$

$$(k_1 - k_2)\frac{dL_1}{d\gamma} + L_1\frac{dk_1}{d\gamma} + L_2\frac{dk_2}{d\gamma} = 0. \tag{18}$$

Solution of (16)-(18) through use of Cramer's rule obtains[2]

$$\frac{dL_1}{d\gamma} = \frac{L_2H_2P(k_1 + \delta)\,E(U_1' \cdot Y) + f_2''(k_2 + \delta)\,E(U_1')\,L_1H_2}{|A|} \tag{19}$$

$$\frac{dk_1}{d\gamma} = -\frac{(k_1 - k_2)\,H_2f_2''(k_2 + \delta)\,E(U_1')}{|A|} \tag{20}$$

$$\frac{dk_2}{d\gamma} = \frac{(k_1 - k_2)\,H_2P(k_1 + \delta)\,E(U_1' \cdot Y)}{|A|} \tag{21}$$

where

$$H_2 = -E(PG_{K1}U_1' \cdot \epsilon + U_1'' \cdot \Pi_{K1}G_1P\epsilon) \tag{22}$$

and $|A| > 0$ (see the Appendix for details).

Several lemmas will be helpful in establishing the qualitative nature of (19)-(21). Lemma 1 can be obtained through application of lemmas derived by Feder [2]; however, since parts (a) and (b) are somewhat different than explicit results which have appeared previously, their derivation is also presented in the Appendix.

LEMMA 1. *If absolute risk aversion is nonincreasing (but positive), then*

(a) $\operatorname{sgn} E[U_1'' \cdot \Pi_{K1}] = \operatorname{sgn} G_{K1}$

$\operatorname{sgn} E[U_1'' \cdot \Pi_{L1}] = \operatorname{sgn} G_{L1}$

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[2] Note that $F_{Lk1} + G_{Lk1}\epsilon = -k_1(f_1'' + g_1''\epsilon)$; $F_{Lk2} = -k_2f_2''$; $F_{Kk1} + G_{Kk1}\epsilon = f_1'' + g_1''\epsilon$ and $F_{Kk2} = f_2''$.

(b) $E[U_1'' \cdot \Pi_{K1}G_{K1}\epsilon], E[U_1'' \cdot \Pi_{L1}G_{L1}\epsilon] < 0$

(c) $E[U_1' \cdot \epsilon] < 0$.

LEMMA 2. *If every realization of the production function satisfies concavity, then*

$$E[U_1' \cdot Y] < 0$$

where $Y = f_1'' + g_1''\epsilon = L_1(F_{KK1} + G_{KK1}\epsilon)$.

Proof. If $F_1 + G_1\epsilon$ is concave for all ϵ, then

$$Y = f_1'' + g_1''\epsilon < 0 \text{ for all } \epsilon.$$

Since $U_1' > 0$, the conclusion of the lemma follows.

LEMMA 3. *Assuming nonincreasing absolute risk aversion,*

$$\operatorname{sgn} H_2 = \operatorname{sgn} g_1' .$$

Proof. Multiplying and dividing (22) by G_{K1} or, equivalently, by g_1' yields

$$H_2 = -\frac{PG_{K1}^2E(U_1' \cdot \epsilon) + PG_1G_{K1}E(U_1'' \cdot \Pi_{K1}\epsilon)}{g_1'} .$$

Assuming risk aversion, the first term in parentheses is negative by Lemma 1(c); the second term in parentheses is negative by Lemma 1(b). Therefore, $\operatorname{sgn} H_2 = \operatorname{sgn} g_1'$.

LEMMA 4. *With concavity of f_1 and g_1 and nonincreasing absolute risk aversion,* $|A| > 0$.

Proof. See the Appendix.

Turning now to equation (20), it is clear from concavity and Lemma 3 that

$$\operatorname{sgn} \frac{dk_1}{dy} = \operatorname{sgn}[(k_1 - k_2)\, g_1'(k_2 + \delta)]. \tag{23}$$

Using this equation and the above lemmas, the following theorem can be derived.

THEOREM 1. *Suppose there is an increase in production uncertainty in uncertain sector. If the uncertain sector is relatively capital intensive, then (i) the capital-labor ratio in the uncertain sector will unambiguously increase if labor marginally increases production risk or capital marginally reduces production risk; but (ii) if labor marginally reduces production risk, the capital-*

labor ratio will decrease if the marginal risk reduction is sufficiently large, specifically,

$$\frac{dk_1}{d\gamma} \gtrless 0 \text{ as } G_{L1} \gtrless -\frac{k_2}{k_1 - k_2}\frac{G_1}{L_1}. \tag{24}$$

Conversely, if the uncertain sector is relatively labor intensive, then (iii) the capital-labor ratio in the uncertain sector will unambiguously decrease if capital marginally increases risk or labor marginally reduces risk; but (iv) the capital-labor ratio will increase if the marginal risk effect of capital is sufficiently large negatively, specifically,

$$\frac{dk_1}{d\gamma} \gtrless 0 \text{ as } G_{K1} \lessgtr \frac{1}{k_1 - k_2}\frac{G_1}{L_1} . \tag{25}$$

Proof. Theorem 1 is proved by determining the signs of terms on the right-hand side of equation (23). To do this, note that

$$k_2 + \delta = k_2 + \frac{G_{L1}}{G_{K1}} = (k_2 - k_1) + \frac{g_1}{g_1'} \tag{26}$$

and that $G_{L1} = g_1 - k_1 g_1' = g_1 - k_1 G_{K1}$; therefore, $G_{L1} < 0$ implies $G_{K1} > 0$ and $G_{K1} < 0$ implies $G_{L1} > 0$. Consider case (i) where either both G_{L1}, $G_{K1} > 0$, or $G_{L1} > 0$, $G_{K1} < 0$. Using the middle expression in (26), it is clear that G_{L1}, $G_{K1} > 0$ implies $k_2 + \delta > 0$; and from (23) sgn $dk_1/d\gamma =$ sgn$(k_1 - k_2)$. If $G_{L1} > 0$ and $g_1' < 0$, then $k_2 + \delta$ and g_1' always have the same sign when $k_1 - k_2 > 0$, thus verifying case (i). Case (iii) is established in a similar manner from (26) by noting that G_{K1}, $G_{L1} > 0$ implies $dk_1/d\gamma < 0$ when $k_1 - k_2 < 0$. If $G_{K1} > 0$ and $G_{L1} < 0$, then $k_2 + \delta > 0$ from (26) if $k_1 - k_2 < 0$, thus verifying the result. Finally, from (23) and (26), note that $dk_1/d\gamma < 0$ if $g_1(k_1 - k_2) - (k_1 - k_2)^2 g_1' < 0$. Case (ii) follows from (23) when $k_1 - k_2 > 0$ by noting that $g_1' = (g_1 - G_{L1})/k_1$ and $g_1 = G_1/L_1$. Case (iv) is established from (23) using (26) where $k_1 - k_2 < 0$ and $g_1' = G_{K1}$.

Remark 1. The implication of the multiplicative disturbance which has been commonly used for simplicity in previous literature implies, in the context of the model of this paper, that G_{L1}, $G_{K1} > 0$. By (26) and Lemma 3, this is a special case where (23) reduces to

$$\operatorname{sgn} \frac{dk_1}{d\gamma} = \operatorname{sgn} (k_1 - k_2)$$

which is the result noted by Batra. Theorem 1 shows that similar results are also obtained in other general cases of qualitative marginal risk effects. Ambiguous results remain only in cases where $G_{L1} < 0$, $k_1 - k_2 > 0$ and $G_{K1} < 0$, $k_1 - k_2 < 0$, i.e., where the marginal risk effects "work against" the relative factor intensities. Nevertheless, Theorem 1 shows that results contrary to Batra's results are possible. For example, if labor sufficiently reduces risk at the margin, then an increase in uncertainty can cause a decline in the capital-labor ratio of a capital-intensive uncertain sector.

Remark 2. Intuition indicates that, as uncertainty is introduced, an input which reduces (increases) uncertainty should become relatively more (less) valuable, *ceteris paribus*, to the sector facing uncertainty. Indeed, this is suggested by (6) and (7). Furthermore, one can note that if risk is increasing in the capital-labor ratio (which implies risk is increasing in capital), then risk must be decreasing in the labor-capital ratio (and thus also be decreasing in labor) and conversely. Thus, if capital marginally reduces risk in case (i) of Theorem 1, then capital becomes relatively more valuable while labor becomes less valuable and, hence, the capital-labor ratio should unambiguously increase since the marginal risk effects tend to strengthen the relative factor intensities. Similar reasoning applies where labor marginally reduces risk in case (iii) of Theorem 1. The intuition where both inputs marginally increase risk follows along earlier lines suggested by Batra. That is, if both inputs increase risk, then introduction of uncertainty causes resources to shift away from the uncertain sector; thus, expected output is reduced along the (concave) contract curve in the Edgeworth-Bowley box causing the capital-labor ratio to increase (decrease) in the capital- (labor-) intensive case.

The results in Theorem 1 and intuition of Remark 2 suggests that the absolute use of risk-reducing inputs may actually increase in the uncertain sector as uncertainty is introduced. In the following theorem it is indeed found that the conditions which govern directional input changes in response to risk are closely tied to cases (i)-(iv) of Theorem 1.

THEOREM 2. (*a*) *With a marginal increase in uncertainty in cases* (*i*) *and* (*iii*) *of Theorem* 1, *both labor input and capital input in the uncertain sector are reduced, while both inputs increase in the certain sector.*

(b) *For cases* (*ii*) *and* (*iv*) *of Theorem* 1, *however, an increase in uncertainty leads to an increase in capital or labor in the uncertain sector if the associated marginal risk reduction of the relevant input is sufficiently large.*

Proof. From (19), one finds that

$$\operatorname{sgn}\frac{dL_1}{d\gamma} = \operatorname{sgn}\frac{\{[L_2PE(U_1'\cdot Y)\,g_1/g_1' + L_1f_2''(k_2+\delta)\,E(U_1')]\}\,H_2}{|A|}. \quad (27)$$

Utilizing Lemma 4 and the fact that $\delta \doteq g_1/g_1' - k_1$, equation (27) becomes

$$\operatorname{sgn}\frac{dL_1}{d\gamma} = \operatorname{sgn}\frac{\{L_2PE(U_1'\cdot Y)\,g_1 + f_2''L_1[(k_2-k_1)\,g_1' + g_1]\,E(U_1')\}\,H_2}{g_1'}. \quad (28)$$

Since $H_2/g_1' > 0$ (by Lemma 3), Lemmas 2 and 4 and concavity of F_2 imply $dL_1/d\gamma < 0$ if $(k_2 - k_1)\, g_1' > 0$. Thus, $g_1' > 0$ and $(k_2 - k_1) > 0$ establishes (a) for case (iii) of Theorem 1. Alternatively, if $G_{L1} > 0$ and $G_{K1} < 0$, then $(k_2 - k_1)\, g_1' > 0$ when $(k_2 - k_1) < 0$; or if $G_{L1} < 0$ and $G_{K1} > 0$, then $k_2 + \delta < 0$ from (27). Thus, (a) follows for case (i) of Theorem 1. To see that capital also decreases under similar conditions, note that $K_1 = L_1 k_1$; hence, using equations (19)–(21),

$$\frac{dK_1}{d\gamma} = L_1 \frac{dk_1}{d\gamma} + k_1 \frac{dL_1}{d\gamma}$$

$$= \frac{[L_1 k_2 (k_2 + \delta) f_2'' E(U_1') + k_1 L_2 PE(U_1' \cdot Y)(g_1/g_1')]\, H_2}{|A|}$$

Again, use of $= g_1/g_1' - k_1$, along with Lemmas 2 and 3, implies that

$$\frac{dK_1}{d\gamma} = \frac{L_1 k_2 [(k_2 - k_1)\, g_1' + g_1]\, f_2'' E(U_1') + k_1 L_2 PE(U_1' \cdot Y)\, g_1 H_2}{g_1' \, |A|} < 0$$

if $(k_2 - k_1)\, g_1' > 0$ which verifies the assertions of (a) relating to capital. The remaining assertions of part (a) thus follow upon additionally using (14) and (15). Finally, part (b) is obtained when $(k_2 - k_1)\, g_1' < 0$ [cases (ii) and (iv) of Theorem 1] such that, for labor,

$$E(U_1') f_2'' L_1 [(k_2 - k_1)\, g_1'] > | f_2'' L_1 g_1 E(U_1') + L_2 PE(U_1' \cdot Y)\, g_1 | \qquad (29)$$

or, for capital,

$$L_1 k_2 (k_2 - k_1)\, g_1' f_2'' E(U_1') > | [L_1 k_2 g_1 f_2'' E(U_1') + k_1 L_2 PE(U_1' \cdot Y)\, g_1] | . \qquad (30)$$

Remark 3. Hence, when the marginal risk effects do not tend to strengthen the relative factor intensity of the uncertain sector [cases (ii) and (iv)], a marginal increase in uncertainty may indeed lead to increased use of the risk-reducing factor. Comparing (29) and (30) with (24) and (25), however, it is clear that a reversal of the effects of uncertainty on absolute factor use (Theorem 2) requires a stronger negative marginal risk effect than is necessary to reverse the effects of uncertainty on relative intensities in the uncertain sector (Theorem 1). Nevertheless, only when both inputs are risk increasing or where the risk-reducing factor is already used intensively does a marginal increase in uncertainty unambiguously lead to a reduction in both input uses. Thus, only in the latter case [corresponding to (i) and (iii) of Theorem 1] is it clear that an increase in uncertainty causes a decrease in expected output in the uncertain sector and an increase in output in the certain sector.

Given the ambiguity of Theorems 1 and 2 relating to the effects of uncertainty on relative factor intensities in the uncertain sector and on absolute factor employment levels in both sectors, it may seem reasonable in a general equilibrium context to expect similar ambiguities with respect to relative factor intensities in the certain sector as well. However, Theorem 3 determines that this is not the case and, in fact, Batra's results relating to certain sector-relative intensities can be generalized.

THEOREM 3. *An increase in uncertainty lowers (raises) the capital-labor ratio in the certain sector if it is relatively more (less) capital intensive regardless of whether factors increase or decrease risk.*

Proof. Using equation (21), it is readily verified that $\operatorname{sgn}[-P(k_1 + \delta) E(U_1' \cdot Y)] = \operatorname{sgn} g_1'$ and, by Lemma 3, that $\operatorname{sgn} H_2 = \operatorname{sgn} g_1'$. Hence, using Lemma 4, one finds $\operatorname{sgn} dk_2/d\gamma = \operatorname{sgn}(k_1 - k_2)$.

In the standard case of risk-increasing inputs, intuition suggests that a factor's real reward declines as uncertainty increases. But given the above cases where use of a risk-reducing input in the uncertain sector can increase with increased uncertainty, the effects of uncertainty on such a factor's real reward are no longer clear. Theorem 4 shows, however, that real rewards decline with increased uncertainty in any case.

THEOREM 4. *An increase in uncertainty causes a decline in the real reward of the factor used intensively by the industry facing uncertainty and a rise in the real reward of the factor used intensively by the certain industry regardless of whether factors marginally reduce or increase risk.*

Proof. From Theorem 3, if $k_2 > k_1$, then $dk_2/dy < 0$ which by concavity implies that f_2' and r rise. Similarly, if f_2' rises, then $F_{L2} = f_2 - k_2 f_2'$ falls and w falls also. If, on the other hand, $k_1 > k_2$, then r falls and w rises by similar reasoning.

Remark 4. Theorem 4 further emphasizes the fact that risk is costly. That is, for example, an increase in risk reduces the social value of each factor used in the uncertain sector no matter how strong its risk-reducing marginal effect may be.

5. APPENDIX

Derivation of Equations (19)-(21). Substitution of $d\Pi_1/d\gamma$ into (16)-(18) and collecting terms give the matrical equation,[3]

$$AX = H \tag{A.1}$$

---

[3] Here Π_1 can be written as

$$\Pi_1 = L_1[P(f_1 + g_1\epsilon) - (f_2 - k_2 f_2') - k_1 f_2'] = L_1 \Pi_{K1}(k_1 + \delta).$$

Hence,

$$\frac{d\Pi_1}{d\gamma} = L_1 \left[\Pi_{K1}\left(\frac{dk_1}{d\gamma}\right) + F_{Kk2}(k_2 - k_1)\left(\frac{dk_2}{d\gamma}\right) + Pg_1\epsilon + (k_1 + \delta)\,\Pi_{K1}\left(\frac{dL_1}{d\gamma}\right)\right].$$

where (T denotes transpose)

$$
\begin{aligned}
H &= (H_1\ H_2\ 0)^T \\
X &= \left(\frac{dL_1}{d\gamma}\ \frac{dk_1}{d\gamma}\ \frac{dk_2}{d\gamma}\right)^T \\
A &= \{A_{ij}\}_{3\times 3} \\
\frac{A_{11}}{\delta} &= A_{21} = E\left(U_1'' \cdot \Pi_{K1}^2 \frac{g_1}{g_1'}\right) \\
A_{12} &= E[\delta L_1 U_1'' \cdot \Pi_{K1}^2 - k_1 P(f_1'' + g_1''\epsilon)\, U_1'] \\
A_{22} &= E[L_1 U_1'' \cdot \Pi_{K1}^2 + P(f_1'' + g_1''\epsilon)\, U_1'] \\
A_{13} &= E[\delta L_1 f_2''(k_2 - k_1)\, U_1'' \cdot \Pi_{K1} + k_2 f_2'' U_1'] \\
A_{23} &= E[L_1 f_2''(k_2 - k_1)\, U_1'' \cdot \Pi_{K1} - f_2'' U_1'] \\
A_{31} &= k_1 - k_2 \\
A_{32} &= L_1 \\
A_{33} &= L_2
\end{aligned}
$$

and

$$\frac{H_1}{\delta} = H_2 = -E[PG_{K1}U_1'\epsilon + U_1'' \cdot \Pi_{K1}G_1P\epsilon].$$

If $|A| > 0$ (as is shown below), use of Cramer's rule in equation (A.1) thus yields

$$
\begin{aligned}
\frac{dL_1}{d\gamma} &= -\frac{L_2H_2(A_{12} - \delta A_{22}) + (\delta A_{23} - A_{13})\, L_1H_2}{|A|} \\
\frac{dk_1}{d\gamma} &= \frac{(k_1 - k_2)\, H_2(\delta A_{23} - A_{13})}{|A|} \\
\frac{dk_2}{d\gamma} &= \frac{(k_2 - k_2)\, H_2(A_{12} - \delta A_{22})}{|A|}.
\end{aligned}
$$

Equations (19)-(21) follow immediately.

Proof of Lemma 1(*a*). Profits in industry 1 can be written as

$$\Pi_1(\epsilon) = E(\Pi_1) + PG_1\epsilon.$$

Let ϵ^* be such that $\Pi_{K1}(\epsilon^*) = 0$. Then, note that

$$U_1'' \gtreqless -R_a(\Pi^*)\, U_1' \text{ as } \epsilon \gtreqless \epsilon^* \tag{A.2}$$

where absolute risk aversion, $R_a(\Pi) = -U''/U_1'$, is nonincreasing and $\Pi^* = \Pi(\epsilon^*)$ (since $d\Pi/d\epsilon > 0$). Multiplying (A.2) by Π_{K1} thus obtains

$$U_1'' \cdot \Pi_{K1} \gtreqless -R_a(\Pi^*)\, U_1' \cdot \Pi_{K1} \text{ as } G_{K1} \gtreqless 0 \tag{A.3}$$

since

$$\Pi_{K1}(\epsilon) - \Pi_{K1}(\epsilon^*) \gtreqless 0 \begin{cases} \text{as } \epsilon \gtreqless \epsilon^* \text{ if } G_{K1} > 0 \\ \text{as } \epsilon \lesseqgtr \epsilon^* \text{ if } G_{K1} < 0. \end{cases}$$

Taking expectations of (A.3) and using first-order conditions in (4) yield the result

$$E(U_1'' \cdot \Pi_{K1}) \gtreqless -R_a(\Pi^*)\, E(U_1' \cdot \Pi_{K1}) = 0 \text{ as } G_{K1} \gtreqless 0.$$

The proof that

$$E(U_1 \cdot \Pi_{L1}) \gtreqless 0 \text{ as } G_{L1} \gtreqless 0$$

follows in an identical manner.

Proof of Lemma 1(*b*).

$$G_{K1}\epsilon = \frac{1}{P}(PF_{K1} + PG_{K1}\epsilon - r) + \frac{1}{P}(r - PF_{K1}) = \frac{1}{P}(\Pi_{K1} + r - PF_{K1});$$

hence,

$$E(U_1'' \cdot \Pi_{K1} G_{K1}\epsilon) = \frac{1}{P}\left[E(U_1'' \cdot \Pi_{K1}^2) + (r - PF_{K1})\, E(U_1'' \cdot \Pi_{K1})\right]. \tag{A.4}$$

With risk aversion, $E(U_1'' \cdot \Pi_{K1}^2) < 0$. Also, using Lemma 1(c) and equation (9) imply $\operatorname{sgn} G_{K1} = \operatorname{sgn}(PF_{K1} - r)$ with risk aversion. Additionally, using Lemma 1(a) in (A.4) thus obtains

$$E(U_1'' \cdot \Pi_{K1} G_{K1}\epsilon) < 0$$

The result for $E(U_1'' \cdot \Pi_{L1} G_{L1}\epsilon) < 0$ is obtained similarly.

Proof of Lemma 4. From (A.1), one finds

$$\begin{aligned} |A| = {} & L_1 A_{21}(A_{13} - \delta A_{23}) + L_2 A_{21}(\delta A_{22} - A_{12}) \\ & + (k_1 - k_2)(A_{23}A_{12} - A_{13}A_{22}). \end{aligned} \tag{A.5}$$

Substituting further from the definition of A into (A.5), the first two terms in (A.5) become

$$(k_1 + \delta)\, E(U_1'' \cdot \Pi_{K1}^2)\left[L_1 f_2''(k_2 + \delta)\, E(U_1') + L_2 P\left(\frac{g_1}{g_1'}\right) E(U_1' \cdot Y)\right] \tag{A.6}$$

while the last term in (A.5) becomes

$$\left(\frac{g_1}{g_1'}\right) L_1 f_2''(k_1 - k_2)^2 PE(U_1' \cdot Y)\, E(U_1'' \cdot \Pi_{K1}) + E(U_1' \cdot Y)\, E(U_1')$$

$$\times f_2''(k_1 - k_2)^2 P + L_1 f_2''(k_2 + \delta)\, E(U_1')\, E(U_1'' \cdot \Pi_{K1}^2)(k_2 - k_1). \quad \text{(A.7)}$$

Combining the first term in (A.6) with the last term in (A.7) yields

$$L_1 f_2''(k_2 + \delta)^2 E(U_1')\, E(U_1'' \cdot \Pi_{K1}^2) > 0$$

assuming risk aversion and concavity of F_2. Also, noting that $E(U'' \cdot \Pi_{k1}^2) < 0$, the second term of (A.6) and the second term of (A.7) are positive by Lemma 2. Thus, $|A| > 0$ if

$$(k_1 + \delta)\, L_1 f_2''(k_1 - k_2)^2 PE(U_1' \cdot Y)\, E(U_1'' \cdot \Pi_{K1}) \geqslant 0. \quad \text{(A.8)}$$

But the inequality in (A.8) is established by Lemmas 1 and 2 and concavity of f_2 since $k_1 + \delta = g_1/g_1'$ and, hence, $\operatorname{sgn} k_1 + \delta = \operatorname{sgn} g_1'$.

REFERENCES

1. R. N. BATRA, Resource allocation in a general equilibrium model of production under uncertainty, *J. Econ. Theory* **8** (1974), 50–63.
2. G. FEDER, The impact of uncertainty in a class of objective functions, *J. Econ. Theory* **16** (1977), 504–512.
3. R. E. JUST AND R. D. POPE, Stochastic specification of production functions and econometric implications, *J. Econometrics* **7** (1978), 67–86.
4. T. J. ROTHENBERG AND K. R. SMITH, The effect of uncertainty on resource allocation in a general equilibrium model, *Quart. J. Econ.* **85** (1971), 440–459.
5. A. SANDMO, On the theory of the competitive firm under price uncertainty, *Amer. Econ. Rev.* **61** (1971), 287–302.
6. J. E. STIGLITZ, Incentives and risk sharing in sharecropping, *Rev. Econ. Stud.* **41** (1974), 219–256.

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A Characterization Result for the Plurality Rule*

Let E be the set of conceivable alternatives and v be any potential nonempty subset of E, where $|v| = n$. $d = (R_1, ..., R_m)$ represents a set of individual linear orders over E for the m voters, where R_i is the linear order of voter i. A social choice function, C, is a function such that for any set of alternatives (v) and set of individual orderings, d, on v, C selects a nonempty subset $C(v, d)$ of v as the social choice set.

A simple scoring function is any social choice function which can be represented by

$$C(v, d) = \left\{ x \in v: \sum_{k=1}^{n} S_k x^k \geq \sum_{k=1}^{n} S_k y^k \forall y \in v \right\}, \tag{1}$$

where S_k is the score assigned an alternative every time it appears in kth place in any individual's preference ordering and x^k is the number of times alternative x appears in kth place in the set of individual orderings.

In [2] Young proved the following result.

*R*1. A social choice function is a simple scoring function if and only if it satisfies the conditions of

$$\textit{Anonymity}: C(v, (R_1, ..., R_m)) = C(v, (R_{\sigma(1)}, ..., R_{\sigma(m)})), \tag{2}$$

where σ is a permutation on the set of voter preferences;

$$\textit{Neutrality}: \lambda C(v, d) = C(\lambda v, d^{\lambda}), \tag{3}$$

where λ is a permutation of the set of alternatives and d^{λ} is a corresponding permutation of preferences;

$$\textit{Consistency}: C(v, d) \cap C(v, d') \neq \emptyset \rightarrow C(v, d) \cap C(v, d') = C(v, d + d') \tag{4}$$

where d, d' are preference profiles for two disjoint sets of voters;

$$\textit{Continuity}: \exists N \in \{1, 2, ...\} \ni C(v, d) = \{x\} \rightarrow C(v, d' + Nd) = \{x\}. \tag{5}$$

One social choice function that is a simple scoring function, satisfying conditions (2)–(5), is the plurality rule, where individuals simply list their

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most preferred candidate and the candidate with the greatest number of votes is selected. Thus, the plurality rule is a case of (1) where $S_1 = 1$ and $S_k = 0\ \forall k \neq 1$. More generally, whenever $S_1 > S_k\ \forall k \neq 1$ and $S_k = S_{k-1}$ for $k = 2, \dots, n-1$ (1) is the plurality rule.

By adding an additional condition to (2)–(5) we can obtain a unique characterization result for the plurality rule. The condition added is the Reduction Principle, formulated by Fishburn [1], which says that removing Pareto-dominated alternatives does not alter the social choice

$$\textit{Reduction}\colon x \in v\ \&\ xP_i y \forall i \to C(v, d) = C(v - \{y\}, d). \qquad (6)$$

We can now state and prove the following theorem:

THEOREM. *There is one and only one social choice function that satisfies Anonymity, Neutrality, Continuity, Consistency, and Reduction—namely, the Plurality rule.*

Proof. Clearly the Plurality rule satisfies the five conditions. We must show that it is the only social choice function to do so. By *R*1 Anonymity, Neutrality, Consistency, Continuity imply that we have a choice function described by Eq. (1).

Reduction implies $S_1 > S_2$ since for the following profile:

$$\forall i\colon x\ y$$

Reduction implies $C(\{x, y\}, d) = C(\{x\}, d) = \{x\}$ so $S_1 > S_2$.

We now show that $S_k = S_{k+1}$ for $k = 2, \dots, n-1$. Let $v = \{x, z, a_1, \dots, a_n\}$ and d be

	S_1	S_2	S_3	S_4	S_{n-1}	S_n
Individual 1	x	z	a_1	$a_2 \cdots$	a_{n-1}	a_n
Individual 2	z	a_1	a_2	$a_3 \cdots$	a_n	x

in which z Pareto dominates each a_n. Anonymity, Neutrality, and Reduction imply $C(v, d) = \{x, z\}$ since Reduction implies $C(v, d) = C(v - \{a_1\}, d) = C(v - \{a_1, a_2\}, d) = \cdots C(\{x, z\}, d)$ while Anonymity and Neutrality imply $C(\{x, z\}, d) = \{x, z\}$. Since C is a scoring function $S_1 + S_2 = S_1 + S_n$ so $S_2 = S_n$.

Given the knowledge that $C(v, d) = \{x, z\}$ we can again cumulatively delete a_i's, beginning with a_n from the profile to conclude our proof. Thus, deleting a_n gives us

	S_1	S_2	S_3	S_{n-1}
Individual 1	x	z	$a_1 \cdots$	a_{n-1}
Individual 2	z	a_1	$a_2 \cdots$	x

so $S_1 + S_2 = S_1 + S_{n-1}$ and $S_2 = S_{n-1}$ since $\sum_{k=1}^{n} S_k x^k$ must be equal to $\sum_{k=1}^{n} S_k z^k$.

Deleting a_{n-1} gives us

	S_1	S_2	S_3	S_{n-2}
Individual 1	x	z	$a_1 \cdots$	a_{n-2}
Individual 2	z	a_1	$a_2 \cdots$	x

so $S_1 + S_2 = S_1 + S_{n-2}$ and $S_2 = S_{n-2}$. Continuing this process yields $S_2 = S_3, \ldots, S_2 = S_{n-3}$.

Hence, we have $S_1 > S_2$ and $S_2 = S_3 \cdots = S_n$ which is exactly the Plurality rule. Q.E.D.

REFERENCES

1. P. C. FISHBURN, "The Theory of Social Choice," Princeton Univ. Press, Princeton, N.J., 1973.
2. H. P. YOUNG, Social choice scoring functions, *SIAM J. Appl. Math.* **28** (1975).

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Some Error Estimates in Matrix Economic Models

In this paper we consider the effects of errors encountered in building matrix economic models on the solutions to these models. Two particular models will be investigated. The first model is the Metzler economic model

$$\frac{dx}{dt} = DAx \qquad [3, \text{p. } 281],$$

while the second is the open Leontief model

$$x(I - A) = d \qquad [5, \text{p. } 30].$$

To develop a study of these models requires some special notions. For this, let $A = (a_{ij})$ be such that $a_{ij} \geqslant 0$ for $i \neq j$. Any such matrix is called an *ml*-matrix [5, p. 40]. If we let $a = \max_i | a_{ii} |$ then $A + aI$ is nonnegative. From this it is seen that the spectrum σ of A is a translation of the spectrum of the nonnegative matrix $A + aI$. Hence, by the Perron Frobenius theorem [5, p. 25] it follows that $r(A) = \max_{\lambda \in \sigma} \operatorname{Re} \lambda$ is an eigenvalue of A. Further, it follows that if A is irreducible, there is a positive eigenvector, called the Perron vector, belonging to $r(A)$ [5, p. 1].

Results. The first results we consider concern the Metzler model. The point of interest here is on $r(DA)$ since when $r(A) < 0$, this number measures the speed at which the system returns to equilibrium as can be seen by viewing the solution [1, p. 124] to the differential equation presented above.

The initial result is on error in D.

THEOREM 1. *Let A be an $n \times n$ ml-matrix with $r(A) < 0$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$ a nonsingular nonnegative matrix. Then*

$$(\max_i d_i)\, r(A) \leqslant r(DA) \leqslant (\min_i d_i)\, r(A).$$

Proof. Suppose first that A is irreducible. Let y be a Perron vector of A belonging to the eigenvalue $r(A)$. Thus $Ay = r(A)\, y$. Set $Y = \operatorname{diag}(y_1, \ldots, y_n)$ and let $a = \max_i d_i | a_{ii} |$. Then $Y^{-1}(DA + aI)\, Y$ is irreducible and non-

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negative. Therefore we can apply to $Y^{-1}(DA + aI)\,Y$ a consequence [5, Corollary 1, p. 6] of the Perron-Frobenius theorem to obtain

$$\min_i[d_i r(A)] + a \leq r[Y^{-1}(DA + aI)Y] \leq \max_i[d_i r(A)] + a.$$

But $r[Y^{-1}(DA + aI)\,Y] = r(DA + aI) = r(DA) + a$ and by assumption, $r(A) < 0$. Hence

$$(\max_i d_i)\, r(A) \leq r(DA) \leq (\min_i d_i)\, r(A).$$

The general case can now be achieved by an elementary continuity argument. ∎

These bounds can in fact be achieved as can be seen by taking $D = uI$. Hence, the bound is best possible. Further, to apply the bounds, it should be noted that if $\mathscr{D} = \operatorname{diag}(\delta_1, \dots, \delta_n)$ is an approximation to $D = \operatorname{diag}(d_1, \dots, d_n)$ and $e_i d_i = \delta_i$ for $i = 1, \dots, n$ the theorem yields that

$$(\max_i e_i)\, r(DA) \leq r(\mathscr{D}A) \leq (\min_i e_i)\, r(DA)$$

and hence some measure of the approximation $r(\mathscr{D}A)$ to $r(DA)$ is obtained.

A second result on Metzler's model is on the error in A.

THEOREM 2. *Let A be an $n \times n$ irreducible ml-matrix which has as Perron vector $y = (y_1, \dots, y_n)^t$, where $\sum_{k=1}^{n} y_k = 1$. Let $E = (\epsilon_{ij})$ be an $n \times n$ matrix where $A + E$ is an ml-matrix and suppose $\epsilon_m \leq \epsilon_{ij} \leq \epsilon_M$ for all $i \neq j$. Then*

$$r(A) + \min_i (\epsilon_{ii} + \epsilon_m y_i^{-1}) \leq r(A + E) \leq r(A) + \max_i (\epsilon_{ii} + \epsilon_M y_i^{-1}).$$

Proof. Set $Y = \operatorname{diag}(y_1, \dots, y_n)$. Then $Y^{-1}AYe = r(A)\,e$, where e is the vector having a one in each entry. Thus $Y^{-1}(A + E)\,Ye = r(A)\,e + Y^{-1}EYe$ Hence, by applying a consequence [5, Corollary 1, p. 6] of the Perron-Frobenius theorem we have

$$r(A + E) = r(Y^{-1}(A + E)\,Y) \leq r(A) + \max_i \left(\epsilon_{ii} + \sum_{k \neq i} y_i^{-1} \epsilon_{ik} y_k\right)$$

$$\leq r(A) + \max_i (\epsilon_{ii} + \epsilon_M y_i^{-1}).$$

Similarly,

$$r(A) + \min_i (\epsilon_{ii} + \epsilon_m y_i^{-1}) \leq r(A + E). \quad ∎$$

These bounds can be achieved by taking $E = \epsilon I$. Hence the bounds are best possible. It should also be noted that when the errors in the measurements of A satisfy $r(A) + \max_i (\epsilon_{ii} + \epsilon_M y_i^{-1}) < 0$, then the Metzler system is clearly stable, i.e., $r(A + E) < 0$. Further, it should be noted that

Theorem 1 and Theorem 2 can be put together to measure the accuracy of $r[\mathcal{U}(A + E)]$ to $r(DA)$.

We now consider the open Leontief model $x(I - A) = d$. We assume here that A is nonnegative with $Ae = (r_1, \dots, r_n)^t$, where $r_k < 1$ for $k = 1, \dots, n$. Thus, by Neumann's theorem [4, p. 26], $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ and so $x = d(I - A)^{-1}$.

Our result on this model is on errors involved in A and d.

THEOREM 3. *Let A be an $n \times n$ nonnegative matrix and $d = (d_1, \dots, d_n)$ a nonnegative vector. Let E be an $n \times n$ matrix so that $A + E$ is nonnegative and if* $\max_{i,j} |\epsilon_{ij}| = \epsilon_M$ *then $n\epsilon_M < 1 - r_M$, where $r_M = \max_i \sum_{k=1}^{n} a_{ik}$. Let $\delta = (\delta_1, \dots, \delta_n)$ be any vector and $\delta_M = \max_i |\delta_i|$. If $x(I - A) = d$ and $y[I - (A + E)] = d + \delta$ then*

$$|y - x| \leq \delta_M e^t (I - A)^{-1} + \epsilon_M (d + \delta_M e^t)(I - A - \epsilon_M F)^{-1} F (I - A)^{-1},$$

where $F = (f_{ij})$ and *$f_{ij} = 1$ for all i and j, and $|y - x| = (|y_1 - x_1|, \dots, |y_n - x_n|)$. Further,*

$$\sum_{i=1}^{n} |y_i - x_i| \leq \frac{n\delta_M}{1 - r_M} + \epsilon_M \frac{(de + n\delta_M)\, n}{(1 - r_M)(1 - r_M - n\epsilon_M)}$$

for $i = 1, \dots, n$.

Proof. Let R be any nonnegative matrix so that $(A + R)\, e = r_M e$. Hence by Neuman's theorem,

$$(I - A)^{-1} e \leq (I - (A + R))^{-1} e = \frac{1}{1 - r_M} \cdot e.$$

Further as $\sum_{k=1}^{n} (a_{ik} + |\epsilon_{ik}|) \leq \sum_{k=1}^{n} (a_{ik} + \epsilon_M) \leq r_M + n\epsilon_M < 1$ it follows from Gerschgorin's theorem [4, p. 47] that $[I - (A + E)]^{-1}$ exists. Now, by combining the equations $x(I - A) = d$ and $y[I - (A + E)] = d + \delta$ we obtain

$$\begin{aligned} y[I - (A + E)] &= x(I - A) + \delta, \\ y[(I - A) - E](I - A)^{-1} &= x + \delta(I - A)^{-1}, \\ y[I - E(I - A)^{-1}] &= x + \delta(I - A)^{-1}, \end{aligned}$$

or

$$y - x = \delta(I - A)^{-1} + (d + \delta)(I - A - E)^{-1} E (I - A)^{-1}.$$

Thus

$$|y - x| \leq \delta_M e^t (I - A)^{-1} + (d + \delta_M e^t)(I - A - \epsilon_M F)^{-1} \epsilon_M F (I - A)^{-1}.$$

Finally, as $(I-A)^{-1}e \le (1/(1-r_M))e$ and $(I-A-\epsilon_M F)^{-1}e \le (1/(1-r_M-n\epsilon_M))e$ it follows that

$$\langle |y-x|, e\rangle \le \frac{n\delta_M}{1-r_M} + \frac{(de+n\delta_M)\,\epsilon_M n}{(1-r_M-n\epsilon_M)(1-r_M)}$$

Hence

$$\sum_{i=1}^{n} |y_i - x_i| \le \frac{n\delta_M}{1-r_M} + \frac{(de+n\delta_M)\,\epsilon_M n}{(1-r_M-n\epsilon_M)(1-r_M)}. \quad \blacksquare$$

The first inequality can be achieved by taking $\delta = \delta_M e$ and $E = \epsilon_M F$. For a particular model, this bound can be computed numerically to assess the effect of error. The second bound is not necessarily achieved for any choices of δ and E. A bound which can be achieved can be found by showing that $(I-A)^{-1}e \ge (1/(1-r_m))e$ and $(I-A-E)^{-1}e \ge (1/(1-r_m-n\epsilon_m))e$, where $r_m = \min_k r_k$ and $\epsilon_m = \min_{i,j}\epsilon_{ij}$. Then by taking $\delta = \delta_M e$ and $E = \epsilon_M F$ we have that

$$\sum_{i=1}^{n} |y_i - x_i| \ge \frac{n\delta_M}{1-r_m} + \frac{(de+n\delta_M)\,\epsilon_M n}{(1-r_m-n\epsilon_M)(1-r_m)}.$$

Thus, in this way, the row sums of A can be used to analyze the error between y and x. Further, it should be noted that this second bound can be used in analyzing error of theoretical models in which A is not given specifically but described in terms of row sums, etc.

REFERENCES

1. J. N. Franklin, "Matrix Theory," Prentice-Hall, Englewood Cliffs, N.J., 1968.
2. M. Marcus and H. Minc, "A Survey of Matrix Theory and Matrix Inequalities," Allyn & Bacon, Boston, 1964.
3. L. A. Metzler, Stability of multiple markets: The Hicks conditions, *Econometrica* **13** (1945), 277–292.
4. J. M. Ortega, "Numerical Analysis," Academic Press, New York, 1972.
5. E. Seneta, "Nonnegative Matrices," Wiley, New York, 1973.

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Utility Theory and Participation in Unfair Lotteries

Since the development of the expected utility maxim by von Neumann and Morgenstern [2], attempts have been made to provide a rational explanation for the observed behavior of individuals under uncertainty. One aspect of such behavior which will concern us in the paper is the willingness of an individual to purchase insurance and lottery tickets. In this connection it has been pointed out that there is an inconsistency between the usual assumption of risk aversion by individuals and their participation in fair and even unfair lotteries. A fair lottery will never be undertaken by an individual with a concave utility function for wealth which implies risk aversion. However, the purchase of insurance is consistent with such a utility function.

Two alternative explanations have been offered to explain the observed purchase by individuals both insurance and unfair lottery tickets. The first one postulates a "specific utility of gambling" which if added to the "negative" expected utility of a lottery yields a positive value. The other approach suggested by Friedman and Savage [1] hypothesises a utility of wealth function that is intially concave, has a convex segment, and becomes concave once more. Thus, there is a range in between where the individual is willing to participate in an unfair lottery. While both these explanations succeed in rationalizing observed behavior without jettisoning the expected utility maxim, they are not in accord with the theory of consumer choice. In this theory, the utility function in commodity space is derived from the axioms of comparability, transitivity, increasing utility, and diminishing marginal utility.

This paper provides an alternative explanation for the observed behavior of individuals that is consistent with consumer choice theory. The basic hypothesis is that there are certain commodities, mainly luxuries, which are available only in integer amounts. Further, the price of even a single unit of these commodities is large in relation to an individual's budget constraint, with the result that consumption of these commodities implies sacrifice of consumption of other more basic commodities. This "lumpiness" of high value commodities creates a situation where an individual is willing to participate in unfair gambles and yet is behaving consistently with the expected utility maxim.

We give below a simple illustration to show that an individual with a monotone increasing concave utility function on commodity space who acts so as to maximize expected utility may choose to participate in certain unfair

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gambles provided at least one commodity is lumpy and his wealth is in a certain range.

Consider an individual with an additive utility function having only two commodities available for consumption. Let one commodity be perfectly divisible and let the other be a luxury commodity which is lumpy. To avoid getting into irrelevant complexities we shall further assume that the lumpy commodity is available only in quantities of zero or one unit.

Let us suppose commodity 1 is indivisible and only 0 or 1 units of it can be consumed. Let us denote the additive utility function by $u_1(c_1) + u_2(c_2)$, where u_1, u_2 are strictly concave continuously differentiable, monotone increasing functions and c_1, c_2 are the amounts consumed of commodities 1 and 2, respectively. To simplify our notation we will assume: (1) Scale and origin for u_1 and u_2 are chosen such that $u_1(0) = u_2(0) = 0$ and $u_1(1) = 1$; (2) units of the commodities are chosen so that the price of each commodity is unity; (3) we define $u(c_2) = u_2(c_2)$. (to avoid carrying along a cumbersome subscript.)

Let $v(B)$ be the utility associated with wealth B. For $B < 1$, the person has no choice but to consume as much commodity 2 as he can. Thus, $v(B) = u(B)$ for $B < 1$. For $B \geq 1$, he can choose between consuming 0 or 1 unit of commodity 1 (spending the remaining wealth on commodity 2). Thus, $v(B) = \text{Max}\{u(B), 1 + u(B - 1)\}$ for $B \geq 1$.

Let us consider the case where $u(1) < 1$ and $1 + u(B - 1) > u(B)$ for B large enough. Both these conditions are reasonable when commodity 1 (the lumpy commodity) is a luxury good. The first condition says that the individual would prefer to consume some positive amount of commodity 2 and not have commodity 1 rather than have commodity 1 but no commodity 2 at all. The second condition says that provided he has enough wealth, he will wish to have commodity 1 and cut back on his consumption of commodity 2.

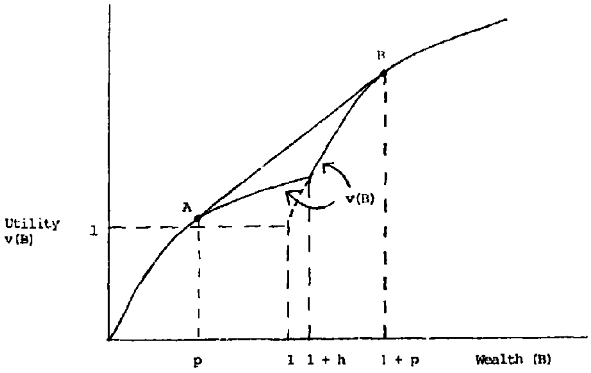

FIGURE 1

If this were not the case, no matter how much wealth he had, the individual would never find commodity 1 attractive.

The utility curve for wealth, $v(B)$ is given by

$$\begin{aligned} v(B) &= u(B), && 0 \leqslant B \leqslant 1 + h. \\ &= 1 + u(B - 1), && B > 1 + h. \end{aligned}$$

For some $h \geqslant 0$. This curve is sketched in Fig. 1.

The utility curve is no longer concave. There is, therefore, a range of wealth $(p, 1 + p)$ in which the individual will find certain fair and even unfair lotteries attractive. (The interval $(p, 1 + p)$ is defined by the unique straight line which is tangent to the utility curve at two points).

REFERENCES

1. M. FRIEDMAN AND L. J. SAVAGE, The utility analysis of choices involving risk, *J. Polit. Econ.* **56** (1948), 279–304.
2. J. VON NEUMANN AND O. MORGENSTERN, "Theory of Games and Economic Behavior," Princeton Univ. Press, Princeton, N.J., 1947.

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Minimal Conditions for Upper-Hemicontinuity of the Choice Correspondence*

We denote by $\varphi(M, r)$ the set of all maximal elements in the constraint set M with respect to a (partial) order r and find the *coarsest* topologies on the domain of φ such that φ is u.h.c.

Let κ be the family of all nonempty compact subsets of a locally compact Hausdorff space X, and denote by ρ the set of all irreflexive binary relations r on X with open graph in $X \times X$. $(x, y) \in r$ is read "x is strictly r-better than y." The *choice correspondence* φ associates with each *constraint* $M \in \kappa$ and each *preference order* $r \in \rho$ the set $\varphi(M, r) = \{z \in M/(x, z) \in r \Rightarrow x \notin M\}$ of all maximal elements in M with respect to r.

φ is usually interpreted as describing the behavior of a rational agent in situation (M, r), at least when $\varphi(M, r) \neq \emptyset$ (e.g., for r transitive); more generally, it associates with each *problem* (M, r) its *solution* $\varphi(M, r)$. If we tolerate a bit of artifice, many problems in economics beyond the usual optimization problems can be put in this format, e.g., problems like finding competitive equilibria or balanced growth paths for an economy, or solutions to algebraic equations or inequalities, fixed points, NASH-equilibria for a game, etc.

The classical question how a change in the problem affects its solution ("sensitivity w.r.t. changes in parameters") can then be studied by studying the continuity properties of φ, or, equivalently, various topologies on the domain and range of φ. In this paper we determine the coarsest topology on the problem space $\kappa \times \rho$ such that φ is upper-hemicontinuous. In other words, we characterize the "most general" modifications to which one can subject a problem (M, r) without significantly enlarging its solution set $\varphi(M, r)$. Intuitively, we shall show in the Theorem below that:

(i) The constraint set M must neither grow nor shrink,

(ii) the preferences r must not become "more vague", i.e., the number of mutually indifferent or incomparable alternatives must not increase (except perhaps "at infinity").

Formally: The *Vietoris* (or "finite") topology T_V on κ has as a basis all sets $[G; G_1, \ldots, G_n] := \{M \in \kappa/M \subset G; M \cap G_i \neq \emptyset, i = 1, \ldots, n\}$ for $G, G_1, \ldots, G_n$

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open in X, $n = 1, 2, \ldots$. The "*compact-open*" topology T_K on ρ has as a basis all sets $\langle K \rangle := \{r \in \rho / r \supset K\}$ for K compact in $X \times X$.

It is easy to see that φ is compact-valued and u.h.c. for the product topology $T_V \times T_K$ on $\kappa \times \rho$ ([2]; the special case for r fixed is known as the "Maximum Theorem" of Berge [1]). Conversely:

THEOREM. (i) *T_V is the coarsest topology on κ such that $\varphi(\cdot, r)$ is u.h.c. on κ for every $r \in \rho$.*

(ii) *Given T_V on κ, T_K is the coarsest topology on ρ such that φ is u.h.c. on $\kappa \times \rho$*

Proof of the theorem. (i) Let $W = [H_0 ; H_1, \ldots, H_n]$ be an open T_V-neighborhood of $M_0 \in \kappa$. For $i = 1, \ldots, n$ choose $M_i \in \kappa$ with $M_i \subset H_i \cap M_0$, and $r_i \in \rho$ with $r_i \supset M_i \times (X \setminus H_i)$. Let $r_0 = \varnothing$.

Then $M_0 \in \{M \in \kappa / \varphi(M, r_i) \subset H_i, i = 0, 1, \ldots, n\} \subset W$.

(ii) We apply the theorem that for a family F of continuous functions on a locally compact Haudorff space there exists a coarsest jointly continuous topology, namely the compact-open topology [3, p. 221, ff.]

$F := \{\varphi(\cdot, r) / r \in \rho\}$ is such a family on the locally compact Hausdorff space (κ, T_V) (cf. [4, Theorem 4.9.12]). Identify F with ρ, define $W(L, G) := \{r \in \rho / \varphi(M, r) \subset G$ for all $M \in L\}$ for G open in X, L compact in (κ, T_V), and choose a nonempty open set $\langle K \rangle$ in (ρ, T_K), K compact in $X \times X$. We have to show that there exist L_i, G_i, such that $\bigcap_{i=1}^{n} W(L_i, G_i) \subset \langle K \rangle$.

By irreflexivity of the preferences and the regularity assumptions made about X, we can cover K by finitely many compact sets of the form $\overline{U}_i \times \overline{V}_i$, each contained in open sets $G_i \times H_i$, with $G_i \cap H_i = \varnothing$. Define $L_i := \{M \in \kappa / M = \{x, y\}$ and either (x, y) or (y, x) is in $\overline{U}_i \times \overline{V}_i\}$. L_i is compact in (κ, T_V) and $r \in W(L_i, G_i)$, all i, implies $r \supset K$. Q.E.D.

Observe that the proof makes essential use of the assumption that X is a locally compact Hausdorff space. The topology T_K on the space of preferences is in general not Hausdorff and is much coarser than the various topologies on ρ that have been considered in general equilibrium theory (for references see [2, p. 108]). Moreover, it can be shown by means of a counterexample that T_K is not, in general, the coarsest topology on ρ such that $\varphi(M, \cdot)$ is u.h.c. on ρ for every $M \in \kappa$.

Remark. Frequently, a preference order r is represented by a continuous real-valued utility or target function u (unique up to strictly monotone increasing transformations). It can be shown that a sequence (r_n) converges to r in (ρ, T_K) iff for some corresponding representation, $u_n \to u$ in the topology of uniform convergence on compacta (i.e. the familiar compact-open topology).

As an application one can show, for example, that "initial segments" of optimal growth programs are insensitive to changes in the length of the planning horizon, without making convexity assumptions on technologies or utilities (Nermuth [5]).

REFERENCES

1. C. BERGE, "Espaces topologiques. Fonctions multivoques," 2ème ed., Dunod, Paris, 1966.
2. W. HILDENBRAND, "Core and Equilibria of a Large Economy," Princeton Univ. Press, Princeton, N. J., 1974.
3. J. L. KELLEY, "General Topology," Van Nostrand, New York, 1955.
4. E. MICHAEL, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951).
5. M. NERMUTH, Sensitivity of optimal growth paths with respect to a change in target stocks or in the length of the planning horizon in a multi-sector model, *J. Math. Econ.* **5** (1978).

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Pitfalls in the Theory of Fairness—Comment

In a recent article, Pazner [2] presents a critique of the theory of fairness. He raises legitimate objections to most of the existing concepts in this theory based on some or all of the following criteria; consistency with Pareto efficiency, horizontal equity, and original position incompatibilities. There are two concepts which meet all these criteria; *egalitarian equivalent* allocations [2] and *just* allocations [1]. Pazner presents a case favoring the former and against the latter. I would like to point out some characteristics of these concepts which could lead to the opposite conclusion.

Prime virtues of the theory of fairness are that it is a symmetric theory of justice and that it requires no externally-imposed interpersonal comparisons. All evaluations of relative advantage and disadvantage are based solely on the preferences of the participants and the solution criteria treat all individuals in a symmetric fashion (an excellent presentation of these properties, relative to alternative conceptions of justice is contained in Varian [4]). The basis of the theory of fairness is the concept of a *fair* allocation—a Pareto efficient allocation at which no individual prefers anyone else's bundle of goods to his own. As Pazner and Schmeidler [3] have shown however, *fair* allocations fail to exist in some economies. Figure 1 contains utility space representations of both two-person and three-person examples for which this is the case. No

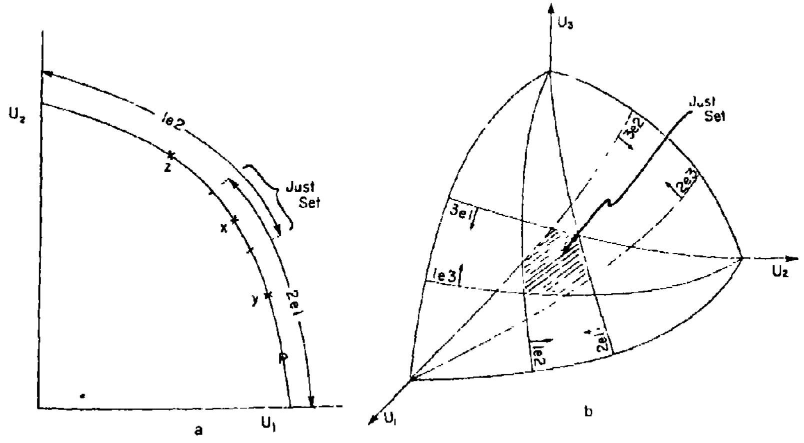

FIG. 1. $i\,e\,j$ indicates that i envies j over the designated portion of the Pareto set.

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matter what concept of equity is used to arrive at an allocation in either case there is going to be at least one instance of someone envying someone else. It does not matter that the concept may not be defined in terms of envy relations—the envy will be present. With this point in mind we address Pazner's arguments.

A CLOSER LOOK AT EGALITARIAN EQUIVALENCE AND JUST ALLOCATIONS

Pazner raises two objections to *just* allocations. The first is based on the informational requirements (individuals must know the consumption bundles and preferences of others to satisfy themselves that an allocation is *just*). However, *egalitarian equivalent* allocations require the same information—there is no difference here.

As a second point, Pazner argues that the "balancedness requirement" of *just* allocations is not normatively appealing; that it may lead in extreme cases to every individual envying every other individual. Figure 1 depicts such extreme cases. The question is; if symmetry is a valued property of justice then what other choice is there? Envy can be reduced only at the expense of treating individuals in an asymmetric fashion. In figure 1(a), allocations y and z both reduce the instances of envy but by placing one or the other of the individuals in a clearly advantageous position. In figure 1(b) or in a general n-person case a similar situation arises. In most cases of course *just* allocations do not require everyone envying everyone else—only that all necessary envy relations be arranged in cycles with no one in an advantageous or disadvantageous position. In fact, *fair* allocations (no envy), when they exist, are always *just*. The same can not be said for *egalitarian equivalence*. Figure 2 shows an example[1] in which both *egalitarian equivalent* and *fair* allocations exist and yet are radically different. This is a four-person exchange economy for which the *egalitarian equivalent* allocation, x, leaves persons 1 and 2 strongly prefering the bundles of persons 3 and 4 respectively. The *fair* and *just* allocation is shown as y.

Besides leading to results such as this, *egalitarian equivalent* allocations are themselves lacking in normative appeal. The justification states that the disadvantaged (persons 1 and 2 in figure 2) should accept their lot because there is a hypothetical economy in which everyone could have equal bundles

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[1] This example originated with Andrew Postlewaite. Vince Crawford made the author aware of it. Individual preferences are chosen so that Pareto efficiency requires persons 1 and 3 to consume only bundles along R_2 and persons 2 and 4 along R_1. Fairness then requires persons 1 and 3 to consume equal bundles and similarily persons 2 and 4. In such a case all four indifference curves can not possibly meet at a common point as required for egalitarian equivalence.

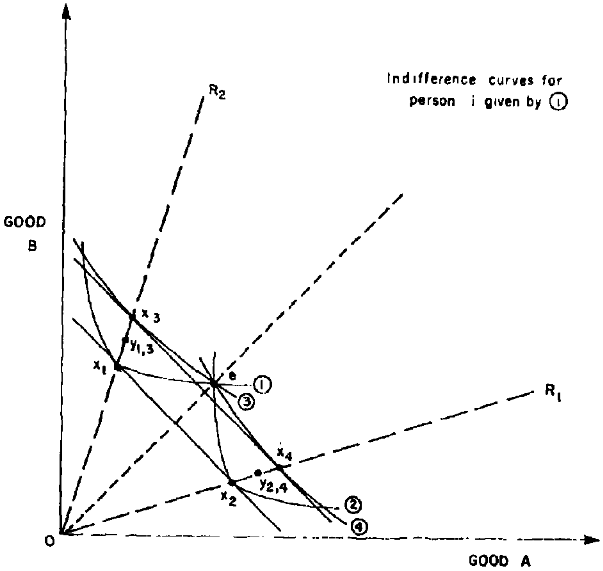

FIGURE 2

of goods giving each the same utility as he now has. In other words it could be *egalitarian equivalent* to leave me in my middle-class life-style envying you living in upper-class splendor if we both were indifferent to trading our positions for a little log cabin in the (now hypothetical) economy of the 1860's. This reasoning would be hard for the disadvantaged to accept when more equitable allocations of goods in the *existing* economy are possible.

SUMMARY

In short, if distributional equity is the goal then *egalitarian equivalence* may lead in an interesting but divergent direction.

REFERENCES

1. T. E. DANIEL, A revised concept of distributional equity, *J. Econ. Theory* **11** (1975), 94–109.
2. E. A. PAZNER, Pitfalls in the theory of fairness, *J. Econ. Theory* 14 (1977), 458–466.
3. E. A. PAZNER AND D. SCHMEIDLER, A difficulty in the concept of fairness, *Rev. Econ. Stud.* **41** (1974), 441–443.

4. H. R. VARIAN, Distributive justice, welfare economics, and the theory of fairness, *Philos. Public Affairs* 4 (1976), 223–247.

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Robustness of the Median Voter Result

In a recent paper [1], Melvin J. Hinich shows that in the presence of a certain kind of stochastic indeterminacy in voter behavior, it need not be a pure strategy equilibrium in a two-candidate election for the candidates to position themselves at the median policy, even if the classical single-peakedness assumptions are otherwise satisfied. He concludes from this that the well-known median voter result is an "artifact." An "artifact," in this context, presumably means small changes or perturbations in the underlying assumptions and parameters of the model can lead to large, discontinuous changes in its behavior, and more specifically that "even a small amount of indeterminateness in voter choices" may lead the candidates to chose strategies far from the median. In Hinich's examples, however, the degree of randomness in the choices of at least some voters seems quite substantial. This is true even of the third example, where it is argued that the candidates will remain at the nonmedian policy $\theta = 0$ even if the parameter δ which characterizes the indeterminism in voter behavior is made arbitrarily small, provided a second parameter ϵ in the utility functions of voters 1 and 2 is "sufficiently small"; for an ϵ which is small enough to keep the candidate equilibrium at $\theta = 0$ will also make the observable choice behavior of voters 1 and 2 nondeterministic over a wide range of possible policy choices, and clearly the relevant measure of the degree of indeterminism present in the example must reflect this behavioral fact, not just the magnitude of the essentially unidentifiable parameter δ. Thus, on any reasonable definition of "smallness," Hinich's three examples all involve considerably more than a "small amount of indeterminateness." The examples demonstrate how different assumptions can lead to different results, but as they stand, they prove nothing one way or the other about the possible artifactuality of the median voter result.

It is still possible, of course, that further analysis of one or another of the examples could yield some insight into the question. In particular, one natural line of attack would be to progressively reduce the degree of randomness in voter choices and examine the limiting behavior of the candidates as voters become "nearly" deterministic. Thus, in the second or third example, we could allow the randomness parameters δ to vary, and then see whether or not the pure strategy equilibria identified by Hinich show any tendency to converge on the median policy as $\delta \to 0$. Unfortunately, however, when the δ become small, the voter response functions P_i eventually become

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nonconcave (over the relevant range), and pure strategy equilibria can no longer be shown to exist. Hence, in order to characterize the strategy choices of candidates for small δ, it becomes necessary to work with some more general solution concept, such as that of a mixed-strategy equilibrium. It is no simple task to characterize such a mixed-strategy equilibrium (or even to establish its existence, in the first example), for this type of general continuous-strategy game, however.

Nevertheless, with one relatively minor and plausible change in one of the assumptions, we can obtain a general result which characterizes the limiting behavior of candidate equilibria in a wide range of situations of the type Hinich considers. In particular, let us suppose the candidates are interested in maximizing their respective probabilities of winning, rather than their expected vote shares. (Hinich himself seems to consider this a more realistic objective, but argues that it is essentially equivalent to his expected-vote criterion.) Then, as the degree of indeterminism (the δ) becomes small, any policy different from the median will eventually become a dominated strategy. We can thus expect the candidates to choose policies close to the median when voter behavior is nearly deterministic. Certainly, no candidate has any positive incentive to do otherwise, since dominated nonmedian strategies can be deleted without affecting the value of the game for either player, and both candidates have optimal mixed strategies in which only the undominated policies close to the median are played with positive probability. These facts would seem to argue strongly for the robustness of the median voter result (at least relative to this class of examples), rather than its artifactuality.

To show that nonmedian policies are dominated strategies, consider a set of two-player games G_λ, $\lambda \in [0, 1]$, in which two candidates attempt to maximize their respective probabilities of winning a majority of votes from a fixed set of voters. There is a fixed interval $I \subset R$ of feasible policies, which is also the set of pure strategies available to each player in every G_λ. Each voter i has a continuous (real-valued, logarithmic) utility function U_i on I, which is single-peaked, i.e., has a unique maximum S_i in I, and is strictly increasing (resp. decreasing) at all feasible $x < S_i$ (resp. $x > S_i$). Let the number n of voters be odd, so there is a unique median $\mu \in I$ of the S_i. The probabilistic choice behavior of each voter i is characterized by a parameter $\delta_i \geqslant 0$, such that in the game G_λ, i votes deterministically (with probability 1) for x over y if $U_i(x) - U_i(y) > \lambda\delta_i$, but may vote stochastically otherwise. Thus as $\lambda \to 0$, voter behavior becomes increasingly deterministic. $\Pi_\lambda^j(x, y)$ is the probability, in game G_λ, that candidate j wins a strict majority of votes when he plays the pure strategy x and his opponent plays the pure strategy y. A policy (or strategy) q is *dominated* in G_λ if for each candidate j there exists a strategy p such that $\Pi_\lambda^j(p, x) \geqslant \Pi_\lambda^j(q, x)$ for all x, with strict inequality for some x. We then have:

Proposition. *For any policy θ distinct from the median μ, there exists a $\lambda^* > 0$ such that θ is dominated in G_λ, for all $\lambda \in [0, \lambda^*]$.*

Proof. Suppose $\theta < \mu$, and consider a small neighborhood $N_\epsilon(\mu) = (\mu - \epsilon, \mu + \epsilon)$ of the median μ, for some $0 < \epsilon < \mu - \theta$. If $S_i \geq \mu$, then from single-peakedness $\inf_{x \in N_\epsilon(\mu)} U_i(x) > U_i(\theta)$, so there exists a $\lambda_i > 0$ such that $[U_i(x) - U_i(\theta)] > \lambda_i \delta_i$ for all $x \in N_\epsilon(\mu)$.

Moreover from the continuity and single-peakedness of the U_i, we can clearly find, for each voter i, a number $\lambda_i' > 0$ such that $[U_i(\mu) - U_i(x)] > \lambda_i' \delta_i$ holds for all $x \leq \mu - \epsilon$ if $S_i \geq \mu$, or for all $x \geq \mu + \epsilon$ if $S_i \leq \mu$. Let λ^* be the smallest of these λ_i and λ_i'.

Now consider the two possible strategies θ and μ for candidate j, against an arbitrary strategy x for the other candidate, in the game G_λ for some $\lambda \in [0, \lambda^*]$. If $x \in N_\epsilon(\mu)$, every voter i for whom $S_i \geq \mu$ will vote deterministically for x over θ, and since μ is the median of the S_i's and n is odd, these voters constitute a majority. Hence $\Pi_\lambda^j(\theta, x) = 0$, so $\Pi_\lambda^j(\theta, x) \leq \Pi_\lambda^j(\mu, x)$. The remaining possibility is $x \notin N_\epsilon(\mu)$; if $x \leq \mu - \epsilon$ (resp. $x \geq \mu + \epsilon$), the set of voters for whom $S_i \geq \mu$ (resp. $S_i \leq \mu$) vote deterministically for μ over x, and since μ is median either of these sets again constitutes a majority, so $\Pi_\lambda^j(\mu, x) = 1$. Thus $\Pi_\lambda^j(\mu, x) \geq \Pi_\lambda^j(\theta, x)$ for any such x, and clearly for some x (for example, $x = \theta$) the inequality will be strict. Hence θ is dominated by μ in G_λ, for all $\lambda \in [0, \lambda^*]$. Q.E.D.

It thus seems somewhat premature to conclude that the median voter result is artifactual or unstable in the presence of small amounts of voter indeterminacy. The available evidence points in the opposite direction.

REFERENCE

1. M. Hinich, Equilibrium in spatial voting: The median voter result is an artifact, *J. Econ. Theory* **16** (1977), 208–219.

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Erratum

Volume **18**, Number 1 (1978) in the article, "A Multistage Game Representation of Sophisticated Voting" by Richard D. McKelvey and Richard G. Niemi, pp. 1–22:

On page 5, the right-hand table of Player 2 should read:

Player 2

1	0	1
0	a_2	Γ_4
1	Γ_4	Γ_4

RICHARD D. MCKELVEY

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[1] Boldface numbers indicate the appropriate volume: lighface numbers indicate pagination.

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Management Production Functions and the Theory of the Firm

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1. INTRODUCTION

Management is often said to be the one factor that limits the size of firms. Now management is itself subject to the economies of the division of labor. Through organization the indivisibility of the single owner–manager may be overcome. If it is believed that increasing size of the organization burdens every productive worker with an ever increasing number of administrators per production worker, then this claim is shown to be wrong both on theoretical [1] and empirical grounds [6]. In fact, looking at business organizations in this way, the opposite result would appear to be true: There is no optimal size of the firm: almost any size is equally efficient in terms of administrative overhead per worker.

There is, however, one piece missing: How is the productivity of workers affected by the size of the organization? Is there possibly a systematic relationship between the span of control and organization size [6]? In any case, how is the optimal span of control determined? Why should it be constant?

To answer these and related questions, an economic theory of management is needed. The purpose of this paper is to supply one. Managerial control will be treated as an intermediate product of the firm. At every administrative level, control from above is used as an input and combined with labor to produce supervision or control for the next lower level. This simple idea will be worked out with that old war horse, the Cobb–Douglas production function, to derive some results which are simple and yet interesting. A solution will be given to the problems of the optimal size of firms, the optimal structure of its administration, and in particular, the optimal span of control.

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2. THE PRODUCTION FUNCTION

Consider an organization with well-defined administrative levels $r = 1, \ldots, R$, where R denotes the presidential or top level. Let $r = 0$ be the level of production workers or operatives.

How shall we describe the output of managers? Managers at level r produce "supervision" or "managerial control" y_r of managers at the next lower level $r - 1$. They produce this managerial control with their own labor x_r and with supervision by their own superiors y_{r+1}.

Each office is a black box whose inputs are, respectively, 1 unit of labor, y_{r+1}/x_r units of supervision by the next level of managers.

The output per office at hierarchical level r is thus

$$x_r f_{r+1}\left(\frac{y_{r+1}}{x_r}\right) = y_r . \tag{2.1}$$

Through successive substitution

$$y_0 = x_0 f_1\left(\frac{x_1}{x_0} f_2\left(\frac{x_2}{x_1} f_3\left(\cdots f_R\left(\frac{x_R}{x_{R-1}}\right)\cdots\right)\right)\right).$$

Here y_0 is management's output of "effective labor units" created through inputs of operative labor x_0 and of supervisory labor $x_1, \ldots, x_R$ at the various supervisory levels r.

In a slightly more general formulation output y_r at each level is a function of labor and supervisory input for that level

$$y_r = F_{r+1}(x_r , y_{r+1}).$$

Operative labor in efficiency units produced by management is then

$$y_0 = F_1(x_0 , F_2(x_0 , \ldots, F_R(x_{R-1} , x_R), \ldots)). \tag{2.2}$$

Final output is the result of a combination of this effective labor y_0 with capital (machines) z.

$$q = z f_0\left(\frac{y_0}{z}\right).$$

Machines are thus in the same relationship to workers as subordinates are in relation to supervisors.

The production function is linear homogeneous

$$q = z f_0\left(\frac{x_0}{z} f_1\left(\frac{x_1}{x_1} f_2\left(\cdots x_{R-1} f\left(\frac{x_R}{x_{R-1}}\right)\cdots\right)\right)\right) \tag{2.3}$$

in the variables $z, x_0 , x_1 , \ldots, x_R$.

The more general production function

$$q = F_0(z, F_1(x_0, F_2(\cdots F_R(x_{R-1}, x_R) \cdots))) \tag{2.4}$$

is linear homogeneous when all the F_r are linear homogeneous. If, however, the presidential input is fixed at unity

$$x_R = 1,$$

the production function for the organization in terms of the variable inputs $z, x_0, x_1, \dots, x_{R-1}$ is not homogeneous.

From now on we specify the functions F_r to be Cobb–Douglas

$$\begin{aligned} F_0 &= b_0 z^{\gamma} y_0^{\delta}, \\ F_r &= b(r)\, x_{r-1}^{\alpha} y_r^{\beta}, \qquad r = 1, \dots, R - 1. \end{aligned} \tag{2.5}$$

Notice that we assume the output elasticities α and β to be the same at all administrative levels. This may be justified by arguing that administration is administration no matter at what level. We allow, however, the output quantities $b(r)$ per unit inputs to be different at different levels. Now the composite production function of the organization is

$$q = b_0 b(1)^{\delta} b(2)^{\delta\beta} \cdots b(R)^{\delta\beta^{R-1}} z^{\gamma} x_0^{\alpha\delta} x_1^{\alpha\delta\beta} \cdots x_{R-1}^{\alpha\delta\beta^{R-1}}$$

or

$$q = b_R \cdot z^{\gamma} \prod_{r=0}^{R-1} x_r^{\alpha\delta\beta^r}, \tag{2.6}$$

where

$$b_R = b_0 \prod_{r=1}^{R} b(r)^{\delta\beta^{r-1}}.$$

The composite Cobb–Douglas production function (2.6) is homogeneous of degree η_R in the variable inputs $z, x_0, \dots, x_{R-1}$ where

$$\eta_R = \gamma + \alpha\delta \sum_{r=0}^{R-1} \beta^r = \gamma + \alpha\delta(1 - \beta^R)/(1 - \beta). \tag{2.7}$$

When all the F_r are linear homogeneous, $\alpha + \beta = 1$, $\gamma + \delta = 1$, then

$$\eta_R = 1 - \delta + \delta(1 - \beta^R) = 1 - \delta\beta^R. \tag{2.8}$$

The degree of homogeneity of the production function is less than one and depends on the organization's top level R (to be called its organizational level). The degree of homogeneity approaches unity as the organizational level is increased.